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Consolidation of a viscoelastic semi-space in the plane state of strain

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WE CONSIDER the consolidation of a viscoelastic semi-space without aging with a boundary permitting filtration and loaded by arbitrary normal and tangential tractions. It is assumed that the rheological properties of the skeleton are different in the processes of dilatational, shear and due to the fluid pressure strains. The constitutive relations of the medium are taken in an integral form. Exact solutions are derived for the equations of the theory of consolidation by means of the Fourier and Laplace integral transforms. The total stresses, the displacements of the skeleton and the pressure of the fluid are presented in the form of improper integrals.

W pracy rozważono konsolidację półprzestrzeni lepko-sprężystej bez starzenia o brzegu przepuszczalnym, obciążonym na brzegu dowolnymi siłami normalnymi i stycznymi. Przyjęto, że własności reologiczne szkieletu w procesie odkształcenia objętościowego, postaciowego i wynikającego z ciśnienia cieczy są odmienne. Równania konstytutywne ośrodka zapisano w postaci całkowej. Otrzymano ścisłe rozwiązanie równań teorii konsolidacji stosując transformacje całkowe Fouriera i Laplace'a. Naprężenia całkowite, przemieszczenia szkieletu i parcie cieczy wyrażone są w postaci całek niewłaściwych.

В работе рассмотрен процесс консолидации без старения вязко-упругого полупространства с проницаемой поверхностью. Краевая нагрузка состоит из произвольных нормальных и касательных усилий. Предполагается, что реологические свойства грунтового скелета различны в процессах объемного деформирования, формоизменения и при воздействии давления жидкости в порах скелета. Определяющие уравнения среды записаны в интегральном виде. Получены точные решения уравнений теории консолидации, основанные на применении интегральных преобразований Фурье и Лапласа. Полные напряжения, перемещения грунтового скелета и напор жидкости выражены в виде несобственных интегралов.

1. Introduction

THE RICH engineering experience and numerous experimental papers dealing with strains in soils subject to action of external loadings prove that all soils exhibit instantaneous strain the magnitude of which depends on the type of the soil and the loading; soils subject to a prolonged loading exhibit an increase of strain in time.

A change of the strain in time may be due both to an outward filtration of the fluid present in the pores of the soil and to the rheological properties of its skeleton.

A theoretical description of the above phenomenon of consolidation was considered by numerous authors. Most of them regarded the skeleton as a linear elastic medium [1, 4].

In the course of development of the theory of consolidation, the rheological properties of the skeleton were taken into account. We mention here first of all BIOT's papers [2, 3], FLORIN's [5] and ZARECKI's [10].

The last author wrote the physical equations of the rheological porous medium by means of Volterra's integral operators of second kind. Assuming the that skeleton of the soil exhibits the same creep properties during the dilatational strain and the strain due to the action of the fluid in the pores, he obtained for the case without aging a solution of the equations by means of successive approximations.

ZAJĄC [8, 9] applied Biot's theory to some selected mechanical problems of rocks in the case of the standard model.

In this paper, we assume that the rheological properties of the skeleton in the course of the dilatational, shear and due to the fluid pressure strains, are different and we shall present an exact solution of the equations of the theory of consolidation. We shall base on the constitutive equations in the integral form.

2. The set of equations of the theory of consolidation

We consider a quasi two-phase medium consisting of a porous viscoelastic skeleton and fluid in the pores. We make the following assumptions:

1. The skeleton is isotropic and homogeneous;

2. The viscoelastic skeleton without aging has different creep properties in the course of dilatational, shear and due to the fluid pressure strains;

3. The porosity of the skeleton is statistically homogeneous;

4. The physical relations are linear;

5. The fluid is filtrated through the pores of the skeleton according to Darcy's law and the filtration coefficient k_φ is constant.

We shall consider the problem of consolidation of the medium in the plane state of strain. For a porous material the skeleton of which exhibits rheological properties without aging, the skeleton physical laws can be written in the form of the following integral relations expressing the Boltzmann hereditary principle [10]:

the law of shear

$$e^s_{ij} = \frac{1}{2G}\left[S_{ij}(t) + \int_0^t K(t-\tau)S_{ij}(\tau)d\tau\right]; \tag{2.1}$$

the law of dilatational strain

$$e^s = \frac{1}{\alpha_v}\left[S(t) + \int_0^t K_v(t-\tau)S(\tau)d\tau\right], \tag{2.2}$$

where e^s_{ij} denotes the components of the skeleton strain deviator, S_{ij} components of the deviator of the total stress, G shear modulus of the skeleton, $K(t-\tau)$ kernels describing the creep during the shear strain, e^s skeleton dilatation, $S = \sigma_{ii}$ (σ_{ij} are the total stresses in the two-phase medium), α_v dilatational modulus, $K_v(t-\tau)$ kernel describing the creep during the dilatational strain.

On the basis of Eqs. (2.1) and (2.2) we obtain relations between the components of the skeleton strain ε^s_{ij} and the stress tensor σ_{ij}, namely

$$\varepsilon^s_{ij}(t) = \frac{1}{2G}\left[\sigma_{ij}(t) + \int_0^t K(t-\tau)\sigma_{ij}(\tau)d\tau\right] + \frac{1}{3}\delta_{ij}\left[\frac{1}{\alpha_v}\left(S(t) + \int_0^t K_v(t-\tau)S(\tau)d\tau\right) - \frac{1}{2G}\left(S(t) + \int_0^t K(t-\tau)S(\tau)\,d\tau\right)\right]. \tag{2.3}$$

In what follows, for brevity, we introduce the integral operators

$$(2.4)\qquad \frac{1}{\tilde{G}}=\frac{1}{G}\Big[1+\int_0^t K(t-\tau)\dots d\tau\Big],\qquad \frac{1}{\tilde{\alpha}_v}=\frac{1}{\alpha_v}\Big[1+\int_0^t K_v(t-\tau)\dots d\tau\Big].$$

Thus the relation (2.3) takes the from

$$(2.5)\qquad \varepsilon^s_{ij}(t)=\frac{1}{2\tilde{G}}\sigma_{ij}(t)+\frac{1}{3}\delta_{ij}\left(\frac{1}{\tilde{\alpha}_v}-\frac{1}{2\tilde{G}}\right)S(t).$$

As a result of the dilatational strain of the soil, there arises a hydrostatic pressure in the fluid in the pores, which acts on the skeleton and tends to increase the pores. This is a rheological process, in general different from the above-considered creep of the skeleton. The dilatation due to the action of the fluid is given by the relation [10]

$$(2.6)\qquad e^p_v(t)=\frac{3}{\alpha_{vp}}\Big[p(t)+\int_0^t K_{vp}(t-\tau)p(\tau)d\tau\Big],$$

where α_{vp} is the dilatation modulus in the process of increase of the pores, $K_{vp}(t-\tau)$ — the creep kernel describing the dilatational strain on the skeleton due to the fluid pressure, p — the pressure of the fluid in the pores, e^p_v — the dilatational strain of the skeleton due to the fluid pressure.

Introducing the operator

$$(2.7)\qquad \frac{1}{\tilde{\alpha}_{vp}}=\frac{1}{\alpha_{vp}}\Big[1+\int_0^t K_{vp}(t-\tau)\dots d\tau\Big],\ \text{we have}\qquad (2.8)\qquad e^p_v(t)=\frac{3}{\tilde{\alpha}_{vp}}p(t).$$

The total state of strain constitutes a sum of the strains due to the action of the stress and due to the action of the fluid pressure in the pores of the soil. This, taking into account (2.5) and (2.8), we obtain for the total strain tensor

$$(2.9)\qquad \varepsilon_{ij}=\frac{1}{2\tilde{G}}\sigma_{ij}+\frac{\delta_{ij}}{3}\left(\frac{1}{\tilde{\alpha}_v}-\frac{1}{2\tilde{G}}\right)S+\frac{\delta_{ij}}{\tilde{\alpha}_{vp}}p$$

or, in the form solved for stresses

$$(2.10)\qquad \sigma_{ij}=2\tilde{G}\varepsilon_{ij}+\tilde{\lambda}\varepsilon_{kk}\delta_{ij}-\tilde{\alpha}_v\frac{1}{\tilde{\alpha}_{vp}}p\delta_{ij},$$

where

$$\tilde{G}=G\Big[1-\int_0^t R(t-\tau)\dots d\tau\Big],\qquad \tilde{\alpha}_v=\alpha_v\Big[1-\int_0^t R_v(t-\tau)\dots d\tau\Big],\qquad \tilde{\lambda}=\frac{1}{3}[\tilde{\alpha}_v-2\tilde{G}],$$

$R(t-\tau)$, $R_v(t-\tau)$ are the resolvents of the kernels $K(t-\tau)$, $K_v(t-\tau)$ of the operators (2.4). The above tensorial relations constitute the most general form of the physical law for a homogeneous, isotropic, linear, invariant in time viscoelastic porous material saturated with fluid. Their generality follows not only from purely physical considerations but constitutes a result of the Riesz-Frechet theorem on the form of a linear functional in the Hilbert space. In particular, for a differential model

$$P_1S_{ij}=Q_1e^s_{ij},\qquad P_2S=Q_2e^s,\qquad P_3p=Q_3e^p_v,$$

where

$$P_i = \sum_{k=0}^{n} a_k^i \frac{\partial^k}{\partial t^k}, \quad Q_i = \sum_{j=0}^{m} b_j^i \frac{\partial^j}{\partial t^j}, \quad i = 1, 2, 3$$

we have

$$2\tilde{G} = \frac{Q_1}{P_1}, \quad \tilde{\lambda} = \frac{1}{3}\left(\frac{Q_2}{P_2} - \frac{Q_1}{P_1}\right), \quad \tilde{\alpha}_v = 3\tilde{\lambda} + 2\tilde{G} = \frac{Q_2}{P_2},$$

when $\tilde{\alpha}_{vp} = 3(Q_3/P_3)$.

For

$$\tilde{\alpha}_v = \alpha_v = \text{const}, \quad \tilde{\alpha}_{vp} = \alpha_{vp} = \text{const}, \quad K_v(t-\tau) = K_{vp}(t-\tau) = 0,$$
$$m = n = 1, \quad a_0^1 = \gamma + \delta, \quad a_1^1 = 1, \quad b_0^1 = 2G\gamma, \quad b_1^1 = 2G$$

(or, equivalently, $K(t-\tau) = \delta e^{-\gamma(t-\tau)}$), we arrive at the standard model investigated in BIOT's [2] and ZAJĄC's [9] papers.

The relations (2.9) and (2.10) constitute the fundamental system of equations for the two-phase medium. The pressure $p(t)$ requires an additional relation. This is supplied by the filtration equation. On the basis of Darcy's law it takes the form

$$\frac{k_\varphi}{\gamma_w} \Delta p = \frac{3n}{\alpha_w} \frac{\partial p}{\partial t} + \frac{\partial}{\partial t}\left[\frac{1}{\tilde{\alpha}_v} S + \frac{3}{\tilde{\alpha}_{vp}} p\right], \tag{2.11}$$

where we have introduced the following notations: k_φ — filtration coefficient, γ_w — specific weight of the fluid, α_w — compressibility modulus dilatations of the fluid, n — porosity.

We observe that Eq. (2.11) is coupled with the relations (2.9) or (2.10). In what follows it is convenient to deal with the uncoupled filtration equation. To derive it, we base on the compatibility equations

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0 \tag{2.12}$$

which, after substitution of (2.9), in view of the equilibrium equations $\sigma_{ij,j} = 0$ and after contraction, yield

$$\Delta S = \frac{1}{2\tilde{L}-1} 4\tilde{G} \frac{1}{\tilde{\alpha}_{vp}} \Delta p, \tag{2.13}$$

where $\tilde{L} = \tilde{\lambda}\frac{1}{\tilde{\alpha}_v}$. Applying to (2.11) the differential operator Δ^x and substituting for ΔS the expression (2.13), we arrive at the required uncoupled filtration equation

$$\frac{k_\varphi}{\gamma_w} \nabla^2\nabla^2 p = \frac{3n}{\alpha_w} \frac{\partial}{\partial t} \nabla^2 p + \frac{\partial}{\partial t}\left[\tilde{L}_1 \frac{1}{\tilde{\alpha}_{vp}} \nabla^2 p\right], \quad \text{where } \tilde{L}_1 = \frac{1}{\tilde{\alpha}_v} \frac{1}{-1+2\tilde{L}} 4\tilde{G} + 3. \tag{2.14}$$

In the considered plane state of strain, we have $\varepsilon_{i3} = 0$, whence, after simple transformations, we obtain from (2.9), (2.10), (2.11) and (2.13)

$$\varepsilon_{\alpha\beta} = \frac{1}{2\tilde{G}}[\sigma_{\alpha\beta} - \delta_{\alpha\beta}\tilde{L}_2 S] + \delta_{\alpha\beta}\frac{\tilde{L}_3}{\tilde{\alpha}_{vp}} p, \quad \alpha, \beta = 1, 2, \tag{2.15}$$

$$\frac{k_\varphi}{\gamma_w} \nabla^2 p = \frac{3n}{\alpha_w} \frac{\partial p}{\partial t} + \frac{\partial}{\partial t}\left[\frac{1-2\tilde{L}_2}{2\tilde{G}} S + \frac{2\tilde{L}_3}{\tilde{\alpha}_{vp}} p\right], \tag{2.16}$$

$$\nabla^2 S = \frac{2\tilde{G}}{-1+2\tilde{L}} \frac{1}{\tilde{\alpha}_{vp}} \nabla^2 p, \quad \tilde{L}_2 = \frac{\tilde{L}}{1-\tilde{L}}, \quad \tilde{L}_3 = \frac{1}{1-\tilde{L}}. \tag{2.17}$$

Repeating transformations leading to (2.14), we finally have

$$\frac{k_\varphi}{\gamma_w}\nabla^2\nabla^2 p = \frac{3n}{\alpha_w}\frac{\partial}{\partial t}\nabla^2 p + \frac{\partial}{\partial t}\left[\tilde{L}_4 \frac{1}{\tilde{\alpha}_{vp}}\nabla^2 p\right], \quad \tilde{L}_4 = \frac{1-2\tilde{L}_2}{-1+2\tilde{L}} + 2\tilde{L}_3. \tag{2.18}$$

Besides the physical relations, we shall use Airy's stress function

$$\sigma_{\alpha\beta} = -F_{,\alpha\beta} + \delta_{\alpha\beta}\Delta F. \tag{2.19}$$

Thus, performing the substitution $S = \sigma_{11}+\sigma_{22} = \Delta F$, we obtain from (2.17) the equation

$$\Delta\Delta F = \frac{2\tilde{G}}{-1+2\tilde{L}}\frac{1}{\tilde{\alpha}_{vp}}\Delta p. \tag{2.20}$$

The relations (2.28) and 2.20) constitute the fundamental uncoupled system of equations of consolidating viscoelastic medium in the plane case. The uncoupling led to a higher order of the equations and therefore their solution (with the appropriate boundary and initial conditions) must satisfy the filtration equation in the form (2.16).

3. The general solution of the problem of consolidation of a semi-space in the plane state of strain

In the Cartesian coordinate system Oxy, we consider a consolidating semi-space in the plane state of strain subject to arbitrary normal and tangential loadings. To solve the problem, we apply the Fourier and Laplace integral transforms. We base on the

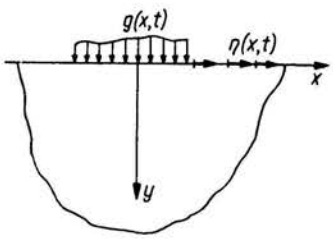

FIG. 1.

filtration Eq. (2.18) over which we perform the Fourier transform with respect to the variable x; thus

$$\frac{k_\varphi}{\gamma_w}(\partial_y^4 - 2\alpha^2\partial_y^2 + \alpha^4)\bar{p} = \frac{3n}{\alpha_w}\frac{\partial}{\partial t}(\partial_y^2-\alpha^2)\bar{p} + \frac{\partial}{\partial t}\left[\tilde{L}_4\frac{1}{\tilde{\alpha}_{vp}}(\partial_y^2-\alpha^2)\bar{p}\right], \tag{3.1}$$

where $\bar{p} = \bar{p}(\alpha, y, t)$ is the Fourier transform of the function p and α is the transform parameter. Now, we perform the Laplace transform with respect to time. Making use of the convolution theorem, we arrive at the ordinary differential equation

$$\begin{aligned}\frac{k_\varphi}{\gamma_w}(\partial_y^4 - 2\alpha^2\partial_y^2 + \alpha^4)\bar{p}^* = \frac{3n}{\alpha_w}(\partial_y^2-\alpha^2)[\lambda\bar{p}^* - \bar{p}(\alpha, y, 0)] \\ + (\partial_y^2-\alpha^2)\left[\lambda\tilde{L}_4^*\frac{1}{\tilde{\alpha}_{vp}^*}\bar{p}^* - \tilde{L}_4(0)\frac{1}{\tilde{\alpha}_{vp}(0)}\bar{p}(\alpha, y, 0)\right],\end{aligned} \tag{3.2}$$

where $\bar{p}^* = \bar{p}^*(\alpha, y, \lambda) = \int\limits_0^\infty \bar{p}(\alpha, y, t)e^{-\lambda t}dt$ is the Laplace transform of the function $\bar{p}$, $\frac{1}{\tilde{\alpha}^*_{vp}}$, $\tilde{L}^*_4$ — Laplace transforms of the operators $\frac{1}{\tilde{\alpha}_{vp}}$, $\tilde{L}_4$; $\tilde{L}^{(0)}_4$, $\frac{1}{\tilde{\alpha}^{(0)}_{vp}}$ — values of the operators at the instant $t = 0$; $\bar{p}(\alpha, y, 0)$ — the value of the Fourier transform of the pressure p at the instant $t = 0$.

Introducing the notations

$$\bar{f}_1(\alpha, y, 0) = -\left(a + \frac{\tilde{L}^{(0)}_4}{w_1}\frac{1}{\tilde{\alpha}^{(0)}_{vp}}\right)(\partial^2_y - \alpha^2)\bar{p}(\alpha, y, 0), \tag{3.3}$$

$$-z^2 = \left[\lambda a + \frac{\lambda \tilde{L}^*_4}{w_1 \tilde{\alpha}^*_{vp}}\right], \quad w_1 = \frac{k_\varphi}{\gamma_w}, \quad a = \frac{3n}{\alpha_w w_1},$$

we obtain the following form of our equation:

$$(\partial^4_y - 2\alpha^2\partial^2_y + \alpha^4)\bar{p}^* + z^2(\partial^2_y - \alpha^2)\bar{p}^* = \bar{f}_1(\alpha, y, 0). \tag{3.4}$$

In view of the condition $\lim\limits_{y\to\infty} p = 0$, the general solution of the above equation has the form

$$\bar{p}^*(\alpha, y, \lambda) = A_1(\alpha, \lambda)e^{-\sqrt{\alpha^2 - z^2}y} + B_1(\alpha, \lambda)e^{-|\alpha|y} + f^*_1(\alpha, y, \lambda), \tag{3.5}$$

where $f^*_1(\alpha, y, \lambda)$ is the particular solution to be determined on the basis of the fluid pressure at the initial instant.

Let us now solve Eq. (2.20); performing over it the Fourier transform, we obtain

$$(\partial^4_y - 2\alpha^2\partial^2_y + \alpha^4)\bar{F}(\alpha, y, t) = \frac{2\tilde{G}}{-1 + 2\tilde{L}}\frac{1}{\tilde{\alpha}_{vp}}(\partial^2_y - \alpha^2)\bar{p}(\alpha, y, t),$$

where $\bar{F}(\alpha, y, t)$ is the Fourier transform of the function $F(x, y, t)$. Moreover, the Laplace transform with respect to the variable t yields

$$(\partial^4_y - 2\alpha^2\partial^2_y + \alpha^4)\bar{F}^* = \frac{2\tilde{G}^*}{-1 + 2\tilde{L}^*}\frac{1}{\tilde{\alpha}^*_{vp}}(\partial^2_y - \alpha^2)\bar{p}^*, \tag{3.6}$$

where $\tilde{G}^*$, $\tilde{L}^*$ are the Laplace transforms of the operators $\tilde{G}$, $\tilde{L}$, respectively.

Taking into account (3.5) and making use of the notation

$$T = \frac{2\tilde{G}^*}{-1 + 2\tilde{L}^*}\frac{1}{\tilde{\alpha}^*_{vp}}, \tag{3.7}$$

we have

$$(\partial^4_y - 2\alpha^2\partial^2_y + \alpha^4)\bar{F}^* = -Tz^2 A_1(\alpha, \lambda)e^{-\sqrt{\ 2 - z^2}y} + f^*_2(\alpha, y, \lambda),$$

where

$$f^*_2(\alpha, y, \lambda) = T(\partial^2_y - \alpha^2)f^*_1(\alpha, y, \lambda).$$

Assuming that the stresses at infinity vanish, we arrive at the following solution of Eq. (3.7):

$$\bar{F}^* = [A(\alpha, \lambda) + B(\alpha, \lambda)|\alpha|y]e^{-|\alpha|y} - \frac{T}{z^2}A_1(\alpha, \lambda)e^{-\sqrt{\alpha^2 - z^2}y} + f^*_3(\alpha, y, \lambda). \tag{3.8}$$

Here $f^*_3(\alpha, y, \lambda)$ is the particular solution corresponding to the inhomogeneity $f^*_2(\alpha, y, \lambda)$.

The functions (3.5) and (3.8) contain four unknown constants A_1, B_1, A and B. They will be determined by means of the boundary conditions (two for the stress function and one for the pressure p), and the filtration Eq. (2.16). We begin from the latter equation and perform over it the Fourier and Laplace transforms. Thus

$$(3.9)\qquad \frac{k_\varphi}{\gamma_w}(\partial_y^2-\alpha^2)\bar{p}^* = \frac{3n}{\alpha_w}\lambda\bar{p}^*+\lambda\left[\frac{1-2\tilde{L}_2^*}{2\tilde{G}^*}\bar{S}^*+\frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\bar{p}^*\right] -\frac{3n}{\alpha_w}\bar{p}(\alpha,y,0)-\frac{1-2\tilde{L}_2^{(0)}}{2\tilde{G}^{(0)}}\bar{S}(\alpha,y,0)-\frac{2\tilde{L}_3(0)}{\tilde{\alpha}_{vp}^{(0)}}\bar{p}(\alpha,y,0),$$

where $\tilde{L}_2^*$, $\tilde{L}_3^*$ are the Laplace transforms of the operators $\tilde{L}_2$ and $\tilde{L}_3$. On the basis of the relation $S=\nabla^2F$, we calculate the transform

$$(3.10)\qquad \bar{S}^* = (\partial_y^2-\alpha^2)\bar{F}^*.$$

Introducing the above expression into (3.9) and substituting for $\bar{p}^*$ and $\bar{F}^*$ from the formulae (3.5) and (3.8), we obtain

$$(3.11)\qquad -\frac{k_\varphi}{\gamma_w}z^2A_1(\alpha,\lambda)e^{-\sqrt{\alpha^2-z^2}y}+\frac{k_\varphi}{\gamma_w}(\partial_y^2-\alpha^2)f_1^*(\alpha,y,\lambda)$$
$$=\frac{3n}{\alpha_w}\lambda[B_1(\alpha,\lambda)e^{-|\alpha|y}+A_1(\alpha,\lambda)e^{-\sqrt{\alpha^2-z^2}y}+f_1^*(\alpha,y,\lambda)]-\frac{3n}{\alpha_w}\bar{p}(\alpha,y,0)$$
$$+\lambda\frac{1-2\tilde{L}_2^*}{2\tilde{G}^*}[-2B(\alpha,\lambda)\alpha^2e^{-|\alpha|y}+TA_1(\alpha,\lambda)e^{-\sqrt{\alpha^2-z^2}y}+(\partial_y^2-\alpha^2)f_3^*(\alpha,y,\lambda)]$$
$$+\lambda\frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}[B_1(\alpha,\lambda)e^{-|\alpha|y}+A_1(\alpha,\lambda)e^{-\sqrt{\alpha^2-z^2}y}+f_1^*(\alpha,y,\lambda)]$$
$$-\frac{-1-2\tilde{L}_2(0)}{2\tilde{G}(0)}(\partial_y^2-\alpha^2)\bar{F}(\alpha,y,0)-\frac{2\tilde{L}_3(0)}{\tilde{\alpha}_{vp}(0)}\bar{p}(\alpha,y,0).$$

After transformations

$$(3.12)\qquad 2\alpha^2B(\alpha,\lambda)\lambda\frac{1-2\tilde{L}_2^*}{2\tilde{G}^*}e^{-|\alpha|y}=\lambda\left[\frac{3n}{\alpha_w}+\frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right]B_1(\alpha,\lambda)e^{-|\alpha|y}+f_4^*(\alpha,y,\lambda),$$

where

$$(3.13)\qquad f_4^*(\alpha,y,\lambda)=\frac{-k_\varphi}{\gamma_w}(\partial_y^2-\alpha^2)f_1^*(\alpha,y,\lambda)+\frac{3n}{\alpha_w}\lambda f_1^*(\alpha,y,\lambda)-\frac{3n}{\alpha_w}\bar{p}(\alpha,y,0)$$
$$+\lambda\frac{1-2\tilde{L}_2^*}{2\tilde{G}^*}(\partial_y^2-\alpha^2)f_3^*(\alpha,y,\lambda)+\lambda\frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}f_1^*(\alpha,y,\lambda)$$
$$-\frac{1-2\tilde{L}_2(0)}{2\tilde{G}(0)}(\partial_y^2-\alpha^2)F(\alpha,y,0)-\frac{2\tilde{L}_3(0)}{\tilde{\alpha}_{vp}(0)}\bar{p}(\alpha,y,0).$$

Applying the operator $\partial_y^2-\alpha^2$ to the function $f_4^*(\alpha,y,\lambda)$ and taking into account the relations between the particular solutions f_1^* and f_2^* with the functions $\bar{p}(\alpha,y,0)$ and $\bar{F}(\alpha,y,0)$, we find that

$$(\partial_y^2-\alpha^2)f_4^*(\alpha,y,\lambda)=0.$$

Hence

$$f_4^*(\alpha, y, \lambda) = C(\alpha, \lambda)e^{-|\alpha|y}.$$

The constant $C(\alpha, \lambda)$ will be determined from the initial conditions. Thus, in view of (3.12),

$$(3.14) \qquad B(\alpha, \lambda) = \left[\frac{3n}{\alpha_w} + \frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right] \frac{2\tilde{G}^*}{1-2\tilde{L}_2^*} \frac{B_1(\alpha, \lambda)}{2\alpha^2} + \frac{C(\alpha, \lambda)2\tilde{G}^*}{2\alpha^2\lambda(1-2\tilde{L}_2^*)}.$$

Introducing, for brevity, the notation

$$\frac{C(\alpha, \lambda)2\tilde{G}^*}{2\alpha^2\lambda(1-2\tilde{L}_2^*)} = C_1(\alpha, \lambda)$$

and introducing (3.14) into the formula (3.8), we finally obtain

$$(3.15) \qquad \bar{F}^* = A(\alpha, \lambda)e^{-|\alpha|y} + \left[\frac{3n}{\alpha_w} + \frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right] \frac{\tilde{G}^*}{1-2\tilde{L}_2^*} \frac{B_1(\alpha, \lambda)}{|\alpha|} y e^{-|\alpha|y}$$
$$+ C_1(\alpha, \lambda)|\alpha| y e^{-|\alpha|y} - \frac{T}{Z^2} A_1(\alpha, \lambda) e^{-\sqrt{\alpha^2 - z^2}y} + f_3^*(\alpha, y, \lambda).$$

Having determined the transform of the stress function, we can use (2.19) to calculate the stresses; thus, inverting the Fourier and Laplace transforms, we have

$$\sigma_y = -\frac{1}{\sqrt{2\pi}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \Bigg\{ A(\alpha, \lambda)\alpha^2 e^{-|\alpha|y} + C_1(\alpha, \lambda)|\alpha|\alpha^2 y e^{-|\alpha|y}$$
$$+ \left[\frac{3n}{\alpha_w} + \frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vy}^*}\right] \frac{\tilde{G}^*}{1-2\tilde{L}_2^*} B_1(\alpha, \lambda)|\alpha| y e^{-|\alpha|y} - \frac{T}{z^2} A_1(\alpha, \lambda)\alpha^2 e^{-\sqrt{\alpha^2 - z^2}y}$$
$$+ f_3^*(\alpha, y, \lambda)\alpha^2 \Bigg\} e^{-i\alpha x - \lambda t} d\lambda d\alpha,$$

$$\sigma_x = \frac{1}{\sqrt{2\pi}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \Bigg\{ A(\alpha, \lambda)\alpha^2 e^{-|\alpha|y} + C_1(\alpha, \lambda)(-2 + |\alpha|y)\alpha^2 e^{-|\alpha|y}$$
$$(3.16) \qquad + \left[\frac{3n}{\alpha_w} + \frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right] \frac{\tilde{G}^*}{1-2\tilde{L}_2^*} B_1(\alpha, \lambda)(-2 + |\alpha|y) e^{-|\alpha|y}$$
$$- \frac{T}{z^2}(\alpha^2 - z^2) A_1(\alpha, \lambda) e^{-\sqrt{\alpha^2 - z^2}y} + \partial_y^2 f_3^*(\alpha, y, \lambda) \Bigg\} e^{-i\alpha x + \lambda t} d\lambda d\alpha,$$

$$\sigma_{xy} = \frac{i}{\sqrt{2\pi}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \Bigg\{ -A(\alpha, \lambda)\alpha|\alpha| e^{-|\alpha|y} + C_1(\alpha, \lambda)\alpha|\alpha|(1 - |\alpha|y) e^{-|\alpha|y}$$
$$+ \frac{T}{z^2} A_1(\alpha, \lambda)\alpha\sqrt{\alpha^2 - z^2} e^{-\sqrt{\alpha^2 - z^2}y} + \frac{\partial}{\partial y} f_3^*(\alpha, y, \lambda)$$
$$+ \left[\frac{3n}{\alpha_w} + \frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right] \frac{\tilde{G}^*}{1-2\tilde{L}_2^*} B_1(\alpha, \lambda) \frac{\alpha}{|\alpha|}(1 - |\alpha|y) e^{-|\alpha|y} \Bigg\} e^{-i\alpha x + \lambda t} d\lambda d\alpha.$$

On the basis of (3.5), we deduce the pressure

(3.17)

$$p = \frac{1}{\sqrt{2\pi}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \{B_1(\alpha, \lambda)e^{-|\alpha|y} + A_1(\alpha, \lambda)e^{-\sqrt{\alpha^2-z^2}y} + f_1^*(\alpha, y, \lambda)\} e^{-i\alpha x+\lambda t} d\lambda d\alpha.$$

In what follows we shall need the formulae for the displacements. In view of (2.15) and the geometric relation $\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|})$, we obtain the vertical displacement in the form

$$(3.18) \quad U_y = \frac{1}{\sqrt{2\pi}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \Bigg\{ \frac{1}{2\tilde{G}^*} A(\alpha, \lambda)|\alpha| e^{-|\alpha|y}$$

$$+ (1-2\tilde{L}_2^* + |\alpha|y) \frac{C_1(\alpha, \lambda)|\alpha|}{2\tilde{G}^*} e^{-|\alpha|y} + \left[\frac{3n}{\alpha_w} + \frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right] \frac{1-2\tilde{L}_2^* + |\alpha|y}{1-2\tilde{L}_2^*} \frac{B_1(\alpha, \lambda)}{2|\alpha|} e^{-|\alpha|y}$$

$$- \frac{\tilde{L}_3^*}{|\alpha|\tilde{\alpha}_{vp}^*} B_1(\alpha, \lambda) e^{-|\ |y} + \frac{z^2\tilde{L}_2^* - \alpha^2}{2\tilde{G}^* z^2 \sqrt{\alpha^2-z^2}} T A_1(\alpha, \lambda) e^{-\sqrt{\alpha^2-z^2}y}$$

$$- \frac{\tilde{L}_3^* A_1(\alpha, \lambda)}{\sqrt{\alpha^2-z^2}\tilde{\alpha}_{vp}^*} e^{-\sqrt{\alpha^2-z^2}y} - \frac{\tilde{L}_2^*}{2\tilde{G}^*} \frac{\partial}{\partial y} f_3^*(\alpha, y, \lambda)$$

$$+ \int \left[\frac{1}{2\tilde{G}^*}(\tilde{L}_2^* f_3^* \alpha^2 - f_3^* \alpha^2) + \frac{\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*} f_1^*\right] dy \Bigg\} e^{-i\alpha x+\lambda t} d\lambda d\alpha + f_{11}(x, t),$$

while the horizontal displacement takes the form

$$(3.19) \quad U_x = \frac{i}{\sqrt{2\pi}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \Bigg\{ \frac{1}{2\tilde{G}^*} [A(\alpha, \lambda)\alpha + (-2+2\tilde{L}_2^* + |\alpha|y)\alpha C_1(\alpha, \lambda)] e^{-|\ |y}$$

$$+ \left[\frac{\tilde{L}_3^* A_1(\alpha, \lambda)}{\tilde{\alpha}_{vp}^* \alpha} - \frac{\alpha^2 - z^2 + z^2\tilde{L}_2^*}{z^2 2\tilde{G}^*} T \frac{A_1(\alpha, \lambda)}{\alpha}\right] e^{-\sqrt{\alpha^2-z^2}y}$$

$$+ \left[\frac{3n}{\alpha_w} + \frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right] \frac{-2+2\tilde{L}_2^* + |\alpha|y}{2\alpha(1-2\tilde{L}_2^*)} B_1(\alpha, \lambda) e^{-|z|y} + \frac{\tilde{L}_3^* B_1(\alpha, \lambda)}{\tilde{\alpha}_{vp}^* \alpha} e^{-|\alpha|y} + \frac{\tilde{L}_3^*}{\alpha\tilde{\alpha}_{vp}^*} f_1^*$$

$$+ \frac{1}{\alpha 2\tilde{G}^*} [\partial_y^2 f_3^*(\alpha, y, \lambda) + \tilde{L}_2^*(\alpha^2 - \partial_y^2) f_3^*(\alpha, y, \lambda)] \Bigg\} e^{-i\alpha x+\lambda t} d\alpha d\lambda + f_{22}(y, t).$$

In view of the above formulae, the compatibility of strain $\varepsilon_{xy} = \frac{1}{2}(U_{x,y} + U_{y,x}) = \frac{1}{2\tilde{G}} \sigma_{xy}$ leads to the relation

$$-\frac{\partial f_{11}}{\partial x} = \frac{\partial f_{22}}{\partial y}.$$

Consequently, similarly to the theory of elasticity, the functions $f_{11}(x,t)$ and $f_{22}(y,t)$ have the form corresponding to a rigid displacement

$$f_{11}(x,t) = -C_x^* + B(t), \quad f_{22}(y,t) = C^* y + D(t).$$

The formulae (3.16)–(3.19) describing the stresses and displacements contain the integration constants A, A_1 and B_1 depending on the transform parameters α and λ. These constants can be determined from the conditions of loading and pressure on the boundary of the semi-space (Fig. 1)

$$\sigma_y(x,0,t) = g(x,t), \quad \sigma_{xy}(x,0,t) = \eta(x,t), \quad p(x,0,t) = p_1(x,t). \tag{3.20}$$

Performing the integral transforms, after simple transformations we obtain

$$A(\alpha,\lambda) = \frac{-\left\{\frac{T}{z^2}\frac{\sqrt{\alpha^2-z^2}}{|\alpha|} - \left[\frac{3n}{\alpha_w} + \frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right]\frac{\tilde{G}^*}{(1-2\tilde{L}_2^*)\alpha^2}\right\}\left(\bar{g}^* + \alpha^2 f_3^*(\alpha,0,\lambda)\right)}{M(\alpha,\lambda)} \tag{3.21}$$

$$-\frac{\frac{T}{z^2}\left\{i\bar{\eta}^*\frac{\alpha}{|\alpha|} + C_1\alpha^2 + |\alpha|\left.\frac{\partial f_3^*}{\partial y}\right|_{y=0} + \left[\frac{3n}{\alpha_w} + \frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right]\frac{\tilde{G}^*}{1-2\tilde{L}_2^*}\left(\bar{p}_1^* - f_1^*(\alpha,0,\lambda)\right)\right\}}{M(\alpha,\lambda)},$$

$$A_1(\alpha,\lambda) = \frac{-\bar{g}^* - i\bar{\eta}^*\frac{\alpha}{|\alpha|} - C_1\alpha^2 - |\alpha|\left.\frac{\partial f_3^*}{\partial y}\right|_{y=0} - \alpha^2 f_3^*(\alpha,0,\lambda)}{M(\alpha,\lambda)}$$

$$+\frac{-\left[\frac{3n}{\alpha_w} + \frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right]\frac{\tilde{G}^*}{1-2\tilde{L}_2^*}\left(\bar{p}_1^* - f_1^*(\alpha,0,\lambda)\right)}{M(\alpha,\lambda)},$$

$$B_1(\alpha,\lambda) = \frac{\bar{g}^* + i\bar{\eta}^*\frac{\alpha}{|\alpha|} + C_1\alpha^2 + |\alpha|\left.\frac{\partial f_3^*}{\partial y}\right|_{y=0} + \alpha^2 f_3^*(\alpha,0,\lambda)}{M(\alpha,\lambda)}$$

$$+\frac{T\frac{|\alpha|\sqrt{\alpha^2-z^2}-\alpha^2}{z^2}\left(\bar{p}_1^* - f_1^*(\alpha,0,\lambda)\right)}{M(\alpha,\lambda)},$$

where

$$M(\alpha,\lambda) = T\frac{|\alpha|\sqrt{\alpha^2-z^2}-\alpha^2}{z^2} - \left[\frac{3n}{\alpha_w} + \frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right]\frac{\tilde{G}^*}{1-2\tilde{L}_2^*}$$

and $\bar{g}^*$, $\bar{\eta}^*$, $\bar{p}^*$ are the transforms of the loadings.

Thus the constants (3.21), besides the boundary data $\bar{g}^*$, $\bar{\eta}^*$ and $\bar{p}^*$, contain also the functions $f_1^*(\alpha,0,\lambda)$, $f_3^*(\alpha,0,\lambda)$ and $C_1(\alpha,\lambda)$. The latter depend on the initial conditions; this fact follows from the formulae which for clarity we present once more:

$$\bar{f}_1(\alpha,y,0) = -\left(a + \frac{\tilde{L}_4^{(0)}}{w_1}\frac{1}{\tilde{\alpha}_{vp}^{(0)}}\right)(\partial_y^2 - \alpha^2)\bar{p}(\alpha,y,0).$$

Here, $f_1^*(\alpha, 0, \lambda)$ is the value of the particular solution (3.4) corresponding to the inhomogeneity $\bar{f}_1(\alpha, y, 0)$ for $y = 0$,

$$f_2^*(\alpha, y, \lambda) = \frac{2\tilde{G}^*}{2\tilde{L}^*-1}\frac{1}{\tilde{\alpha}_{vp}^*}(\partial_y^2-\alpha^2)f_1^*(\alpha, y, \lambda),$$

$f_3^*(\alpha, 0, \lambda)$ is the value of the particular solution (3.7) corresponding to the inhomogeneity $f_2^*(\alpha, y, \lambda)$ for $y = 0$,

$$C_1(\alpha, \lambda) = \frac{\tilde{G}^*}{\alpha^2\lambda(1-2\tilde{L}_2^*)}C(\alpha, \lambda),$$

$$C(\alpha, \lambda)e^{-|\alpha|y} = \left[-\frac{k_\varphi}{\gamma_w}(\partial_y^2-\alpha^2)+\frac{3n}{\alpha_w}\lambda+\lambda\frac{2\tilde{L}_3^*}{\tilde{\alpha}_{vp}^*}\right]f_1^*(\alpha, y, \lambda)$$
$$+\frac{1-2\tilde{L}_2^*}{2\tilde{G}^*}(\partial_y^2-\alpha^2)\lambda f_3^*(\alpha, y, \lambda)-\left[\frac{3n}{\alpha_w}+\frac{2\tilde{L}_3^{(0)}}{\tilde{\alpha}_{vp}^{(0)}}\right]\bar{p}(\alpha, y, 0)$$
$$-\frac{1-2\tilde{L}_2(0)}{2\tilde{G}(0)}(\partial_y^2-\alpha^2)\bar{F}(\alpha, y, 0).$$

If we assume that at the initial instant $t = 0$, the body is in its natural state, i.e., $p(x, y, 0) = F(x, y, 0) = 0$, then $\bar{p}(\alpha, y, 0) = \bar{F}(\alpha, y, 0) = 0$ and, consequently,

$$f_1^*(\alpha, 0, \lambda) = f_3^*(\alpha, 0, \lambda) = C_1(\alpha, \lambda) = 0.$$

The formulae (3.16)–(3.19) completed by (3.21) yield therefore the general exact solution of the problem. The total stresses in the two-phase medium, the fluid pressure and the skeleton displacements are given in the form of non-elementary proper and improper integrals a computation of which for definite functions $g(x, t)$, $\eta(x, t)$ and $p_1(x, t)$ can be cumbersome but does not lead to any difficulties. On the basis of the Krylov interpolation method [6, 7], the inversion of the Fourier and Laplace transforms is basically reduced to the calculation of the values of the integrands at the interpolation nods. For various cases of loadings and material (various types of soils), we can therefore effectively determine the distributions of displacements and stresses, to estimate the influence of the viscoelasticity of the medium on the phenomenon of consolidation and to describe quantitatively the process in time. Results of the above numerical analysis will constitute the subject of another paper.

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