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Stability synthesis of a plane dynamic system

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KNOWING the differential second-order equation with two parameters we state two problems of synthesis of a dynamical system on the plane having a rest point which is globally asymptotically stable. The first problem consists in the determination of a class of one of these parameters, the class of the second being given; while the second problem consists in the determination of the optimal parameter in the sense of the maximum decrease of a positive definite function prescribed on the plane.

Dla danego równania różniczkowego drugiego rzędu o dwu parametrach zbadano dwa zadania syntezy układu dynamicznego na płaszczyźnie, mającego punkt spoczynku globalnie asymptotycznie stateczny. Pierwsze z tych zadań polega na doborze klasy jednego z tych parametrów przy zadanej klasie drugiego, drugie zadanie — na wyborze parametru optymalnego w sensie największego malenia pewnej dodatnio określonej funkcji zadanej na płaszczyźnie.

Задание дифференциального уравнения второго порядка с двумя параметрами определяет две задачи синтеза динамической системы на плоскости, обладающей особой точкой асимптотически устойчивой в целом. Первая из этих задач состоит в подборе класса одного из этих параметров при задании класса второго из них. Вторая задача сводится к выбору оптимального параметра по критерию наибольшего уменьшения некоторой положительно определенной функции, заданной на плоскости.

1. THE AIM of the paper is the choice of characteristics of a mechanical system described by a mathematical model, so that the global asymptotic stability of the state of rest is ensured, and the determination of some of the above characteristics ensuring that the above global asymptotic stability is optimally realised to within an assumed function (the Lapunov function).

The mathematical model is taken in the form of the differential equation

$$(*) \qquad \ddot{y}+F(y, \dot{y}, \alpha, \beta) = 0,$$

where F is a scalar function, α and β are parameters. In what follows, we shall make use of a system of differential equations of first order.

2. Assume that the equation

$$(1) \qquad \dot{x} = f(x, \alpha, \beta), \quad x \in \mathscr{R}^2,$$

has the form (*), where

$$f_1 = x_2, \quad f_2 = \varphi(x, \alpha)+\psi(x, \beta),$$

i.e., we assume that f_2 is a sum of the functions φ and ψ each depending on one parameter only. We assume, moreover, that φ and ψ are continuous functions satisfying Lipschitz condition with respect to x and of class C^1 with respect to α and β.

Let h be a function defined on $\mathscr{R}^2$ with the following properties:

(i) $h \in C^1$;

(ii) $h(x) \geqslant 0$ for every x and $h(x) = 0 \Leftrightarrow x = (0, 0)$;

(iii) Consider the equation $h(x) = C$, $C > 0$, and let $D' \overset{\text{df}}{=} \{x : h(x) = C\}$. Then there exists a compact connected set D such that a) $D' = Fr\, D$, b) $\mathring{D} \ni (0, 0)$, where $\mathring{D}$ denotes the interior of the set D;

(iv) $D_1 \subset \mathring{D}_2$ for arbitrary C_1 and C_2 such that $C_1 < C_2$.

3. We denote by $\mathscr{F}$ the set of all functions g defined on $\mathscr{R}^2$ with values in $\mathscr{R}$ such that g belongs to the class of piecewise continuous functions and satisfying the Lipschitz condition in the continuity domains.

Consider two functions $u_1, u_2 \in \mathscr{F}$ such that

$$-\infty < u_1(x) \leqslant u_2(x) < \infty$$

for every x. We defined the family of sets

$$\mathfrak{A} : x \to A_x, \quad A_x \overset{\text{df}}{=} \langle u_1(x), u_2(x) \rangle. \tag{2}$$

Similarly

$$\mathfrak{B} : x \to B_x, \quad B_x \overset{\text{df}}{=} \langle v_1(x), v_2(x) \rangle, \tag{3}$$

where also $v_1, v_2 \in \mathscr{F}$,

$$-\infty < v_1(x) \leqslant v_2(x) < \infty$$

for every x.

DEFINITION 1. We say that α and β are admissible functions if they belong to the set $\mathscr{F}$ and if $\alpha(x) \in A_x$ and $\beta(x) \in B_x$ for every $x \in \mathscr{F}^2$; A_x and B_x are defined by (2) and (3), respectively.

4. The synthesis problem mentioned in Sec. 1 is described by the following two conditions:

$$\bigwedge_x \frac{dh(x)}{dt}\bigg|_{\dot{x}=f(x,\alpha(x),\beta(x))} \leqslant 0, \tag{4}$$

where $\alpha \in \mathscr{F}$, β is an arbitrary admissible function (the synthesis condition) and

$$\bigwedge_x \frac{dh(x)}{dt}\bigg|_{\dot{x}=f(x,\alpha^*(x),\beta(x))} = \inf_\alpha \frac{dh(x)}{dt}\bigg|_{\dot{x}=f(x,\alpha(x),\beta(x))}, \tag{5}$$

where α and β are arbitrary admissible functions, α satisfying condition (4) (the condition of optimal synthesis).

5. The synthesis problem 1. Consider a family of sets $\mathfrak{B}$. It is required to determine a family of sets

$$\mathfrak{P} : x \to P_x, \quad P_x \subset \mathscr{R}$$

such that for every function $\alpha \in \mathscr{F}$ and such that for every x $\alpha(x) \in P_x$, the condition (4) is satisfied (the synthesis condition) independently of the choice of β (β is an admissible function). This α will be called the function solving Problem 1 of the synthesis.

Consider the Eq. (1) and let α and β be two functions such that α and $\beta \in \mathscr{F}$ and let $\bar{f}$ be the following function:

$$\bar{f}(x) \overset{df}{\equiv} f(x, \alpha(x), \beta(x)).$$

Suppose that G is the set of all points of discontinuity of $\bar{f}$ in $\mathscr{R}^2$ and consider a condition in the form

$$H(x, \alpha(x), \beta(x)) \leqslant 0, \quad x \in \mathscr{R}^2 - G. \tag{6}$$

We say that condition (6) is determined on $\mathscr{R}^2$ if for every $x \in \mathscr{R}^2$ we have

$$H(x, \lim_{x_i \to x} \alpha(x_i), \lim_{x_i \to x} \beta(x)) \leqslant 0, \quad x_i \in \mathscr{R}^2 - G, \tag{7}$$

and the inequality (7) has to hold for all possible $\lim\limits_{x_i \to x} \alpha(x_i)$, $\lim\limits_{x_i \to x} \beta(x_i)$ in x; we shall write

$$H(x, \alpha(x), \beta(x)) \leqslant 0, \quad x \in \mathscr{R}^2.$$

THEOREM 1. *Consider the Eq.* (1) *and assume that the following data is given*: *the family of sets* $\mathfrak{B}$ *in accordance with* (3) *and a function h with the properties* (i)–(iv). *Let* β *be an admissible function* (Definition 1); *then a necessary and sufficient condition that the function* α, $\alpha \in \mathscr{F}$, *satisfies the synthesis condition* (4) *is the following*(1)

$$\alpha(x) \in \{z\colon\ h_{|1}(x)x_2 + h_{|2}(x)\varphi(x, z) + h_{|2}(x)\psi(x, \beta^*(x)) \leqslant 0, \tag{8}$$

where

$$h_{|2}(x)\psi(x, \beta^*(x)) = \sup_{\beta} \left(h_{|2}(x)\psi(x, \beta(x))\right), \tag{9}$$

$x \in \mathscr{R}^2, \quad x \neq (0, 0)$.

P r o o f. N e c e s s i t y. Let α ($\alpha \in \mathscr{F}$) be a function satisfying the synthesis condition (4) on $\mathscr{R}^2$ and $\bar{x}$ an arbitrary fixed point. Defining the value of the function h at $\bar{x}$,

$$h(\bar{x}) = C, \quad C > 0,$$

we obtain the set

$$D' = \{x\colon h(x) = C\},$$

constituting the boundary of a subset $D \in \mathscr{R}^2$ [see the properties (ii) and (iii) of the function h]. In view of (i), at every point of $x \in D'$, grad $h(x)$ is defined and (4) can be written in the form of the scalar product

$$\operatorname{grad} h(x) \cdot f(x, \alpha(x), \beta(x)) \leqslant 0,$$

where β is an admissible function, $x \in D'$, $C > 0$, or else

$$\sup_{\beta} \left(\operatorname{grad} h(x) \cdot f(x, \alpha(x), \beta(x))\right) \leqslant 0, \quad x \in \mathscr{R}^2, x \neq (0, 0). \tag{10}$$

In view of the form of the function f, we obtain from (10)

$$h_{|1}(x)x_2 + h_{|2}(x)\varphi(x, \alpha(x)) + \sup_{\beta} \left(h_{|2}(x) \cdot \psi(x, \beta(x))\right) \leqslant 0, \quad x \in \mathscr{R}^2, \quad x \neq (0, 0),$$

i.e., the condition of the Theorem.

S u f f i c i e n c y. Let $\bar{\beta}$ be an admissible function and let α, $\alpha \in \mathscr{F}$, be a function satisfying at every point x condition (8) and not satisfying the synthesis condition (4), i.e., there is a point $\bar{x}$ such that

$$\left.\frac{dh(x)}{dt}\right|_{\dot{x} = f(\bar{x}, \alpha(\bar{x}), \bar{\beta}(\bar{x}))} = e > 0, \quad \alpha \in \mathscr{F}. \tag{11}$$

(1) $h_{|i} = \partial h / \partial x_i$

where $\overline{\beta}$ is admissible. As before, expressing (11) in the form

$$\operatorname{grad} h(\overline{x}) f(\overline{x}, \alpha(\overline{x}), \overline{\beta}(\overline{x})) = e_1 > 0,$$

we obtain, taking into account the form of f,

$$h_{|1}(\overline{x})\overline{x}_2 + h_{|2}(\overline{x})\varphi(\overline{x}, \alpha(\overline{x})) + h_{|2}(\overline{x})\psi(\overline{x}, \overline{\beta}(\overline{x})) = e_1.$$

For an arbitrarily selected admissible function $\overline{\beta}$, we have

$$h_{|2}(\overline{x})\psi(\overline{x}, \overline{\beta}(\overline{x})) \leqslant \sup_{\beta} \big(h_2(\overline{x})\psi(\overline{x}, \beta(\overline{x}))\big),$$

whence, in view of (9),

$$h_{|1}(\overline{x})\overline{x}_2 + h_{|2}(\overline{x})\varphi(\overline{x}, \alpha(\overline{x})) + h_{|2}(\overline{x})\psi(\overline{x}, \beta^*(\overline{x})) \geqslant e_1 > 0,$$

which contradicts (8). This proves the sufficiency of condition (8) and ends the proof of the theorem.

Remark. β^* is, in the sense of the synthesis condition, "the worst" function among the admissible functions β.

It was assumed in Sec. 1 that ψ has a continuous first derivative in β. Hence, in view of (9), we have

COROLLARY. If for every x and every $\beta \in \langle v_1(x), v_2(x)\rangle$ we have

$$\frac{\partial\psi(x, \beta)}{\partial\beta^*} \neq 0,$$

then

$$\beta^*: x \to v(x), \quad \text{where} \quad v(x) = v_1(x) \quad \text{or} \quad v_2(x).$$

If the inequality (8) has a solution (with respect to z) for every x, then Theorem 1 states the conditions defining the considered family of sets $\mathfrak{P}$. In fact, denoting for every x the set of all z satisfying (8) by P_x, we obtain the family of sets $\mathfrak{P}$. In general, P_x may be a class of disjoint sets

$$P_x = \bigcup_i P_x^i, \tag{12}$$

elements of which are closed numerical intervals

$$P_x^i = \langle p_1^i(x), p_2^i(x)\rangle. \tag{13}$$

Their boundaries satisfy the equations

$$\left.\begin{aligned} h_{|1}(x)x_2 + h_{|2}(x)\varphi(x, p_1^i(x)) + h_{|2}(x)\psi(x, \beta^*(x)) = 0\\ h_{|1}(x)x_2 + h_{|2}(x)\varphi(x, p_2^i(x)) + h_{|2}(x)\psi(x, \beta^*(x)) = 0\end{aligned}\right\}\text{(a)} \quad \text{or} \quad \left.\begin{aligned} p_1^i(x) = -\infty,\\ p_2^i(x) = \infty,\end{aligned}\right\}\text{(b)}. \tag{14}$$

Thus, if the set of function $\mathfrak{P}$ is defined [by (8)] and there exists a function α such that

1 $\qquad \alpha(x) \in P_x \quad$ for $\quad$ every x,

2 $\qquad \alpha \in \mathscr{F}$,

then the synthesis problem has a solution.

6. The synthesis problem 2. Consider two given families of sets $\mathfrak{A}$ and $\mathfrak{B}$. It is required to determine the optimal, in the sense of the condition (5) (the optimal synthesis), function α.

Consider a family of sets $\mathfrak{P}$ defined by Theorem 1 and let $\mathfrak{A}$ be a family of sets defined by (2). We assume

$$(15)\qquad \mathfrak{R}: x \to R_x, \quad R_x = A_x \cap P_x, \quad x \in \mathscr{R}^2.$$

In accordance with (12) and (13)

$$(16)\qquad R_x = \bigcup_i R_x^i, \quad R_x^i = \langle r_1^i(x), r_2^i(x)\rangle, \quad R_x^i \cap R_x^j = \emptyset, \quad j \neq i.$$

Moreover, we assume

$$(17)\qquad r_1(x) \overset{\text{df}}{=} \sup_i r_1^i(x), \quad r_2(x) \overset{\text{df}}{=} \inf_i r_2^i(x).$$

THEOREM 2. *Consider Eq.* (1) *and let there be given families of sets* $\mathfrak{A}$ *and* $\mathfrak{B}$ *and a function h. Assume that the conditions of Theorem* 1 *are satisfied and the set* R_x *is non-empty for every* x. *Then the necessary and sufficient condition that* α^* *be optimal on* $\mathscr{R}^2$ *is that*

$$(18)\qquad \alpha^* = \begin{cases} r_1(x), & x: h_{|2}(x)x_2 < 0, \\ r_2(x), & x: h_{|2}(x)x_2 > 0; \end{cases}$$

the above is a suffcient condition that α^* *be one of the functions solving the synthesis problem* 1.

P r o o f. Necessity. Let α^* be a function defined on $\mathscr{R}^2$ such that $\alpha^*(x) \in R_x$ for every x; the set R_x is by assumption non-empty. Let β be an arbitrary admissible function. If α^* satisfies condition (5) then

$$(19)\qquad \left.\frac{dh(x)}{dt}\right|_{\dot{x}=f(x,\alpha^*(x),\beta(x))} = \inf_\alpha \left.\frac{dh(x)}{dt}\right|_{\dot{x}=f(x,\alpha(x),\beta(x))}, \quad x \in \mathscr{R}^2, \quad x \neq (0,0),$$

or, equivalently (see the proof of Theorem 1)

$$(20)\qquad \operatorname{grad} h(x)\cdot f(x, \alpha^*(x), \beta(x)) = \inf_\alpha\left(\operatorname{grad} h(x)\cdot f(x, \alpha(x), \beta(x)\right), \quad x \in \mathscr{R}^2, x \neq (0,0).$$

In view of the form of f in Eq. (1), we obtain from (20)

$$(21)\qquad -h_{|2}(x)x_2\,\alpha^*(x) = \inf_\alpha\left(-h_{|2}(x)x_2\,\alpha(x)\right), \quad x \in \mathscr{R}^2, x \neq (0,0).$$

Thus, if α^* satisfies (21), it has the form (18). This fact proves the necessity.

The sufficiency of condition (18) follows immediately. In fact, if α^* has the form (18), it satisfies (21) and hence also (20) and (19).

Thus (18) is the necessary and sufficient condition of the optimality on $\mathscr{R}^2$ of the function α^*.

The regularity conditions for functions appearing in Eqs. (14a) imply that p_j^i ($i = 1, 2, \ldots, j = 1, 2$) belong to the class of piecewise continuous functions. Thus it follows from the definition of the family of sets $\mathfrak{A}$, $u_1, u_2 \in \mathscr{F}$ and the definition of $\mathfrak{R}$ [cf. (15), (16) and (17)] that $r_1, r_2 \in \mathscr{F}$ and therefore $\alpha^* \in \mathscr{F}$. Since $\alpha^*(x) \in P_x$ for every x [cf. (15)], α^* is one of the functions solving the synthesis problem 1 Q.E.D.

COROLLARY. α^* defined by (18) is optimal in the sense of condition (5) function α for every admissible function β.

7. The method of formulation of the synthesis problems in Secs. 5 and 6 implies the statement of the above Theorems and the order in the synthesis procedure: 1 — determination of the set of all functions α satisfying the synthesis condition (the synthesis problem), 2 — the choice of the optimal function α^* from this set (the problem of optimal synthesis).

Problem 2 can be formulated independently of Problem 1: having Eq. (1), the families of sets $\mathfrak{A}$ and $\mathfrak{B}$ and the function h, it is required to determine an admissible function α^* satisfying the condition

$$\frac{dh(x)}{dt}\bigg|_{\dot{x}=f(x,\alpha^*(x),\beta^*(x))} = \inf_{\alpha}\sup_{\beta} \frac{dh(x)}{dt}\bigg|_{\dot{x}=f(x,\alpha(x),\beta(x))}, \qquad x \in \mathscr{R}^2, x \neq (0,0),$$

where α, β are admissible functions. If, moreover,

$$\frac{dh(x)}{dt}\bigg|_{\dot{x}=f(x,\alpha^*(x),\beta^*(x))} \leqslant 0, \qquad x \in \mathscr{R}^2, x \neq (0,0),$$

where β is admissible, then α^* is optimal in the sense of condition (5).

8. Independently of the solution of both problems of synthesis there remains an open question whether Eq. (1) completed by the functions α and β, where α is a function solving the problem 1 or 2 of the synthesis and β is an admissible function, defines on $\mathscr{R}^2$ a dynamical system [1] — the set G of the discontinuity points of $\bar{f}$ (cf. Sec. 5) can have a structure such that general conditions ensuring that (1) defines a dynamical system are not satisfied.

In what follows we assume the following condition.

C o n d i t i o n. 1. If α is a function solving the synthesis problem (1 or 2) and β is an arbitrary admissible function, then Eq. (1) defines on $\mathscr{R}^2$ a dynamical system.

9. Let [cf. (iii)]

$$D'_c \stackrel{\mathrm{df}}{=} \{x\colon h(x) = C, C > 0\}$$

and

$$\mathscr{A}_1 \stackrel{\mathrm{df}}{=} \{\alpha\colon \{x\colon \frac{dh(x)}{dt}\bigg|_{\dot{x}=f(x,\alpha(x),\beta(x))} = 0, \quad \alpha-\text{solution}, \quad \beta-\text{adm}\} \supset D'_c\},$$

where $\mathscr{A}_1$ may be an empty set [cf. (iii)] moreover, is the complement of $\mathscr{A}_1$ to the set of functions solving problem 1 or 2.

Assume that condition 1 is satisfied. Then, if $\alpha \in \mathscr{A}$ and β is admissible, then the set $h = 0$ does not contain entire trajectories of Eq. (1).

A function h with the properties (i)–(iv) is a positive definite function with the derivative $\dot{h}$ [in view of (1)] which is non-positive [cf. conditions (4) and (5)].

Thus, if $\alpha \in \mathscr{A}$ and β is admissible, then the conditions of La Salle's theorem [2] on the global asymptotic stability of the zero solution are satisfied. Moreover, the following theorems implied by it and by Theorems 1 and 2, are true.

THEOREM 3. *Consider Eq.* (1) *and let there be given: the family of sets* $\mathfrak{B}$ *in accordance with* (3) *and a function h with the properties* (i)–(iv). *Suppose that* β *is an arbitrary admissible function. Then the necessary and sufficient condition that the zero solution* (0, 0) *be globally asymptotically stable is that the function* α *satisfies condition* (8) *and* $\alpha \in \mathscr{A}$.

THEOREM 4. *Consider Eq.* (1) *and let there be given: the families of sets* $\mathfrak{A}$ *and* $\mathfrak{B}$ *and a function h. Assume that the conditions of Theorem* 3 *are satisfied and the set* R_x *is nonempty for every x. Then, if* $\alpha^* \in \mathscr{A}$ *where* α^* *is the function defined by* (18), *then the zero solution* (0, 0) *of Eq.* (1) *is globally asymptotically stable and this global asymptotic stability is optimal in the sense of condition* (5).

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