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BRIEF NOTES

Overstable convection in a viscoelastic fluid layer at large Chandrasekhar number

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THE EFFECT of a *very strong* magnetic field on the overstable mode of convection in a viscoelastic fluid layer heated from below is explored using an asymptotic procedure As in the case of low or moderate field strengths [1, 2] a *stabilizing* effect is found.

THE EFFECT of a uniform vertical magnetic field $\mathbf{H} = (0, 0, H)$ on the onset of thermal convection as overstability in an infinite horizontal layer of a Maxwell viscoelastic fluid of depth d which is heated uniformly from below and confined by two free, isothermal and non-deformable boundaries has been recently studied by BHATIA and STEINER [1, 2]. Using the normal mode stability analysis on the linearized, Boussinesq simplified, thermo-magneto-elastic equations governing small perturbations, an eighth-order double characteristic differential eigenvalue system for neutral instability was derived and readily converted by means of an exact solution into the following algebraic eigenvalue equation

$$(\pi^2+a^2)(\pi^2+a^2+\sigma p_1)[(\pi^2+a^2+\sigma p_2)\{\pi^2+a^2+\sigma(1+\Gamma\sigma)\}+(1+\Gamma\sigma)\pi^2 Q] = Ra^2(1+\Gamma\sigma)(\pi^2+a^2+\sigma p_2), \tag{1}$$

where a is the horizontal wave number, σ is the growth rate, $p_1 = \nu/\varkappa$ is the Prandtl number, $p_2 = \nu/\eta$ is the magnetic Prandtl number, $\Gamma = t_0\nu/d^2$ is the elastic parameter, $R = g\alpha\beta d^4/\varkappa\nu$ is the Rayleigh number, and

$$Q = \frac{\mu_e H^2 d^2}{4\pi\varrho\nu\eta} \text{ is the "Chandrasekhar" number,} \tag{2}$$

while ϱ, g, t_0, α, β, η, $\varkappa$, μ_e and ν denote, respectively, the density, gravity, relaxation time, coefficient of volume expansion, adverse temperature gradient, resistivity, thermometric conductivity, magnetic permeability and kinematic viscosity. The stabilizing effect of the magnetic field on the overstable mode of convection was then obtained numerically. As a numerical investigation is clearly limited, it is necessary to derive the asymptotic behaviour in the limit of very large Chandrasekhar number. Since the algebra involved is very long and messy, the derivation will be briefly described for the case $p_1 < p_2$ only; the same procedure can be employed for the alternative case.

Remembering that σ can be complex and letting

$$\sigma = i\sigma_1, \quad x = a^2, \quad b = \pi^2 + a^2. \tag{3}$$

it is clearly seen from (1) that for an arbitrary σ_1, R will be complex. But, from physical considerations, R must be real, thus implying a relation between the real and imaginary parts of σ_1. But as we are interested in specifying the critical Rayleigh number for the onset of overstability via a state of purely oscillatory motion, we shall suppose that σ_1 is real in the above equation and try to obtain the conditions for such solutions to exist.

Assuming then that σ_1 is real, Eq. (1) can be separated into real and imaginary parts, both of which must vanish separately. This leads to the following pair of equations:

$$R = \frac{b[b^2 + (\Gamma b - 1)p_1\sigma_1^2 - p_1\Gamma^2\sigma_1^4]}{x(1+\Gamma^2\sigma_1^2)} + \frac{\pi^2 Q b(b^2 + p_1 p_2 \sigma_1^2)}{x(b^2 + p_2^2\sigma_1^2)}, \tag{4}$$

and the quadratic in σ_1^2

$$A_0\sigma_1^4 + A_1\sigma_1^2 + A_2 = 0, \tag{5}$$

where

$$\begin{aligned} A_0 &= p_2^2\Gamma^2, \\ A_1 &= b^2\Gamma^2 + p_2^2(p_1 + 1 - \Gamma b) + \pi^2\Gamma^2 Q(p_1 - p_2), \\ A_2 &= b^2(p_1 + 1 - \Gamma b) + \pi^2 Q(p_1 - p_2). \end{aligned} \tag{6}$$

Remembering that $p_1 < p_2$, the positive root of (5) can be represented, in the limit $Q \to \infty$, as:

$$\sigma_1^2 = \frac{\pi^2(p_2 - p_1)}{p_2^2}Q + \frac{p_2^2(\Gamma b - p_1) - b^2\Gamma^2}{p_2^2\Gamma^2} + \frac{(\Gamma b - p_1)(p_2^2 - b^2\Gamma^2)}{\pi^2\Gamma^4(p_1 - p_2)}Q^{-1} + 0(Q^{-2}). \tag{7}$$

Substituting this relation into (4) gives after lengthy calculations:

$$R = \frac{b}{x}\left[\frac{\pi^2 p_1^2}{p_2^2}Q + \frac{p_1^2 p_2^2 + b^2\Gamma^2(p_1 + p_2)}{p_2^2\Gamma^2} - \frac{b^3\Gamma^3(p_1 + p_2) - b^2\Gamma^2(p_1^2 + p_1 p_2 + p_2^2) + p_1^2 p_2^2}{\pi^2\Gamma^4(p_2 - p_1)}Q^{-1} + 0(Q^{-2})\right]. \tag{8}$$

The critical wave number (a_c) can now be obtained by minimizing R with respect to x. This leads after some simplifications a quartic in x whose solution at sufficiently large Q is

$$x_c \equiv a_c^2 = y\left\{1 - \frac{\pi^2}{2}y^{-1} + \left[\frac{\pi^4}{4} + \frac{\pi^2 p_1^2 p_2^2}{4(p_2^2 - p_1^2)\Gamma}\right]y^{-2} + 0(y^{-3})\right\}, \tag{9}$$

where

$$y = \left[\frac{\pi^4 p_1^2 Q}{2(p_1 + p_2)}\right]^{1/3}. \tag{10}$$

Substituting for x_c from (9) into (7) and (8) gives finally:

$$(11)\qquad (\sigma_1^2)_c = y^3\left[\frac{2(p_2^2-p_1^2)}{p_1^2p_2^2\pi^2} - \frac{1}{p_2^2}y^{-1} + \left(\frac{1}{\Gamma} - \frac{\pi^2}{p_2^2}\right)y^{-2} + \left\{\frac{\pi^2(p_2^2-2p_1^2)}{2(p_2^2-p_1^2)\Gamma} - \frac{3\pi^4}{4p_2^2} - \frac{p_1}{\Gamma^2}\right\}y^{-3} + 0(y^{-4})\right],$$

and

$$(12)\qquad R_c = y^3\left(\frac{p_1+p_2}{p_2^2}\right)\left[\frac{2}{\pi^2} + 3y^{-1} + 3\pi^2y^{-2} + 0(y^{-3})\right].$$

Equations (9)–(12) are the required analytic expressions for the critical wave number (a_c), frequency $(\sigma_1)_c$, and Rayleigh number (R_c) for the onset of overstability in the limit as $Q \to \infty$, for the case $p_1 < p_2$. The values of the critical constants obtained from these expressions are in excellent agreement with the computed results obtained by the method described in [1, 2]. This is typically illustrated for the case $p_1 = 1$, $p_2 = 100$, $\Gamma = 1$ and a few values of Q in the Table 1.

Table 1

Q	Numerical solution			Asymptotic solution		
	a_c	$\log_{10}(\sigma_1)_c$	$\log_{10} R_c$	a_c	$\log_{10}(\sigma_1)_c$	$\log_{10} R_c$
10^5	5.6806	1.9959	2.1868	5.6745	1.9959	2.1753
10^6	8.5925	2.4951	3.0793	8.5922	2.4951	3.0780
10^7	12.811	2.9950	4.0330	12.813	2.9950	4.0328
10^8	18.951	3.4950	5.0121	18.951	3.4950	5.0121

It is clearly seen from Eq. (12) that as Q increases, R_c increases and ultimately becomes independent of Γ. Thus, a very strong magnetic field has a stabilizing effect on the overstable mode of convection, as expected. In the presence of a vertical magnetic field any convective motion must involve the fluid crossing the field and we would certainly expect a stabilizing influence when the field is very strong.

References

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2. P. K. BHATIA and J. M. STEINER, *Thermal instability in a viscoelastic fluid layer in hydromagnetics*, JMAA, **41**, 271–283, 1973.

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