

Mixtures of fluids and isotropic solids

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NOLL's isotropy group is generalized to cover second grade materials. The particular cases of fluids and isotropic solids are covered in detail, for both single materials and mixtures.

Uogólniono grupę izotropii Nolla w celu uwzględnienia materiałów drugiego rzędu. Dokładniej omówiono przypadki szczególne dotyczące cieczy i izotropowych ciał stałych zarówno w przypadku materiałów prostych jak i mieszanin.

Дано обобщение понятия группы изотропии, введенного Ноллом. Целью этого обобщения является учет свойств материалов второго порядка. Более подробно обсуждены частные случаи, относящиеся к жидкостям и изотропным твердым телам, как в случае простых материалов, так и смесей.

1. Introduction

NOLL's [1] fundamental definition of the isotropy group of a simple material has been adapted in a variety of ways to apply to rigid heat conductors [4], to materials of second grade [7], to thermodynamic materials [2], and to mixtures [3, 6, 8, 9].

TOUPIN [7] dealt with materials of second grade, using as independent variables a pair $(\mathbf{F}, \nabla\mathbf{F})$ of first and second deformation gradients, but for the symmetry group he considered only rigid deformations [7, Eqs. (8.10), (10.24), (10.25), and § 13]. In a series of papers [3, 6, 8, 9], BOWEN, BOWEN & WIESE, and BOWEN & GARCIA adapted the idea of symmetry groups to mixtures, for mixtures of materials of second grade. While the symmetry group is not restricted to the orthogonal group in these papers, it is restricted to linear transformations.

It seems that possible elements in the isotropy group should be of the same type as those considered in the constitutive relation. Since the response function R of a material of second grade depends on the pair $(\mathbf{F}, \nabla\mathbf{F})$, where \mathbf{F} is the deformation gradient and $\nabla\mathbf{F}$ the second gradient, its isotropy group⁽¹⁾ should be formed by pairs (\mathbf{H}, \mathbf{G}) , \mathbf{H} corresponding to the deformation gradient and \mathbf{G} to the second gradient, such that

$$(1.1) \quad R(\mathbf{F}, \nabla\mathbf{F}) = R((\mathbf{F}, \nabla\mathbf{F}) \circ (\mathbf{H}, \mathbf{G})),$$

where the composition rule is induced by the chain rule for first and second gradients.

This approach is consistent with the work of NOLL [1] and of GURTIN & WILLIAMS [2].

⁽¹⁾ The isotropy group defined by the condition (1.1) has been considered recently by MORGAN in as yet unpublished paper on material bodies of second and higher grade. The results of the present work concern constitutive relations for materials and mixtures of materials of second grade and are independent of the work of MORGAN.

The isotropy group of [3, 6, 7, 8, 9] would be only a subgroup of the present group, expressed as elements of the form $(\mathbf{H}, 0)$.

The condition (1.1) for the isotropy group is not only more natural for materials of second-grade, but it also leads to more explicit forms of the response function in some cases. In § 5, we prove that the forms of constitutive relations assumed by MÜLLER [10] and DUNWOODY [11] are necessary from material symmetry if suitable isotropy groups defined by (1.1) are used. In § 6, we prove that the constitutive relations assumed by GURTIN & VARGAS [12] and GURTIN [13] are necessary from material symmetry in a similar way. The key section is § 4 and its apparent simplicity owes much to C.-C. WANG. The results of §§ 4–6 are prepared by the basic sections, §§ 2–3, which introduce second-order classes and isotropy groups.

We do not treat the effect of frame indifference on the constitutive relations for two reasons: first, conditions on several different types of functions and variables are involved; some of these conditions have already been solved in general and it only takes an easy application of these general results to obtain the corresponding special forms [12, 13, 14]. The other reason is that no polar decomposition is known for the third-order tensors which appear at vital places in the theory of second-grade materials.

Notation. The notation ϕ_{**p} by itself would suggest that this quantity is intrinsic to the body manifold. It is obvious that it is not intrinsic, and that it varies in a non-tensorial way under coordinate transformations. The pair (ϕ_{*p}, ϕ_{**p}) , however, is intrinsic; the notation adopted is used in lieu of the “jet theory” notation, where this pair is a second-order jet. A formulation of a theory of materials in jet theory language may be seen in the reports of MORGAN referred to in the footnote; we intend to frame a theory of membranes and shells in this language in a further paper, using the idea of the isotropy group developed here.

2. Second-grade classes

Let \mathcal{A} be a body [5]: a connected, orientable, three-dimensional, differentiable manifold which can be covered by a single coordinate neighbourhood; i.e., there exist C^∞ — diffeomorphisms $\phi, \kappa, \lambda, \dots$, called (*global configurations*), which map \mathcal{A} into \mathcal{R}^3 ,

$$\phi: \mathcal{A} \rightarrow \mathcal{R}^3.$$

We define a second-grade equivalence class of configuration at a point $p \in \mathcal{A}$ by saying that $\phi, \kappa: \mathcal{A} \rightarrow \mathcal{R}^3$ are *second-grade equivalent* at p , if there is a coordinate system (x^1, x^2, x^3) for \mathcal{A} such that

$$(2.1) \quad \left. \frac{\partial \phi^k}{\partial x^i} \right|_p = \left. \frac{\partial \kappa^k}{\partial x^i} \right|_p \quad \text{and} \quad \left. \frac{\partial^2 \phi^k}{\partial x^i \partial x^j} \right|_p = \left. \frac{\partial^2 \kappa^k}{\partial x^i \partial x^j} \right|_p$$

for all $i, j, k = 1, 2, 3$, where ϕ^k, κ^k are the coordinate representations of ϕ and κ .

LEMMA 1. *The second-grade equivalence classes are well-defined, i.e., they are independent of the choice of coordinate system.*

P r o o f. In coordinate form, let $\phi: \mathcal{A} \rightarrow \mathcal{R}^3$ be written as

$$\phi(p) \equiv (\phi^1(x^1, x^2, x^3), \phi^2(x^1, x^2, x^3), \phi^3(x^1, x^2, x^3))$$

for all $p \in \mathcal{A}$, and similarly $\varkappa : \mathcal{A} \rightarrow \mathcal{R}^3$ be written as

$$\varkappa(p) \equiv (\varkappa^1(x^1, x^2, x^3), \varkappa^2(x^1, x^2, x^3), \varkappa^3(x^1, x^2, x^3)).$$

Under any change of coordinates from (x^i) to (X^k) on \mathcal{A} , we have

$$x^k = x^k(X^1, X^2, X^3), \quad k = 1, 2, 3.$$

Hence we have two new coordinate representations for the maps ϕ and \varkappa :

$$\phi(p) \equiv (\Phi^1(X^1, X^2, X^3), \Phi^2(X^1, X^2, X^3), \Phi^3(X^1, X^2, X^3))$$

and

$$\varkappa(p) \equiv (K^1(X^1, X^2, X^3), K^2(X^1, X^2, X^3), K^3(X^1, X^2, X^3)).$$

Then, by the chain rule for functions of several variables,

$$\left. \frac{\partial \Phi^k}{\partial X^i} \right|_p = \left. \frac{\partial \phi^k}{\partial x^j} \right|_p \left. \frac{\partial x^j}{\partial X^i} \right|_p = \left. \frac{\partial \varkappa^k}{\partial x^j} \right|_p \left. \frac{\partial x^j}{\partial X^i} \right|_p = \left. \frac{\partial \varkappa^k}{\partial X^i} \right|_p,$$

since, for example,

$$\Phi^k(X^1, X^2, X^3) = \phi^k(x^1(X^1, X^2, X^3), x^2(X^1, X^2, X^3), x^3(X^1, X^2, X^3)),$$

and similarly for \varkappa . The expression for the second gradient transforms is as follows:

$$\begin{aligned} \left. \frac{\partial^2 \Phi^k}{\partial X^i \partial X^j} \right|_p &= \left. \frac{\partial}{\partial X^i} \left\{ \left. \frac{\partial \Phi^k}{\partial X^j} \right| \right\} \right|_p \\ &= \left. \frac{\partial^2 \phi^k}{\partial x^s \partial x^t} \right|_p \left. \frac{\partial x^s}{\partial X^i} \right|_p \left. \frac{\partial x^t}{\partial X^j} \right|_p + \left. \frac{\partial \phi^k}{\partial x^t} \right|_p \left. \frac{\partial^2 x^t}{\partial X^i \partial X^j} \right|_p \\ &= \left. \frac{\partial^2 \varkappa^k}{\partial x^s \partial x^t} \right|_p \left. \frac{\partial x^s}{\partial X^i} \right|_p \left. \frac{\partial x^t}{\partial X^j} \right|_p + \left. \frac{\partial \varkappa^k}{\partial x^t} \right|_p \left. \frac{\partial^2 x^t}{\partial X^i \partial X^j} \right|_p \\ &= \left. \frac{\partial^2 K^k}{\partial X^i \partial X^j} \right|_p, \end{aligned}$$

on using (2.1). Hence the lemma is proved. The transformation of the second gradient uses the equality of the first gradients as an essential step when the coordinate change is non-linear.

We denote a typical second-grade equivalence class at $p \in \mathcal{A}$ by (ϕ_{**p}, ϕ_{**p}) . Note that ϕ_{**p} is symmetric, and that ϕ_{*p} is a linear isomorphism,

$$(2.2) \quad \phi_{*p} : \mathcal{A}_p \rightarrow \mathcal{R}^3,$$

where \mathcal{A}_p is the tangent space of \mathcal{A} at p . A second-grade equivalence class is also called a (second-grade) local configuration of p . Clearly, a configuration ϕ induces a field of local configuration on \mathcal{A} , but a field of local configurations on \mathcal{A} generally need not be an induced field.

LEMMA 2. Every local configuration (\mathbf{K}, \mathbf{M}) at $p \in \mathcal{A}$ can be regarded as the induced local configuration at p of a configuration \varkappa .

PROOF. Choose some coordinate system (X^i) near p ; choose $\varkappa(p)$ to be the origin of \mathcal{R}^3 . Let K^i_j and M^i_{jk} be the components of \mathbf{K} and \mathbf{M} relative to (X^m) . Since the matrix

$[K^i]$ is nonsingular, the inverse function theorem guarantees a neighbourhood \mathcal{N} of p on which the equation

$$(2.3) \quad \varkappa^i(q) = K^i_j X^j + \frac{1}{2} M^i_{jk} X^j X^k, \quad i = 1, 2, 3; \quad q \in \mathcal{N},$$

defines a diffeomorphism \varkappa with the desired properties. Outside \mathcal{N} , \varkappa is arbitrary.

We define a *local reference configuration* of p to be a distinguished local configuration of p , say $(\mathbf{K}, \mathcal{M})$. Relative to such a local configuration, we may define a *local deformation* $(\mathbf{F}, \nabla\mathbf{F})$ by using the following lemma.

LEMMA 3. Suppose $\varkappa: \mathcal{A} \rightarrow \mathcal{R}^3$ is a configuration, and let $\mathbf{f}: \mathcal{R}^3 \rightarrow \mathcal{R}^3$ be a C^∞ — diffeomorphism. Then we have

$$(2.4) \quad (\mathbf{f}\varkappa)_{*p} = \mathbf{f}_{*\varkappa(p)}\varkappa_{*p} \quad \text{and} \quad (\mathbf{f}\varkappa)_{**p} = \mathbf{f}_{**\varkappa(p)}\varkappa_{**p} + \mathbf{f}_{***\varkappa(p)}(\varkappa_{*p}, \varkappa_{*p}),$$

(where the last composition is defined in coordinate form in the proof below).

PROOF. Choose coordinate systems (X^i) on \mathcal{A} , (y^j) on \mathcal{R}^3 , and note that

$$y^k = \varkappa^k(X^1, X^2, X^3), \quad k = 1, 2, 3,$$

for all $\varkappa(p)$. Then, from the chain rule, we have

$$\left. \frac{\partial(\mathbf{f}\varkappa)^k}{\partial X^i} \right|_p = \left. \frac{\partial f^k}{\partial y^j} \right|_{\varkappa(p)} \left. \frac{\partial \varkappa^j}{\partial X^i} \right|_p,$$

and from the product and chain rules we get

$$\left. \frac{\partial^2(\mathbf{f}\varkappa)^k}{\partial X^i \partial X^j} \right|_p = \left. \frac{\partial^2 f^k}{\partial y^m} \right|_{\varkappa(p)} \left. \frac{\partial^2 \varkappa^m}{\partial X^i \partial X^j} \right|_p + \left. \frac{\partial^2 f^k}{\partial y^m \partial y^n} \right|_{\varkappa(p)} \left. \frac{\partial \varkappa^m}{\partial X^i} \right|_p \left. \frac{\partial \varkappa^n}{\partial X^j} \right|_p,$$

which establish the formulae and prove the lemma. We have used the fact that \mathcal{R}^3 is a body in the sense of [5].

The *local deformation* $(\mathbf{F}, \nabla\mathbf{F})$ from the configuration \varkappa to the configuration $\phi \equiv \mathbf{f} \circ \varkappa$ is defined as the pair $(\mathbf{f}_{*\varkappa(p)}, \mathbf{f}_{**\varkappa(p)})$.

Now, suppose that $(\mathbf{K}, \mathcal{M})$ at p corresponds to some configuration \varkappa , i.e. $(\mathbf{K}, \mathcal{M}) = (\varkappa_{*p}, \varkappa_{**p})$. Denote by $(\mathbf{F}, \nabla\mathbf{F})$ the local deformation from $\varkappa(\mathcal{A})$ to $\phi(\mathcal{A})$ corresponding to $\mathbf{f} \equiv \phi \circ \varkappa^{-1}$, where ϕ is some configuration of \mathcal{A} . Then, from Lemma 3,

$$(2.5) \quad \begin{aligned} \phi_{*p} &= \mathbf{F}\mathbf{K}, \\ \phi_{**p} &= \mathbf{F}\mathcal{M} + \nabla\mathbf{F}(\mathbf{K}, \mathbf{K}), \end{aligned}$$

so that

$$(2.6) \quad \begin{aligned} \mathbf{F} &= \phi_{*p} \circ \mathbf{K}^{-1}, \\ \nabla\mathbf{F} &= \phi_{**p}(\mathbf{K}^{-1}, \mathbf{K}^{-1}) - \phi_{*p} \circ \mathbf{K}^{-1} \mathcal{M}(\mathbf{K}^{-1}, \mathbf{K}^{-1}), \end{aligned}$$

i.e.,

$$(2.7) \quad (\mathbf{F}, \nabla\mathbf{F}) = (\phi_{*p}, \phi_{**p}) \circ (\mathbf{K}, \mathcal{M})^{-1}.$$

$\mathbf{F}: \mathcal{R}^3 \rightarrow \mathcal{R}^3$ is a linear isomorphism, $\nabla\mathbf{F}$ is symmetric.

If (x^i) are coordinates for ϕ , (X^A) for κ , then

$$(2.8) \quad \begin{aligned} F^i_A &= \left. \frac{\partial x^i}{\partial X^A} \right|_p, \\ (\nabla F)^i_{AB} &= \left. \frac{\partial^2 x^i}{\partial X^A \partial X^B} \right|_p = (\nabla F)^i_{BA}. \end{aligned}$$

Since \mathbf{F} is an isomorphism, \mathbf{F}^{-1} exists; further,

$$(2.9) \quad (\mathbf{F}^{-1})^A_i = \left. \frac{\partial X^A}{\partial x^i} \right|_p$$

in the chosen coordinates.

R e m a r k. Every pair of values $(\mathbf{F}, \nabla \mathbf{F})$, where \mathbf{F} is nonsingular and $\nabla \mathbf{F}$ is symmetric, can be taken as the value of a diffeomorphism from \mathcal{R}^3 to \mathcal{R}^3 , as in (2.8).

The *response function* r on the body \mathcal{A} is a map from the space of local configurations to the response space (stress, entropy, heat flux, etc.), such that for the configuration ϕ , the response R at the point $p \in \mathcal{A}$ is

$$(2.10) \quad R \equiv r(\phi_{*p}, \phi_{**p}, p).$$

Relative to a given local configuration (\mathbf{K}, \mathbb{M}) , we may define

$$(2.11) \quad R = r(\phi_{*p}, \phi_{**p}, p) \equiv R(\mathbf{F}, \nabla \mathbf{F}, p)$$

by

$$(2.12) \quad r(\phi_{*p}, \phi_{**p}, p) = r(\mathbf{F}\mathbf{K}, \mathbf{F}\mathbb{M} + \nabla \mathbf{F}(\mathbf{K}, \mathbf{K}), p) \equiv R(\mathbf{F}, \nabla \mathbf{F}, p).$$

R is called the *response function relative to* (\mathbf{K}, \mathbb{M}) . If p is equipped with a response function, then it is called a *second-grade material particle*; a body composed of such particles is called a *second-grade body*.

If p and q are second-grade particles, then they are said to be *materially isomorphic* if there exist local reference configurations $(\mathbf{K}(p), \mathbb{M}(p))$, $(\mathbf{K}(q), \mathbb{M}(q))$ such that

$$(2.14) \quad R(\mathbf{F}, \nabla \mathbf{F}, p) = R(\mathbf{F}, \nabla \mathbf{F}, q)$$

for all $(\mathbf{F}, \nabla \mathbf{F})$.

LEMMA 4. p , q are materially isomorphic if, and only if, there exist maps $\mathbf{K}(p, q)$ and $\mathbb{M}(p, q)$, where \mathbf{K} is a linear isomorphism from \mathcal{A}_p to \mathcal{A}_q and \mathbb{M} is symmetric such that

$$(2.15) \quad r(\phi_{*q}, \phi_{**q}, q) = r(\phi_{*q} \mathbf{K}, \phi_{*q} \mathbb{M} + \phi_{**q}(\mathbf{K}, \mathbf{K}), p)$$

for all configurations ϕ .

Naturally, we call such a pair $(\mathbf{K}(p, q), \mathbb{M}(p, q))$ a *material isomorphism* from p to q .

LEMMA 5. If $(\mathbf{K}(p), \mathbb{M}(p))$, $(\mathbf{K}(q), \mathbb{M}(q))$ are local reference configurations such that

$$(2.16) \quad (\mathbf{K}(p, q), \mathbb{M}(p, q)) = (\mathbf{K}(q), \mathbb{M}(q))^{-1} \circ (\mathbf{K}(p), \mathbb{M}(p))$$

is a material isomorphism, then the corresponding relative response functions of p and q satisfy (2.14). Conversely, if (2.14) holds for some $(\mathbf{K}(p), \mathbb{M}(p))$ and $(\mathbf{K}(q), \mathbb{M}(q))$, then the mapping $(\mathbf{K}(p, q), \mathbb{M}(p, q))$ of (2.16) constructed above is a material isomorphism from p to q .

The proof of Lemmas 4 and 5 is similar to that in [1, 5].

3. Isotropy group

Let p be a second-grade particle. Then the *isotropy group* $g(p)$ of p is the group of all material automorphisms of p , i.e., if $h: \mathcal{A}_p \rightarrow \mathcal{A}_p$ is an isomorphism and g is symmetric then $(h, g) \in g(p)$ if, and only if,

$$(3.1) \quad r(\phi_{*p}, \phi_{**p}, p) = r((\phi_{*p}, \phi_{**p}) \circ (h, g), p) = r(\phi_{*p}h, \phi_{*p}g + \phi_{**p}(h, h), p)$$

for all configurations ϕ . As we shall see from Lemma 6 below, $g(p)$ is a group. Clearly, elements of the form $(h, 0)$ in $g(p)$ form a (linear) subgroup of $g(p)$, but in general $g(p)$ is not a linear group. However, if r is continuous, then $g(p)$ is a Lie group [5].

Let (\mathbf{K}, \mathbf{M}) be a local reference configuration of p and R the corresponding relative response function. Then (\mathbf{H}, \mathbf{G}) is called a material automorphism relative to (\mathbf{K}, \mathbf{M}) if $\mathbf{H}: \mathcal{R}^3 \rightarrow \mathcal{R}^3$ is a linear isomorphism and \mathbf{G} is symmetric, and for all $(\mathbf{F}, \nabla \mathbf{F})$,

$$(3.2) \quad R(\mathbf{F}, \nabla \mathbf{F}, p) = R((\mathbf{F}, \nabla \mathbf{F}) \circ (\mathbf{H}, \mathbf{G}), p) = R(\mathbf{F}\mathbf{H}, \mathbf{F}\mathbf{G} + \nabla \mathbf{F}(\mathbf{H}, \mathbf{H}), p).$$

We denote the group of all material automorphisms relative to (\mathbf{K}, \mathbf{M}) by $\mathcal{G}(p)$.

LEMMA 6. $g(p)$ and $\mathcal{G}(p)$ are groups, and

$$(3.3) \quad \mathcal{G}(p) = (\mathbf{K}, \mathbf{M}) \circ g(p) \circ (\mathbf{K}, \mathbf{M})^{-1},$$

where (\mathbf{K}, \mathbf{M}) is a local reference configuration for p .

P r o o f. Closure of each set under the product rule

$$(3.4) \quad (h, g) \circ (h_1, g_1) = (hh_1, hg_1 + g(h_1, h_1))$$

and its analogue for pairs (\mathbf{H}, \mathbf{G}) are consequences of (3.1) and (3.2), respectively. Associativity comes from the fact that the elements are transformations of a set. The identities are $(i, 0)$ and $(\mathbf{I}, 0)$, where $i: \mathcal{A}_p \rightarrow \mathcal{A}_p$, $\mathbf{I}: \mathcal{R}^3 \rightarrow \mathcal{R}^3$ are the identity maps. The inverses are

$$(3.5) \quad \begin{aligned} (h, g)^{-1} &= (h^{-1}, -h^{-1}g(h^{-1}, h^{-1})), \\ (\mathbf{H}, \mathbf{G})^{-1} &= (\mathbf{H}^{-1}, -\mathbf{H}^{-1}\mathbf{G}(\mathbf{H}^{-1}, \mathbf{H}^{-1})), \end{aligned}$$

and these are elements of their respective sets, since, for example,

$$\begin{aligned} R((\mathbf{F}, \nabla \mathbf{F}) \circ (\mathbf{H}, \mathbf{G})^{-1}, p) &= R((\mathbf{F}, \nabla \mathbf{F}) \circ (\mathbf{H}, \mathbf{G})^{-1} \circ (\mathbf{H}, \mathbf{G}), p) \\ &= R((\mathbf{F}, \nabla \mathbf{F}) \circ (\mathbf{I}, 0), p) = R(\mathbf{F}, \nabla \mathbf{F}, p), \end{aligned}$$

when (\mathbf{H}, \mathbf{G}) is an isotropy element or material isomorphism.

The relation (3.3) is proved from (3.1), (3.2) and (2.7):

$$(3.6) \quad \begin{aligned} R((\phi_{*p}, \phi_{**p}) \circ (\mathbf{K}, \mathbf{M})^{-1}, p) &= R((\phi_{*p}, \phi_{**p}) \circ (h, g) \circ (\mathbf{K}, \mathbf{M})^{-1}, p) \\ &= R((\phi_{*p}, \phi_{**p}) \circ (\mathbf{K}, \mathbf{M})^{-1} \circ \{(\mathbf{K}, \mathbf{M}) \circ (h, g) \circ (\mathbf{K}, \mathbf{M})^{-1}\}, p). \end{aligned}$$

LEMMA 7. If $(\mathbf{K}_1, \mathbf{M}_1)$ and $(\mathbf{K}_2, \mathbf{M}_2)$ are local reference configurations of p , then

$$(3.7) \quad \mathcal{G}_2(p) = \mathbf{C}\mathcal{G}_1(p)\mathbf{C}^{-1},$$

where

$$(3.8) \quad \mathbf{C} = (\mathbf{K}_2, \mathbf{M}_2) \circ (\mathbf{K}_1, \mathbf{M}_1)^{-1}.$$

If $c = (\mathbf{K}(p, q), \mathcal{M}(p, q))$ is a material isomorphism of p, q , then

$$(3.9) \quad g(q) = cg(p)c^{-1}.$$

Consequently, if $(\mathbf{K}(p), \mathcal{M}(p)), (\mathbf{K}(q), \mathcal{M}(q))$ are as in Lemma 5,

$$(3.10) \quad \mathcal{G}(q) = \mathcal{G}(p),$$

and if C of (3.8) is a material automorphism, then

$$(3.11) \quad \mathcal{G}_2(p) = \mathcal{G}_1(p).$$

The proof of this lemma is obvious from (3.3), (3.6) and Lemma 5.

4. Simple fluids and isotropic solids

4.1. Second-grade fluids

To define a second-grade fluid point as a particle whose isotropy group with respect to all reference configurations is the set²

$$\{(\mathbf{H}, \mathbf{G}): \mathbf{H} \in \mathcal{U}\mathcal{M}(3), \mathbf{G} \text{ arbitrary but symmetric}\}$$

would remove all the dependence of R on the second gradient $\nabla\mathbf{F}$ (see § 6 below).

To avoid this over-simplification, we seek some restrictions on the pairs (\mathbf{H}, \mathbf{G}) in the isotropy group. Let $(\mathbf{K}, \mathcal{M})$ be a given local reference configuration of p . Then $(\mathbf{K}, \mathcal{M})$ can be regarded as the induced local configuration of a configuration κ , by Lemma 2. Suppose that (\mathbf{H}, \mathbf{G}) is any given local deformation. Then $(\mathbf{H}, \mathbf{G}) \circ (\mathbf{K}, \mathcal{M})$ defines a new local configuration of p and can be regarded as the induced local configuration of a configuration λ . For the deformation $\mathbf{f} \equiv \lambda \circ \kappa^{-1}$ from $\kappa(\mathcal{A})$ to $\lambda(\mathcal{A})$, we then have

$$(4.1) \quad (\mathbf{f}_{**p}, \mathbf{f}_{***p}) = (\mathbf{F}, \nabla\mathbf{F})|_p = (\mathbf{H}, \mathbf{G}).$$

We now choose the following restriction on (\mathbf{H}, \mathbf{G}) : we require that (\mathbf{H}, \mathbf{G}) be an element of $\mathcal{G}(p)$ if, and only if, the deformation $\mathbf{f} = \lambda \circ \kappa^{-1}$ be second-grade volume preserving, i.e.,

$$(4.2) \quad |\det\mathbf{F}| = 1 \quad \text{and} \quad \nabla|\det\mathbf{F}|_p = \mathbf{0}.$$

Two choices were made in obtaining \mathbf{f} , so we must now show that $\mathcal{G}(p)$ is well-defined, i.e., independent of the choices for κ and λ . Indeed, if κ_1 is another configuration such that

$$(\kappa_{1**p}, \kappa_{1***p}) = (\mathbf{K}, \mathcal{M})$$

and, similarly, if λ_1 is such that

$$(\lambda_{1**p}, \lambda_{1***p}) = (\mathbf{H}, \mathbf{G}) \circ (\mathbf{K}, \mathcal{M}),$$

then the deformation $\mathbf{f}_1 \equiv \lambda_1 \circ \kappa_1^{-1}$ from $\kappa_1(\mathcal{A})$ to $\lambda_1(\mathcal{A})$ also obeys the condition

$$(4.3) \quad (\mathbf{F}_1, \nabla\mathbf{F}_1)|_p = (\mathbf{H}, \mathbf{G}).$$

It remains to show that if (4.2) is valid, so too is

$$(4.4) \quad |\det\mathbf{F}_1|_p = 1 \quad \text{and} \quad \nabla|\det\mathbf{F}_1|_p = \mathbf{0},$$

(²) We denote the unimodular group over \mathcal{R}^3 by $\mathcal{U}\mathcal{M}(3)$, and the orthogonal over \mathcal{R}^3 by $\mathcal{O}(3)$.

and *vice versa*. (4.4)₁ is obvious, since $\mathbf{F}(p) = \mathbf{H} = \mathbf{F}_1(p)$ from (4.1) and (4.3). (4.4)₂ then follows from the chain rule and the fact that $\nabla \mathbf{F}(p) = \mathbf{G} = \nabla \mathbf{F}_1(p)$. Thus $\mathcal{G}(p)$ is well-defined by (4.2).

Having stated the desired restrictions on $\mathcal{G}(p)$, we now examine their implications for the pair (\mathbf{H}, \mathbf{G}) . Let (X^A) and (Y^B) be coordinate systems for \mathfrak{x} and \mathfrak{y} , respectively. Then, from (4.1), we have

$$(4.5) \quad H^A_B = \left. \frac{\partial Y^A}{\partial X^B} \right|_p, \quad G^A_{BC} = \left. \frac{\partial^2 Y^A}{\partial X^B \partial X^C} \right|_p.$$

From (4.2)₁, we see that

$$|\det \mathbf{H}| = |\det \partial Y^A / \partial X^B|_p = 1,$$

and from (4.2)₂, we obtain

$$(4.6) \quad \frac{\partial}{\partial X^E} |\det \partial Y^A / \partial X^B|_p = 0.$$

If we expand (4.6) by the chain rule and use the formula $\partial |\det \mathbf{K}| / \partial \mathbf{K} = |\det \mathbf{K}| (\mathbf{K}^{-1})^T$ for any invertible \mathbf{K} , we obtain

$$(4.7) \quad \left. \frac{\partial X^C}{\partial Y^D} \right|_p \left. \frac{\partial^2 Y^D}{\partial X^C \partial X^E} \right|_p = 0,$$

or equivalently, from (4.5),

$$(4.8) \quad (\mathbf{H}^{-1})^C_D G^D_{CE} = 0.$$

We write (4.7) in the form

$$(4.9) \quad \text{tr}(\mathbf{H}^{-1} \mathbf{G}) = \mathbf{0}$$

which defines the operator tr as used below.

The conditions $|\det \mathbf{H}| = 1$ and $\text{tr}(\mathbf{H}^{-1} \mathbf{G}) = \mathbf{0}$ may now be taken as the defining conditions for the isotropy group of a second-grade fluid: p is called a (*second-grade fluid particle*), if there exists a local reference configuration (\mathbf{K}, \mathbf{M}) relative to which the isotropy group $\mathcal{G}(p)$ consists of all pairs (\mathbf{H}, \mathbf{G}) such that

- (a) \mathbf{H} is unimodular, and
- (b) $\text{tr}(\mathbf{H}^{-1} \mathbf{G}) = \mathbf{0}$.

Note that if (a) is satisfied relative to one local reference configuration, it is satisfied for all [1]. But (b) is not independent of the choice of local reference configuration. We call a local reference configuration in which (b) is satisfied an *undistorted reference*.

R e m a r k. Pairs (\mathbf{H}, \mathbf{G}) satisfying (a) and (b) do form a group. We need only check the product of two elements $(\mathbf{H}, \mathbf{G}) \circ (\mathbf{P}, \mathbf{S})$ for condition (b), and

$$\begin{aligned} \{(\mathbf{HP})^{-1}\}^A_B \{\mathbf{HS} + \mathbf{G}(\mathbf{P}, \mathbf{P})\}^B_{AC} &= (\mathbf{P}^{-1})^A_E (\mathbf{H}^{-1})^E_B (H^B_R S^R_{AC} + G^B_{RT} P^R_A P^T_C) \\ &= (\mathbf{H}^{-1})^E_B G^B_{ET} P^T_C + (\mathbf{P}^{-1})^A_E S^E_{AC} = 0, \end{aligned}$$

since each part is separately zero.

We have now come to the first of our basic results.

PROPOSITION 8. *The response function $R(\mathbf{F}, \nabla\mathbf{F}, p)$ relative to an undistorted reference of a fluid particle p can be represented by a function of $|\det\mathbf{F}|$ and $\text{grad}|\det\mathbf{F}|$; i.e., $(\mathbf{F}, \nabla\mathbf{F})$ and $(\bar{\mathbf{F}}, \bar{\nabla}\bar{\mathbf{F}})$ satisfy*

$$(4.10) \quad |\det\mathbf{F}| = |\det\bar{\mathbf{F}}|$$

and

$$(4.11) \quad \text{grad}|\det\mathbf{F}| = \text{grad}|\det\bar{\mathbf{F}}|$$

if, and only if, the local deformation (\mathbf{H}, \mathbb{G}) connecting them, viz.,

$$(4.12) \quad (\mathbf{H}, \mathbb{G}) = (\mathbf{F}, \nabla\mathbf{F})^{-1} \circ (\bar{\mathbf{F}}, \bar{\nabla}\bar{\mathbf{F}}),$$

belongs to the isotropy group $\mathcal{G}(p)$.

PROOF OF NECESSITY. We must show that the pair (\mathbf{H}, \mathbb{G}) of (4.12) satisfies (a) and (b).

Lemmas 2 and 3 guarantee that there are configurations $\boldsymbol{\kappa}$, $\boldsymbol{\lambda}$, ϕ , and $\bar{\phi}$ such that the diffeomorphisms $\mathbf{f} = \phi \circ \boldsymbol{\lambda}^{-1}$, $\bar{\mathbf{f}} = \bar{\phi} \circ \boldsymbol{\kappa}^{-1}$, and $\mathbf{h} = \boldsymbol{\lambda} \circ \boldsymbol{\kappa}^{-1}$ give rise to $(\mathbf{F}, \nabla\mathbf{F})$, $(\bar{\mathbf{F}}, \bar{\nabla}\bar{\mathbf{F}})$, and (\mathbf{H}, \mathbb{G}) when their gradients are evaluated at p . We omit this evaluation in what now follows.

From (4.12), $\mathbf{H} = \mathbf{F}^{-1}\bar{\mathbf{F}}$ and so (4.10) obviously implies that (a) holds. Then, since the chain rule gives

$$(4.13) \quad \text{grad}|\det\mathbf{F}|_j = |\det\mathbf{F}|(\nabla\mathbf{F})^i_{AB}(\mathbf{F}^{-1})^A_i(\mathbf{F}^{-1})^B_j$$

together with a similar formula for $\text{grad}|\det\bar{\mathbf{F}}|_j$, (4.10) and (4.11) combine to give

$$(4.14) \quad (\bar{\nabla}\bar{\mathbf{F}})^i_{AB}(\bar{\mathbf{F}}^{-1})^A_i(\bar{\mathbf{F}}^{-1})^B_j = (\nabla\mathbf{F})^i_{AB}(\mathbf{F}^{-1})^A_i(\mathbf{F}^{-1})^B_j.$$

We now use the formulae for inverse and product of second-grade isotropy elements to obtain

$$(4.15) \quad (\mathbf{H}^{-1})^A_B G^B_{AC} = \{(\mathbf{F}^{-1}\bar{\mathbf{F}})^{-1}\}^A_B \{-\mathbf{F}^{-1}\nabla\mathbf{F}(\mathbf{F}^{-1}\bar{\mathbf{F}}, \mathbf{F}^{-1}\bar{\mathbf{F}}) + \mathbf{F}^{-1}\bar{\nabla}\bar{\mathbf{F}}\}^B_{AC} \\ = \{-\nabla\mathbf{F}\}^i_{MN}(\mathbf{F}^{-1})^M_i(\mathbf{F}^{-1})^N_j + (\bar{\nabla}\bar{\mathbf{F}})^i_{MN}(\bar{\mathbf{F}}^{-1})^M_i(\bar{\mathbf{F}}^{-1})^N_j\} \bar{\mathbf{F}}^j_C = 0, \quad \text{from (4.14).}$$

Hence (b) holds also.

PROOF OF SUFFICIENCY. If (a) and (b) hold for (\mathbf{H}, \mathbb{G}) , then (4.10) and (4.11) are trivial consequences of the reverse algebra via (4.12), (4.14) and (4.15). Hence the proposition is proved.

From balance of mass it is well known that the densities ρ_0 and ρ in the local reference configuration $(\mathbf{K}, \mathcal{M})$ and in the present configuration $(\mathbf{F}, \nabla\mathbf{F}) \circ (\mathbf{K}, \mathcal{M})$, respectively, are related by

$$(4.16) \quad \rho_0 = \varrho|\det\mathbf{F}|,$$

which implies also

$$(4.17) \quad \text{grad}\rho_0 = |\det\mathbf{F}|\text{grad}\rho + \rho\text{grad}|\det\mathbf{F}|.$$

Hence $|\det\mathbf{F}|$ and $\text{grad}|\det\mathbf{F}|$ are determined by ρ and $\text{grad}\rho$ and *vice versa*, at any point p , since a full specification of a second grade particle requires that ρ_0 and $\text{grad}\rho_0$ (or $\nabla\rho_0$) be given in the local reference configuration. From this simple remark we see that the preceding proposition is equivalent to:

COROLLARY 9. The response function $R(\mathbf{F}, \nabla\mathbf{F}, p)$ relative to an undistorted reference of a fluid particle p can be represented by a function $\tilde{R}(\rho, \text{grad } \rho, p)$ of ρ and $\text{grad } \rho$, i.e.

$$(4.18) \quad R(\mathbf{F}, \nabla\mathbf{F}, p) = \tilde{R}(\rho, \text{grad } \rho, p).$$

4.2. Second-grade solids

To obtain suitable restriction on pairs (\mathbf{H}, \mathbf{G}) for solid particles, let us look at the local reference configuration induced by κ and λ once again. Then as above, we have

$$(\mathbf{F}, \nabla\mathbf{F})|_p = (\mathbf{H}, \mathbf{G}).$$

But we now choose the following restriction on (\mathbf{H}, \mathbf{G}) : we require that (\mathbf{H}, \mathbf{G}) be an element of $\mathcal{G}(p)$ if, and only if, the deformation $\lambda \circ \kappa^{-1}$ be *second-grade inner-product preserving at p* , i.e., that

$$(4.19) \quad (\mathbf{F}^T\mathbf{F})|_p = 1 \quad \text{and} \quad \nabla(\mathbf{F}^T\mathbf{F})|_p = \mathbf{0}.$$

As before, we can show that $\mathcal{G}(p)$ is well defined by this restriction. Further, if (X^A) and (Y^B) are coordinate systems corresponding to κ and λ as before, then the restriction (4.19) has the coordinate form

$$\left. \frac{\partial Y^B}{\partial X^A} \right|_p \left. \frac{\partial Y^B}{\partial X^C} \right|_p = \delta_{AC}$$

and

$$(4.19') \quad \left. \frac{\partial^2 Y^B}{\partial X^E \partial X^A} \right|_p \left. \frac{\partial Y^B}{\partial X^C} \right|_p + \left. \frac{\partial Y^B}{\partial X^A} \right|_p \left. \frac{\partial^2 Y^B}{\partial X^E \partial X^C} \right|_p = 0.$$

This last equation, after cyclic interchanges of the indices A, C, E and a suitable combination, gives

$$\left. \frac{\partial^2 Y^B}{\partial X^A \partial X^E} \right|_p \left. \frac{\partial Y^B}{\partial X^C} \right|_p = 0,$$

or

$$(4.20) \quad \left. \frac{\partial^2 Y^B}{\partial X^A \partial X^E} \right|_p = 0.$$

(4.19')₁ implies that $[\partial Y^B / \partial X^E]_p$ is orthogonal. So the restrictions imply that

(c) $H^B_E = \partial Y^B / \partial X^E|_p$ is orthogonal, and

(d) $G^B_{AE} = \partial^2 Y^B / \partial X^A \partial X^E|_p = 0$.

So we define a *second-grade solid particle* to be a particle p whose isotropy group $\mathcal{G}(p)$, relative to some local reference configuration, is a linear group, a subgroup of the orthogonal group $\mathcal{O}(3)$ under the identification

$$(\mathbf{H}, \mathbf{O}) \rightarrow \mathbf{H}.$$

Notice that both (c) and (d) depend on the choice of local reference configuration. As before, we call a local reference configuration an *undistorted reference* if it satisfies these conditions. An *isotropic solid particle* is a particle p for which $\mathcal{G}(p)$ is the orthogonal group $\mathcal{O}(3)$ for some local reference configuration.

PROPOSITION 10. *Relative to an undistorted reference, the response function $R(\mathbf{F}, \nabla\mathbf{F}, p)$ of an isotropic solid particle p can be represented by a function of the left Cauchy-Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and of $\text{grad } \mathbf{B} = \text{grad}(\mathbf{F}\mathbf{F}^T)$, i.e., $(\mathbf{F}, \nabla\mathbf{F})$ and $(\bar{\mathbf{F}}, \bar{\nabla}\bar{\mathbf{F}})$ are such that*

$$(4.21) \quad \mathbf{B} = \bar{\mathbf{B}},$$

$$(4.22) \quad \text{grad } \mathbf{B} = \text{grad } \bar{\mathbf{B}}$$

if, and only if, the second-grade local deformation (\mathbf{H}, \mathbf{G}) defined by

$$(4.23) \quad (\mathbf{H}, \mathbf{G}) = (\mathbf{F}, \nabla\mathbf{F})^{-1} \circ (\bar{\mathbf{F}}, \bar{\nabla}\bar{\mathbf{F}})$$

belongs to the isotropy group $\mathcal{G}(p)$.

Proof of necessity. We must show that (\mathbf{H}, \mathbf{G}) satisfies (c) and (d). (4.23) gives $\mathbf{H} = \mathbf{F}^{-1}\bar{\mathbf{F}}$, and (4.21) implies that

$$(4.24) \quad \mathbf{H}\mathbf{H}^T = \mathbf{F}^{-1}\bar{\mathbf{F}}\bar{\mathbf{F}}^T(\mathbf{F}^T)^{-1} = \mathbf{I},$$

so (c) holds. We have

$$(4.25) \quad B^{ij} = F^i_A F^j_A,$$

and

$$(4.26) \quad (\text{grad } \mathbf{B})^{ij}_k = (\nabla\mathbf{F})^i_{AB} F^j_A (\mathbf{F}^{-1})^B_k + (\nabla\mathbf{F})^j_{AB} F^i_A (\mathbf{F}^{-1})^B_k = M^i_{km} B^{mj} + M^j_{km} B^{mi},$$

where

$$(4.27) \quad M^i_{jk} = (\nabla\mathbf{F})^i_{AB} (\mathbf{F}^{-1})^A_j (\mathbf{F}^{-1})^B_k, \quad \mathbb{M} \equiv \nabla\mathbf{F}(\mathbf{F}^{-1}, \mathbf{F}^{-1}).$$

From differentiation of

$$(4.28) \quad (\mathbf{B}^{-1})_{ik} B^{jk} = \delta^j_i,$$

we get

$$(4.29) \quad \frac{\partial(\mathbf{B}^{-1})_{ik}}{\partial x^m} = -(\mathbf{B}^{-1})_{is} \frac{\partial B^{sj}}{\partial x^m} (\mathbf{B}^{-1})_{kj}$$

and hence from (4.26), we get

$$(4.30) \quad B^{mj} (\text{grad}(\mathbf{B}^{-1}))_{ikm} = -B^{mj} (\mathbf{B}^{-1})_{is} M^s_{mk} - B^{mj} (\mathbf{B}^{-1})_{ks} M^s_{mi}.$$

Now form

$$(4.31) \quad (\mathbf{B}^{-1})_{is} \partial B^{sj} / \partial x^k + (\mathbf{B}^{-1})_{ks} \partial B^{sj} / \partial x^i = 2M^j_{ik} - B^{mj} \partial(\mathbf{B}^{-1})_{ik} / \partial x^m,$$

using (4.26), (4.27) and (4.29).

Now, evaluate these expressions at p . Then (4.21) gives

$$(4.32) \quad \mathbf{B}(p) = \bar{\mathbf{B}}(p), \quad \text{and} \quad \mathbf{B}^{-1}(p) = \bar{\mathbf{B}}^{-1}(p),$$

while (4.22), (4.29) and (4.32) give

$$(4.33) \quad \{\text{grad}(\mathbf{B}^{-1})\}(p) = \{\text{grad}(\bar{\mathbf{B}}^{-1})\}(p),$$

as well as

$$(4.34) \quad \text{grad } \mathbf{B}(p) = \text{grad } \bar{\mathbf{B}}(p).$$

Hence (4.31), (4.32) and (4.33) give

$$(4.35) \quad \mathbb{M} = \bar{\mathbb{M}}.$$

But (4.23) and (4.27) give

$$(4.36) \quad \mathbf{F} \mathbf{G} (\bar{\mathbf{F}}^{-1}, \bar{\mathbf{F}}^{-1}) = \bar{\mathbf{M}} - \mathbf{M} = 0.$$

Therefore, since $\bar{\mathbf{F}}$ and \mathbf{F} are invertible,

$$\mathbf{G} = 0.$$

Proof of sufficiency. This is now trivial. \mathbf{H} orthogonal implies $\mathbf{B}(p) = \bar{\mathbf{B}}(p)$; and $\mathbf{G} = 0$ implies $\mathbf{M} = \bar{\mathbf{M}}$ from (4.36), and thus $\text{grad} \mathbf{B}(p) = \text{grad} \bar{\mathbf{B}}(p)$ from (4.26).

Therefore, the relative response function for an isotropic solid particle can be reduced from a function of $(\mathbf{F}, \nabla \mathbf{F})$ to $(\mathbf{B}, \text{grad} \mathbf{B})$, and we can write

$$(4.37) \quad R = R(\mathbf{F}, \nabla \mathbf{F}, p) \equiv \bar{R}(\mathbf{B}, \text{grad} \mathbf{B}, p).$$

5 Mixtures without chemical reactions

Suppose we have a collection of bodies \mathcal{A}_α , $\alpha = 1, 2, \dots, N$, such that for some region \mathcal{V} in \mathcal{R}^3 at time t ,

$$\chi_\alpha: \mathcal{A}_\alpha \rightarrow \mathcal{V}$$

is a diffeomorphism for each α . Then \mathcal{V} is a *mixture*. To each point \mathbf{x} in \mathcal{V} , there correspond N particles, say $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_N$ for reference configurations $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$.

We consider the following type of response function for the α -th constituent particle of a mixture at (\mathbf{x}, t) :

$$(5.1) \quad R_\alpha = R_\alpha(\mathbf{x}, t) = R_\alpha(\theta_\beta, \mathbf{g}_\beta, \mathbf{F}_\beta, \nabla_\beta \mathbf{F}_\beta, \dot{\mathbf{x}}_\beta; \mathbf{X}_\beta), \quad \alpha = 1, 2, \dots, N.$$

Here, R_α may depend on \mathbf{X}_β , $\beta = 1, 2, \dots, N$, so that (5.1) is a point condition, and all variables relate to a particular set of points. For this reason, we will suppress this explicit dependence on \mathbf{X}_β , both in this section and the next.

The independent variables in (5.1) are the $34N$ values below:

θ_β is the temperature of the β -th constituent; $\mathbf{g}_\beta = \text{grad} \theta_\beta$ is the temperature gradient of the β -th constituent; $(\mathbf{F}_\beta, \nabla_\beta \mathbf{F}_\beta)$ is the local deformation of the β -th constituent corresponding to some local reference configuration $(\mathbf{K}_\beta, \mathbf{M}_\beta)$ at \mathbf{X}_β in \mathcal{A}_β and $\dot{\mathbf{x}}_\beta = \partial \chi_\beta(\mathbf{X}_\beta, t) / \partial t$ is the velocity of the β -th constituent. The subscript β under the arguments in (5.1) indicates a dependence on all N sets of arguments for each constituent.

While $\mathbf{F}_\beta, \nabla_\beta \mathbf{F}_\beta$ may be regarded as gradients (Lemma 3, and the following Remark), it is not necessary that they be induced from the configuration \mathbf{x}_β . In fact, we have

$$(\mathbf{F}_\beta, \nabla_\beta \mathbf{F}_\beta) = (\chi_{\beta**}, \chi_{\beta***}) \circ (\mathbf{K}_\beta, \mathbf{M}_\beta)^{-1}$$

at \mathbf{X}_β , for $\beta = 1, 2, \dots, N$.

In this section, we treat mixtures in which no chemical reactions take place. The mass balance law for the constituents of a mixture [e.g. 3, (2.32)₁, (2.35)] is

$$(5.2) \quad \dot{\rho}_\alpha + \rho_\alpha \text{div} \dot{\mathbf{x}}_\alpha = \rho \hat{c}_\alpha,$$

where \hat{c}_α is the supply of mass to the α -th constituent due to chemical reactions between the other $N-1$ constituents. In the absence of chemical reactions, we have $\hat{c}_\alpha = 0$ for $\alpha = 1, 2, \dots, N$, and we can integrate (5.2) to obtain

$$(5.3) \quad \rho_\alpha |\det \mathbf{F}_\alpha| = \rho_{\alpha 0}$$

for $\alpha = 1, 2, \dots, N$, where the $\rho_{\alpha 0}$ are constants. (5.3) is the basic idea behind the requirement in § 4 that the linear transformations be unimodular, and it corresponds to (4.16).

For each constituent particle at \mathbf{x} at t we may define an isotropy group $\mathcal{G}_\alpha(\mathbf{x}, t)$ as follows: $(\mathbf{H}, \mathbf{G}) \in \mathcal{G}_\alpha$ if, and only if, for $\beta = 1, 2, \dots, N$,

$$(5.4) \quad R_\beta(\theta_\sigma, \mathbf{g}_\sigma, \mathbf{F}_1, \dots, \mathbf{F}_\alpha, \dots, \mathbf{F}_N, \nabla_1 \mathbf{F}_1, \dots, \nabla_\alpha \mathbf{F}_\alpha, \dots, \nabla_N \mathbf{F}_N, \mathbf{x}_\sigma) \\ = R_\beta(\theta_\sigma, \mathbf{g}_\sigma, \mathbf{F}_1, \dots, \mathbf{F}_\alpha \mathbf{H}, \dots, \mathbf{F}_N, \nabla_1 \mathbf{F}_1, \dots, \mathbf{F}_\alpha \mathbf{G} + \nabla_\alpha \mathbf{F}_\alpha(\mathbf{H}, \mathbf{H}), \dots, \nabla_N \mathbf{F}_N, \dot{\mathbf{x}}_\sigma).$$

It is clear that \mathcal{G}_α is a group of the same type as that defined in § 3.

It is natural, in view of (5.3), to ask that, relative to some undistorted reference, \mathcal{G}_α preserve the volume of the α -th constituent to the second grade in the reference configuration. We call an *inert* (or *non-reacting*) *constituent particle* a fluid if its isotropy group satisfies the requirements on a fluid in § 4A, and an (isotropic) solid if its group satisfies the requirements on an (isotropic) solid in § 4B.

Suppose we have a mixture of inert constituents at a point in \mathcal{V} at t , a mixture of inert fluid and isotropic solids, say, fluids for $\alpha = 1, 2, \dots, \phi$, and isotropic solids for $\alpha = \phi + 1, \phi + 2, \dots, N$. Then the arguments of § 4 apply and we obtain immediately that (5.1) reduces to

$$(5.5) \quad R_\alpha(\mathbf{x}, t) = R_\alpha(\theta_\beta, \mathbf{g}_\beta, \rho_\omega, \text{grad } \rho_\omega, \mathbf{B}_\tau, \text{grad } \mathbf{B}_\tau, \dot{\mathbf{x}}_\beta)$$

for $\alpha, \beta = 1, 2, \dots, N$; $\omega = 1, 2, \dots, \phi$; $\tau = \phi + 1, \phi + 2, \dots, N$.

Thus the Eqs. (6.11) of [3] are simplified; the awkward tensors \mathbf{M}_α are replaced by $\text{grad } \mathbf{B}_\alpha$, a fact later known to Bowen; similarly, (5.10) of [6] is simplified. Application of the constraints $\theta_\alpha \equiv \theta$, $\mathbf{g}_\alpha \equiv \mathbf{g}$ does not affect our (5.5), nor does the inclusion of the extra variable $\dot{\mathbf{F}}_\alpha$, as these may be immediately replaced by

$$\dot{\mathbf{F}}_\alpha \mathbf{F}_\alpha^{-1},$$

which is obviously not affected by local reference configuration changes.

PROPOSITION 11. *For a mixture of inert fluid and isotropic solids, the response function relative to an undistorted reference can be represented by a function of density and density gradient for the fluid constituents, and by a function of the left Cauchy-Green tensor and its gradient for the isotropic solid constituents, together with the temperatures, temperature gradients and the velocities of the constituents, i.e., from (5.1) to (5.5).*

6. Mixtures with chemical reactions

For mixtures of second-grade materials with chemical reactions, it seems pointless to require that the change in local reference configuration preserve the density or volume in the reference configuration, since (5.3) is no longer valid.

We adopt the (slightly modified) list of independent variables of BOWEN [8],

$$(6.1) \quad \theta_\alpha, \mathbf{g}_\alpha, \mathbf{F}_\alpha, \nabla_\alpha \mathbf{F}_\alpha, \dot{\mathbf{x}}_\alpha, \rho_\alpha J_\alpha, \text{grad}(\rho_\alpha J_\alpha),$$

where $J_\alpha = |\det \mathbf{F}_\alpha|$. The variables are independent if the \hat{c}_α are non-zero in (5.2), i.e., $\rho_\alpha J_\alpha$ is independent of \mathbf{F}_α , and $\text{grad}(\rho_\alpha J_\alpha)$ is independent of \mathbf{F}_α and $\nabla_\alpha \mathbf{F}_\alpha$, if the mass of the α -th constituent is not conserved.

Since the density field in the reference configuration is now not restricted by (5.3) or (4.16), we may drop the requirement that the linear part of the isotropy group be a subgroup of the unimodular group \mathcal{UM} (3), as in [8]. So let us define a *reacting fluid* to be a particle whose linear group is the full second-order group, i.e., if (\mathbf{H}, \mathbf{G}) is an element of the isotropy group for a reacting fluid if \mathbf{H} is invertible and \mathbf{G} is arbitrary but symmetric.

In general, $(\mathbf{H}, \mathbf{G}) \in \mathcal{G}_\alpha(\mathbf{x}, t)$ if, and only if, for all values of the argument,

$$(6.2) \quad R_\beta(\mathbf{x}, t) = R_\beta(\theta_\sigma, \mathbf{g}_\sigma, \mathbf{F}_1, \dots, \mathbf{F}_\alpha, \dots, \mathbf{F}_N, \nabla_1 \mathbf{F}_1, \dots, \nabla_\alpha \mathbf{F}_\alpha, \dots, \nabla_N \mathbf{F}_N, \dot{\mathbf{x}}_\sigma, \\ \rho_1 J_1, \dots, \rho_\alpha J_\alpha, \dots, \rho_N J_N, \text{grad}(\rho_1 J_1), \dots, \text{grad}(\rho_\alpha J_\alpha), \dots, \text{grad}(\rho_N J_N)) \\ = R_\beta(\theta_\sigma, \mathbf{g}_\sigma, \mathbf{F}_1, \dots, \mathbf{F}_\alpha \mathbf{H}, \dots, \mathbf{F}_N, \nabla_1 \mathbf{F}_1, \dots, \mathbf{F}_\alpha \mathbf{G} + \nabla_\alpha \mathbf{F}_\alpha (\mathbf{H}, \mathbf{H}), \dots, \nabla_N \mathbf{F}_N, \dot{\mathbf{x}}_\sigma, \\ \rho_1 J_1, \dots, \rho_\alpha J_\alpha |\det \mathbf{H}|, \dots, \rho_N J_N, \text{grad}(\rho_1 J_1), \dots, \text{grad}(\rho_\alpha J_\alpha |\det \mathbf{H}|), \dots, \text{grad}(\rho_N J_N)),$$

where $\mathbf{H} \in \mathcal{GL}(3)$, and $\beta = 1, 2, \dots, N$.

A necessary consequence of (6.2) is that the dependence on the reacting fluid variables is reduced as follows, by noting that $(\mathbf{H}, \mathbf{G}) = (\mathbf{F}_\alpha, \nabla_\alpha \mathbf{F}_\alpha)^{-1}$ is an allowable choice:

$$(6.3) \quad \rho_\alpha J_\alpha |\det \mathbf{H}| = \rho_\alpha |\det \mathbf{F}_\alpha \mathbf{F}_\alpha^{-1}| = \rho_\alpha, \quad \text{grad}(\rho_\alpha |\det \mathbf{F}_\alpha \mathbf{H}|) = \text{grad} \rho_\alpha, \\ (\mathbf{F}_\alpha, \nabla_\alpha \mathbf{F}_\alpha) \circ (\mathbf{F}_\alpha, \nabla_\alpha \mathbf{F}_\alpha)^{-1} = (\mathbf{1}, \mathbf{0})$$

and $\theta_\alpha, \mathbf{g}_\alpha, \dot{\mathbf{x}}_\alpha$ are unchanged. Hence, for a mixture of reacting fluids,

$$(6.4) \quad R_\beta(\mathbf{x}, t) = R_\beta(\theta_\sigma, \mathbf{g}_\sigma, \mathbf{1}, \mathbf{1}, \dots, \mathbf{1}, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0}, \dot{\mathbf{x}}_\sigma, \rho_\sigma, \text{grad} \rho_\sigma),$$

for $\beta = 1, 2, \dots, N$; i.e., we can define new functions \tilde{R}_β and write for $\beta = 1, 2, \dots, N$,

$$(6.5) \quad R_\beta(\mathbf{x}, t) = \tilde{R}_\beta(\theta_\sigma, \mathbf{g}_\sigma, \dot{\mathbf{x}}_\sigma, \rho_\sigma, \text{grad} \rho_\sigma).$$

Note further that the same reduced form (6.5) holds for a mixture of inert and reacting fluids, since (5.3) is a constraint which reduces the set $(\mathbf{F}, \nabla \mathbf{F}, \rho J, \text{grad}(\rho J))$ to $(\mathbf{F}, \nabla \mathbf{F})$ for an inert fluid and reduces the linear part of the isotropy group $\mathcal{UM}(3)$.

The definitions of inert and reacting fluids are compatible; for in the case of a reacting fluid, the density in the reference configuration need not be preserved, but for an inert fluid, preservation of the densities in both configurations, under (5.3), implies that $|\det \mathbf{H}| = 1$.

For a reacting solid particle, or a reacting isotropic solid particle, we leave the definition unchanged, i.e., the isotropy group has elements of the form $(\mathbf{H}, \mathbf{0})$, $\mathbf{H} \in \mathcal{O}(3)$. Then the analogous definition for the α -th constituent particle to be an isotropic solid will have $\mathbf{H} \in \mathcal{O}(3)$, $\mathbf{G} = \mathbf{0}$, in (6.2). Then, exactly as in Proposition 10, we can reduce the dependence from $(\mathbf{F}, \nabla \mathbf{F}, \rho J, \text{grad}(\rho J))$ to $(\mathbf{B}, \text{grad} \mathbf{B}, \rho, \text{grad} \rho)$, where the reduction follows from

$$(6.6) \quad \rho |\det \mathbf{F}| = \rho |\det \mathbf{B}|^{1/2}, \\ \text{grad}(\rho |\det \mathbf{F}|) = |\det \mathbf{B}|^{1/2} \text{grad} \rho + \frac{1}{2} \rho |\det \mathbf{B}|^{1/2} \text{tr}(\mathbf{B}^{-1} \text{grad} \mathbf{B})$$

for reacting isotropic solids. Note that for *inert* isotropic solids, (5.3) reduces the dependence to $(\mathbf{B}, \text{grad} \mathbf{B})$ as before, since ρ and $\text{grad} \rho$ can be expressed in terms of \mathbf{F} and $\nabla \mathbf{F}$, as in (4.16) and (4.17).

So for a mixture of ϕ fluids, σ reacting isotropic solids, and $N - \phi - \sigma$ inert solids, we have the response functions

$$(6.7) \quad R_\alpha(\mathbf{x}, t) = R_\alpha(\theta_\beta, \mathbf{g}_\beta, \dot{\mathbf{x}}_\beta, \rho_\omega, \text{grad } \rho_\omega, \mathbf{B}_\tau, \text{grad } \mathbf{B}_\tau),$$

where $\alpha, \beta = 1, 2, \dots, N$, $\omega = 1, 2, \dots, \phi + \sigma$, and $\tau = \phi + 1, \dots, N$.

THEOREM 12. *For a mixture of second-grade fluids and isotropic solids, the response function relative to an undistorted reference can be represented by a function of the density and density gradient for the fluids, whether they be inert or reacting, to a function only of the left Cauchy-Green tensor and its gradient for the inert isotropic solids, and to a function of all four of these variables if the isotropic solid is reacting, together with the temperature gradients and the velocities of all the constituents, i.e., from (6.1) to (6.7).*

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