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On the duality of foundations of mechanics of discrete elastic systems

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THE PAPER contains a discussion of dual aspects of the minimum energy theorems by Lagrange and Castigliano, their relations to the dual Legendre transformation and the concept of construction of a kineto-static potential representing a generalization of the notion of energy.

Praca zawiera omówienie dwoistych aspektów zasad minimalnych Lagrange'a i Castigliano, ich związku z dualną transformacją Legendre'a oraz koncepcji utworzenia potencjału kineto-statycznego jako uogólnienia pojęcia energii.

В работе рассматриваются двойственные аспекты принципов минимума Лагранжа и Кастильяно, их связи с двойственном преобразованием Лежандра, а также обсуждается вопрос о кинетостатическом потенциале, обобщающем понятие энергии.

1. Preliminaries

THE PAPER sets out to discuss the fundamental questions of dual relations occurring in mechanics of elastic discrete systems subject to nodal loads. Attention is focused mainly on problems connected with the dual Legendre transformation, on its mechanical interpretation and on its relation to the two general kineto-static, energy-type potentials L_1, L_2 introduced in Sec. 6. It is shown that, owing to the Legendre transformation, these potentials may, if necessary, be represented in the two equivalent forms of functions of kinetic or static variables. In this manner, the dual character of energy notions is accentuated. Many of the facts indicated in the paper are not to be considered as original; but the author hopes that placing them in a new perspective may contribute to a better understanding of the inter-connections and dual relations in mechanics.

The paper is a result of a search for a systematic and the most complete approach possible to the dual extremum problems connected with the determination of states of equilibrium of forces and compatibility of displacements in discrete elastic systems with bilateral constraints. The idea of a dual problem is usally associated with mathematical programming in which it plays a fundamental role. Consequently, it should be stressed here that confining this idea and its application to mathematical programming only excessively reduces the content of the notion of dualism, which may be successfully applied e.g. in the classical problem of Lagrange multipliers [3]. To avoid misunderstanding, it should further be stressed that in this paper the notion of dual problems is not associated with mathematical programming, the discussion being exclusively confined to classical problems of mechanics.

In spite of the fact that there are to be found in the literature accessible to the author various formulations and approaches to the energetistic problems of discrete and continuous elastic systems, still lacking is a full and completely satisfying approach — an approach including all the aspects of the problem, even if concerning the discrete systems only. In the author's opinion such an approach should contain the determination and interpretation of the relations which hold true between the dual Legendre transformation [7, 14] and the mutually dual extremum principles by LAGRANGE and CASTIGLIANO. This duality appears to require a special attention since it is not so far generally accepted.

ARGYRIS [19], in his papers devoted to energy theorems in structural mechanics, mentioned the mini-max theorems; the formulation of them is, however, somewhat vague and superficial. In the book by FUNK [2] may be found an exact formulation of extremum problems and theorems in mechanics (based on what is called the Friedrichs principle), though the considerations are confined to linearly elastic systems. Interpretation of the Lagrange and Castigliano theorems based on the Legendre transformation is presented by LURIE [14], no conclusions being drawn, however, with respect to extremum problems. In the book [12] by COURANT and HILBERT, the mini-max variational problems are treated in a purely mathematical manner, and their applications are limited to the Castigliano theorem. In the majority of handbooks dealing with elasticity [10, 13, 15], the minimum theorems concerning the total actual and complementary energies are formulated and applied, but a complete analysis of their mutual relations and reference to the Legendre transformation is usually lacking. The difficulties which arise in solving problems involving mixed boundary conditions constitute a typical manifestation of a principle which is more general — and hence more complicated — than the two minimum principles. REISSNER [18] and ABOVSKII [9] derive the general form of energy functionals for the problems of linear elasticity, and the Lagrange, Castigliano and Reissner functionals follow as special cases from that general form, no reference being made, however, to dual problems. In the theory of load carrying capacity (e.g. [4, 6, 20]), the extremum theorems are used for two-sided estimation of the load capacity and for optimization of the structural properties, also without going more deeply into the common theoretical foundations of the theorems. Similarly, in a number of papers dealing with applications of variational methods in structural mechanics (e.g. [5, 11, 16]) use is made of the minimum theorems without referring to the dual relations between them. GOLDENBLAT'S book [11] contains a discussion of extremum theorems and introduces the generalized mixed potentials of structural mechanics, but his purely formal considerations lead him to an erroneous formulation of the variational principle.

In the present author's opinion, the most complete and self-consistent approach to the problems is to be found in the paper by SEWELL [7] in which the general theory is illustrated by examples from various fields of continuum mechanics. The theory of optimization and dual approximation is constructed on the basis of mathematical programming formulated, from the very beginning, in a dual approach. The paper is characterized by seeking a new, common approach to the problems both of mathematical programming and continuum mechanics.

Several conclusions of a rather essential nature may be drawn from the short and certainly incomplete review presented above. The consistently dual approach to the foundations and principles of mechanics is relatively new, though the way for it was cleared long ago. This approach is not used particularly frequently despite the really urgent needs: on the one hand, application of approximate methods, both analytical and numerical, necessitates the estimation of errors which determine the applicability of the corresponding methods and algorithms. Two-sided — usually the most valuable — estimation may — in the majority of cases — be realized only by means of a dual approach. On the other hand, considering both the minimum energy theorems as particular cases of a more general principle creates a convenient basis for a more universal and through analysis of mechanical problems.

2. Introduction

The notion of duality will be considered in what follows to express the action of two principles, mutually independent and qualitatively different but complementing each other. This action is manifested in mechanics of elastic systems in the form of theorems by LAGRANGE (1788) and CASTIGLIANO (1875) [17], which were formulated on the basis of considerations of mainly linearly elastic systems and served for many years as satisfactory tools for solving systems of that type.

The necessity of revision and generalization of these methods of analysis when applied to engineering structures made of materials exhibiting physical non-linearity effects was first observed by ENGESSER [1] in 1889. He found the Menabrea-Castigliano theorem on minimum of elastic energy (the least work of deformation principle [1, 17]) to be not a general theorem and to lose its validity in the case of structures made of materials which do not conform to Hooke's law. The least work of deformation principle was then replaced by the more general principle of *least complementary work* which remains true also in the case of arbitrary relation $\varepsilon = f(\sigma)$.

Engesser's formulation of the new principle of mechanics was a result of the introduction of a new notion of *complementary work* as opposed to the *actual work*. In spite of the fact that the notion was introduced in a purely formal manner (being defined as the quantitative "difference between the virtual and actual works"), it contributed to the construction of a new basis for a dual formulation of principles of mechanics.

Without going into details, it may be observed that, in general, in the two mutually dual notions of work forces and displacements exchange their roles in the following manner: in the case of actual work, the force is expressed as a function of displacement, while in the case of complementary work, displacement plays the role of a function depending on the value of the force applied. It should be noted that the latter situation is in fact closer to our "everyday experience" — in the contrary to the first impression which might follow from the names of the two notions of work. The fact that the two notions are complementary is the reason why a full and complete approach to mechanical problems requires them being treated as inseparable. If only one of the notions is used, the approach will remain one-sided and incomplete.

Dual interpretation of the virtual work principle is closely connected with the dual notions of work presented above. It is known that this principle may be formulated either on the basis of a virtual kinematical state (Lagrange's virtual displacements principle), or on the basis of a virtual statical state (Castigliano's virtual forces principle). According to the mechanical sense of the principles, the first one should be called the principle of actual virtual work, the second — the complementary virtual work principle. One way or another, each of the principles involves a different mechanical meaning and thus they should be treated as completely independent.

The Lagrange principle reveals a statical sense and may serve for deriving the equations of static equilibrium. The Castigliano theorem has a kinematic sense (as was established in 1936 by SOUTHWELL [13]) and may be used for deriving the equations of kinematic compatibility (St. Venant's equations and boundary conditions in terms of displacements).

The necessity of an entirely independent consideration of the qualitatively different theorems of LAGRANGE and CASTIGLIANO is not yet generally recognized, even today; this is evident from inspection of numerous examples encountered in literature, which attribute the same mechanical sense to the two theorems, stressing their equivalence etc.

Application of the Castigliano theorem, typically kinematical in character, to the determination of states of static equilibrium contradicts the very sense of that theorem. In this manner, kinematical aspects of the problem, constituting the dual and necessary complement to the statical approach, become entirely lost. The only adequate application of Castigliano's theorem is for the analysis of compatibility of kinematical states. Mechanical meanings of the Lagrange and Castigliano theorems are identified with each other mostly in those cases in which the corresponding dual notions of strain energy (actual and complementary) are not properly interpreted and discerned.

The situation described seems to result from two reasons. First of all, the principal domains of application of energy methods always were (and still usually are) the linear problems in which the two notions of energy are apparently similar: they are expressed by similar formulae containing the same factor 1/2, and their numerical values are identical. This makes it difficult to distinguish between the two types of energy and to establish and analyze all the dual relations. Of certain importance may also be the fact that the static approach is usually preferred, especially in structural mechanics. Such an approach may frequently be found in literature on the subject, which proves that the approach to these problems still remains too one-sided.

The proper and logical interpretation of the Lagrange and Castigliano theorems, consistent with their physical sense resulting from the two notions of work, leads to the determination of certain relations of a general nature. Both theorems are found to constitute a pair of mutually dual relations called the dual Legendre transformation. It was shown by SEWELL [7] in 1969 that this transformation constitutes one of the common foundations of mechanics of continua and mathematical programming.

3. Actual and complementary work

In view of the fundamental meaning of the concept of distinguishing between the two qualitatively different but complementary notions of work — actual and complementary — it appears to be useful to present two parallel mechanical interpretations of these notions.

For the sake of simplicity, let us consider a simple model in the form of a single bar (Fig. 1a) made of a material with non-linear elastic properties (Fig. 2). Horizontal displacement s of its upper end will be measured from the point 0 corresponding to the undeformed state.

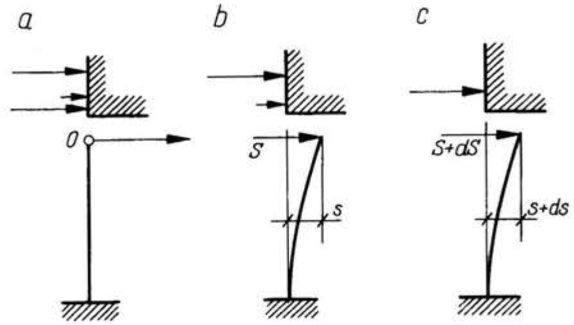

FIG. 1.

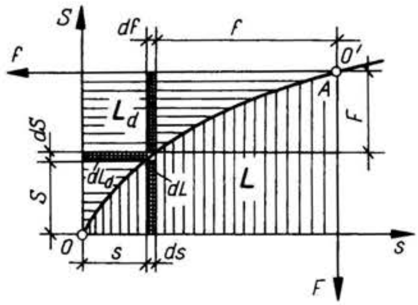

FIG. 2.

Let us imagine that the horizontal load to be applied to the bar has been "stored" alongside the bar, at the reference line $s = 0$, and is then quasi-statically transferred, step by step, to the bar. A certain intermediate state of loading is shown in Fig. 1b. The state is characterized by the action of a force S producing a corresponding deflection s of the bar. In order to apply some additional load dS (Fig. 1c), we have to perform, at first, the complementary work $dL_d = s \cdot dS$ along the displacement measured from the reference line to the line determined by the coordinate s, and then — the actual work $dL = S \cdot ds$ due to the actual loading which makes the bar deflect by ds together with the load S already existing. The total work done during the additional loading is equal to

$$dL_c = dL_d + dL + ds \cdot dS = s \cdot dS + S \cdot ds + ds \cdot dS = d(sS) + ds \cdot dS. \tag{3.1}$$

Summing up, we may conclude that in the process of loading, the current force increments do the complementary work on displacement already produced in the bar, and the force previously applied to the bar does the actual work on current increments of displacement.

From the interpretation of the dual notions of the work presented, it follows that the actual, work is a function of displacement s, while the complementary work — a function of the force S. The situation may sometimes be obscured by the fact of existence of a one-to-one correspondence between the variables s and S due to which any of the two variables may be used as an independent one.

Integration of the Eq. (3.1) for an elementary total work within the limits determined by the original and final states of the loading process yields in the case considered (Fig. 2):

$$L_c(s, S) = L(s) + L_d(S) = sS. \tag{3.2}$$

To shed some additional light on the subject, let us discuss the possibility of another interpretation of the notions of work. To that end, let us consider the same system as before.

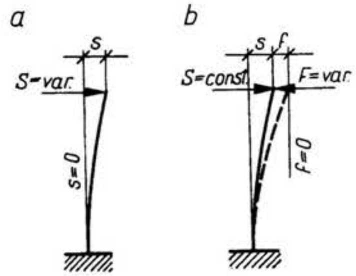

FIG. 3.

In the process of loading (Fig. 3a), force S performs (on displacement s) the actual work which is numerically equal to the area L in Fig. 2. Assume the value of force S to remain constant after loading and try to bring the bar back to the unloaded state by means of an additional force F (Fig. 3b). In the process of unloading, force F performs the actual work on displacement f produced by itself and measured from the deformed state; the work is equal to the area L_d shown in Fig. 2. In this manner, it becomes evident that the complementary work of the loading force S is numerically equal to the actual work done by the unloading force F. The inverse relation also holds true.

To conclude the considerations, let us moreover observe that the simplest expression for the total work is obtained in the case in which both the displacement s and force S are measured from the natural state — i.e., from the strain-free and stress-free state of the system [Fig. 2 and Eq. (3.2)]. In the case in which s and S are measured otherwise — e.g., from the state in which they assume the values s_0 and S_0 — we obtain

$$L_c(s_0+s, S_0+S) = L(s_0+s)+L_d(S_0+S) = (s_0+s)(S_0+S) =$$
$$= s_0 S_0+sS+s_0 S+sS_0 = L_c(s_0, S_0)+L_c(s, S)+s_0 S+sS_0 .$$

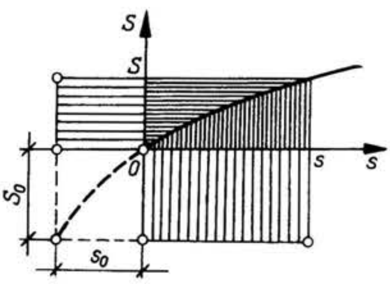

FIG. 4.

The increment of the total work value of the system due to passing from the original state to the actual one is then:

$$\Delta L_c(s, S; s_0, S_0) = L_c(s_0+s, S_0+S)-L_c(s_0, S_0) =$$
$$= L_c(s, S)+s_0 S+sS_0 = sS+s_0 S+sS_0,$$

as is shown in Fig. 4; ΔL_c corresponds to the sum of shaded areas.

4. Potential of forces and displacements. Dual Legendre transformation

Let us consider an arbitrary elastic element with displacements and loads which may be described by suitable single-column matrices s and S with components s_α and $S_\alpha(\alpha = 1, 2, \dots, k)$. Assume that there exists a one-to-one (not necessarily linear) correspondence between the loads and displacements of the element. Returning to the example of Fig. 1a, where $k = 1$, we may represent that correspondence in the manner shown in Fig. 5.

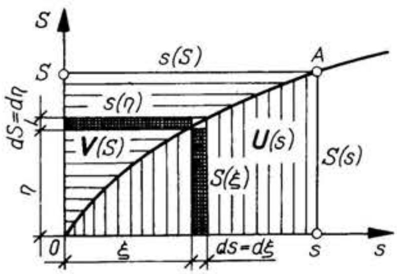

FIG. 5.

Let us denote the areas OAs and OAS by $U(s)$ and $V(S)$, respectively. From Fig. 5 it immediately follows that

$$U(s)+V(S) \equiv sS. \tag{4.1}$$

If $k > 1$, the notation $U(s)$ and $V(S)$ should be identified with the notation $U(s_1, s_2, \dots, s_k)$ and $V(S_1, S_2, \dots, S_k)$, the product sS being understood as a product $s^T S$ of matrices s^T and S.

From the definition of $U(s)$ and $V(S)$ it follows that

$$U(s) = \int_0^s S(\xi)\,d\xi, \qquad V(S) = \int_0^S s(\eta)\,d\eta. \tag{4.2}$$

For $k > 1$, the operations of integration are to be understood in a suitably generalized sense. Thus we have

$$\partial U(s)/\partial s = S(s), \qquad \partial V(S)/\partial S = s(S), \tag{4.3}$$

that is $S(s) = \operatorname{grad} U(s)$, $\quad s(S) = \operatorname{grad} V(S)$.

It follows from the Eqs. (4.2) and (4.3) and from Figs. 2 and 5 that $U(s)$ is a potential of forces $S(s)$ and its numerical value is equal to the actual work of these forces done on the corresponding displacements s treated as independent variables. The function $U(s)$ is called the actual potential strain energy.

An entirely similar reasoning leads to the conclusion that $V(S)$ is the potential of displacements $s(S)$ and its numerical value is equal to the complementary work performed on these displacements by the forces S treated as independent variables. The function $V(S)$ will be called the complementary potential strain energy.

The relations (4.3) derived in a purely formal manner and resulting from the definitions of functions U and V represent the analytical, well-known version of the Lagrange and

Castigliano theorems. In the case of linear relations between s and S and for corresponding values of the variables, the values of U and V are *always* equal. It should be stressed here that the Castigliano theorem in its original form: $\partial U/\partial S = s$ is, in general, not true and yields good results only for linearly elastic systems in which $U \equiv V$. This fact was observed by ENGESSER who proposed an improved, correct form $\partial V/\partial S = s$ which remains true for any non-linear relation between S and s. This fundamental observation made by ENGESSER remained virtually unknown for a period of half a century — until 1941, when WESTERGAARD [8] revived the idea of complementary work.

From the mathematical standpoint the relation $(4.3)_1$ written in terms of components

$$S_\alpha = \partial U/\partial s_\alpha, \quad \alpha = 1, 2, \ldots, k, \tag{4.4}$$

constitutes a certain transformation of old variables s_α into the new ones S_α by means of the generating function U. In the case in which the function is of the class C^2, it is possible to construct a square matrix $H(U)$ with elements $\partial^2 U/\partial s_\alpha \partial s_\beta$, $(\alpha, \beta = 1, 2 \ldots, k)$ called the Hess matrix, identical with the Jacobi matrix with elements $\partial S_\alpha/\partial s_\beta$ $(\alpha, \beta = 1, 2, \ldots, k)$ constructed for the set of functions S_α.

The system of equations (4.4) may be solved for the old variables s_α in those regions of the space in which $\det H(U) \neq 0$. This means that in these regions there exists a transformation inverse to (4.4); it may be proved to have the form of the relation $(4.3)_2$ or

$$s_\alpha = \partial V/\partial S_\alpha, \quad \alpha = 1, 2, \ldots, k. \tag{4.5}$$

The two mutually inverse transformations (4.4) and (4.5) are characterized by a typical symmetry in variables s_α and S_α and by the fact that their generating functions U and V satisfy the relation (4.1); they bear the common name of dual Legendre transformation [7, 14].

From these remarks it follows that an arbitrary elastic element with a one-to-one correspondence between the displacements s and forces S, may always be interpreted as a certain mechanical "Legendre transformer" realizing that correspondence. The Hess matrices $H(U)$ and $H(V)$ play here the roles of the stiffness and compliance matrices, respectively.

5. States of compatibility of displacements and equilibrium of forces

Let us imagine an arbitrary nodally loaded system "assembled" from certain definite elastic elements joined together at the nodes. Let us treat the elements and nodes as completely independent components of the structure, though exerting certain forces of interaction. In such a case, the states of displacement and loading of all the elements may be uniquely described by two single-column matrices s and S with elements s_α and $S_\alpha (\alpha = 1, 2, \ldots, m)$. Similarly, the states of displacement and loading of all the nodes may be determined by means of single-column matrices r and R with elements r_β and R_β $(\beta = 1, 2, \ldots, n)$. The system is assumed to be kinematically and statically linear and made of a material with arbitrary though strictly increasing elastic characteristics ensuring a one-to-one correspondence between the variables s and S. The loads are assumed to be sufficiently small to exclude the unstability of the equilibrium states.

In the case when the system considered is statically determinate, in the full set of its states determined by the variables s, r, S, R only one of these states is actual and satisfies the equations of

$$\text{(5.1)}\qquad \begin{aligned} &\text{(1) compatibility of displacements} && s = Ar,\\ &\text{(2) equilibrium of forces} && S = CR. \end{aligned}$$

Let us apply the principle of virtual work to the actual state of the system: $S^T s = R^T r$. In view of the Eq. (5.1), we have

$$S^T s = (CR)^T Ar = R^T C^T Ar,$$

hence

$$C^T A = E = A^T C,$$

and it follows that

$$\text{(5.2)}\qquad C^T = A^{-1} \quad \text{and} \quad A^T = C^{-1},$$

and also $AC^T = E = CA^T$.

Multiplying the left-hand sides of the Eqs. (5.1) by C^T and A^T, respectively, we obtain in view of the Eqs. (5.2) another, *equivalent* form of these equations:

$$\text{(5.3)}\qquad \begin{aligned} &\text{(1) compatibility of displacements} && r = C^T s,\\ &\text{(2) equilibrium of forces} && R = A^T S. \end{aligned}$$

The relations (5.1) and (5.3) determine the unique [due to the Eqs. (5.2)], one-to-one correspondence between the variables r, s and R, S characteristic for all statically determinate systems. On the basis of these relations, the following interpretation of columns and rows of the matrices A, C may be given.

The columns of $A(C)$ constitute a set of n states of displacements s (loads S) produced by consecutive unit states of displacements $r_\beta = 1$ (loads $R_\beta = 1$) for $\beta = 1, 2, \ldots, n$. The rows of $A(C)$ constitute a set of m states of loads R (displacements r) corresponding to the consecutive unit states of loads $S_\alpha = 1$ (displacements $s_\alpha = 1$) for $\alpha = 1, 2, \ldots, m$.

Let us now assume that the system considered is statically indeterminate and denote by r_0 and R_0 the displacements and forces occurring in the system. Using the Lagrange concept, we may transform the system into a statically determinate one by releasing it from the hyperstatic constraints. As a result, certain displacements r_* appear which did not exist in the original system, and also the corresponding forces R_* replacing the action of the constraints removed. The released system is statically determinate, therefore the relations (5.1)–(5.3) hold true except for the fact that

$$\text{(5.4)}\qquad A = (A_0 A_*), \quad C = (C_0 C_*), \quad r = \begin{pmatrix} r_0 \\ r_* \end{pmatrix}, \quad R = \begin{pmatrix} R_0 \\ R_* \end{pmatrix}.$$

Satisfaction of the Eqs. (5.1) or (5.3) has now a sense somewhat different from the previous one since it is not equivalent to satisfying the complete set of conditions of compatibility of displacements and equilibrium of forces in the original, statically indeterminate system; it merely concerns the conditions valid for the released auxiliary system.

6. Kineto-static potentials

6.1. Statically determinate systems

Let us pass to the problem of determining the state of equilibrium of a statically determinate system loaded by given forces P and satisfying the condition of compatibility $(5.1)_1$ of displacements s and r. According to the Lagrange principle, the problem may be replaced by an equivalent problem of constrained minimization of the corresponding potential of forces with respect to the independent displacements. This potential is represented by the total actual energy of the system:

$$F_1 = U(s) - P^T r \tag{6.1}$$

whose minimum is sought for under the constraint $s = Ar$. The necessary (and under the assumptions made in Sec. 5 — also sufficient) condition of existence of $\min F_1$

$$\frac{\partial F_1}{\partial r} = \frac{\partial U}{\partial s}\frac{\partial s}{\partial r} - P = A^T S - P = 0 \tag{6.2}$$

is the equation of equilibrium of forces. Owing to the uniqueness of $S(s)$, the Eq. (6.2) has exactly one solution

$$\hat{S} = CP, \quad \hat{s} = s(\hat{S}) = s(CP) = \hat{s}(P), \quad \hat{r} = C^T \hat{s}(P) = \hat{r}(P), \tag{6.3}$$

uniquely determining the state of equilibrium sought for.

The same problem may also be solved by the method of Lagrange multipliers owing to which the constrained minimization of the potential (6.1) is transformed into an equivalent problem of unconstrained determination of the stationary point of another potential of forces

$$F_1' = U(s) - P^T r + S^T(Ar - s), \tag{6.4}$$

in which the role of a Lagrange multiplier is played by the force $S = S(s)$. Function F_1' may be interpreted as a properly generalized potential energy of the system subdivided into separate elements and nodes, taking into account the actual work of forces S done on the kinematic incompatibilities $Ar - s$ which appear as a result of absence of nodal joints in the system. In the case considered the variables s and r are not subject to any constraints and thus they remain independent. Two necessary and sufficient conditions of existence of a stationary point of F_1' appear:

$$\frac{\partial F_1'}{\partial s} = \frac{\partial U}{\partial s} + \frac{\partial S^T}{\partial s}(Ar - s) - S = H(U)\cdot(Ar - s) = 0,$$

$$\frac{\partial F_1'}{\partial r} = -P + A^T S = 0.$$

On the basis of the assumptions quoted in Sec. 5, $\det H(U) \neq 0$, and thus the necessary and sufficient conditions mentioned above take the form:

$$Ar - s = 0, \quad A^T S - P = 0. \tag{6.5}$$

It follows that the stationary point of F_1' satisfies simultaneously the conditions of compatibility of displacements and of equilibrium of forces. Since we are dealing with a statically determinate system in which the unique state of compatibility of displacements is, at the same time, the only state of equilibrium of forces, both the Eqs. (6.5) have exactly one solution identical with (6.3).

Let us now remark that, in view of the identity (4.1), function F_1' may be represented in the *completely equivalent* two forms:

$$F_1' = U(s) - P^T r + S^T(Ar - s) = -V(S) + r^T(A^T S - P). \tag{6.6}$$

If we take into account that $s = s(S)$ and use S and r instead of s and r, then the necessary and sufficient conditions take the form of the equations:

$$\frac{\partial F_1'}{\partial S} = -\frac{\partial V}{\partial S} + Ar = Ar - s = 0, \quad \frac{\partial F_1'}{\partial r} = A^T S - P = 0,$$

entirely identical with (6.5).

The formally derived identity (6.6) is of a fundamental significance owing to its applications. Let us observe that, on the basis of it, the problem of seeking the stationary point of F_1' in the set of states of displacements compatibility determined by the condition $Ar = s$ is reduced to the formerly considered minimization of F_1 with the constraint $Ar = s$. If our considerations are confined to the set of states of forces equilibrium determined by the condition $A^T S = P$, it is readily observed that F_1' is transformed into the negative complementary strain energy of the system. The problem may then be viewed from another standpoint — namely, through the Castigliano theorem.

To that end let us formulate the problem of determination of the state of compatibility in the system satisfying the equation of equilibrium $(5.3)_2$ of forces S and $R = P$. According to the Castigliano theorem, the problem may be replaced with the equivalent problem of constrained minimization of a corresponding displacement potential with respect to S. This potential is represented by the complementary strain energy of the system $V(S)$ whose minimum should be found with the constraint $A^T S = P$. Owing to the fact that the system is statically determinate and loaded by given forces P, there exists exactly one state of equilibrium and thus the force S can not be considered as an independent variable. The triviality of the problem is now obvious provided that we are dealing with statically determinate system; it disappears when passing to statically indeterminate systems possessing the entire set of solutions $\hat{S}$.

Using the concept of Lagrange multipliers, constrained minimization of the function $V(S)$ may be transformed into a unconstrained problem of determination of a stationary point of another displacement potential

$$F_1'' = V(S) - r^T(A^T S - P), \tag{6.7}$$

in which the role of Lagrange multiplier is played by the displacement r. On comparing the Eqs. (6.7) with (6.6), it is seen that $F_1'' = -F_1'$. Consequently, the problem of minimization of $V(S)$ with the constraint $A^T S = P$ is identical with the problem of maximization of F_1' with the same constraint.

From the considerations presented, it follows that F_1' represents a kind of kineto-static potential with a saddle point corresponding to the actual state of the system in which the conditions of compatibility and equilibrium are satisfied. The potential is transformed, in the set of compatibility states, into the total actual energy, and in the set of states of equilibrium — into the negative complementary strain energy of the system. Owing to that property, the problems of Castigliano and Lagrange may be considered as a pair of dual problems:

$$\begin{aligned}\min_{D_1}[U(s)-P^T r] &= \min_{D_1} F_1', \\ \min_{D_2} V(S) = \max_{D_2}[-V(S)] &= \max_{D_2} F_1', \end{aligned} \tag{6.8}$$

where $D_1 = \{s, r: \quad Ar = s\}$, $D_2 = \{S: \quad A^T S = P\}$.

It turns out that the problem of calculation of a statically determinate system subject to a static load P is accompanied by an analogous problem arising in the case of a kinematic load p. To stress that analogy let us regard in a slightly different manner the problem of determination of the state of equilibrium which constituted the starting point of our previous considerations.

Let us confine ourselves to the states of compatibility of displacements s and r, and to the states of equilibrium of forces S and R in a statically determinate system — that is, assume a one-to-one correspondence between the variables s, r, S, R defined by the identities

$$\begin{aligned} s = Ar, \quad S = CR, \quad s = s(S), \ \text{or} \\ r = C^T s, \quad R = A^T S, \quad S = S(s). \end{aligned} \tag{6.9}$$

The identities yield $R = R(r)$ and $r = r(R)$, which indicates the existence of a dual Legendre transformation between the variables r and R. Indeed, on the basis of (4.3) and (6.9):

$$\frac{\partial U}{\partial r} = \frac{\partial U}{\partial s}\cdot\frac{\partial s}{\partial r} = A^T S = R, \quad \frac{\partial V}{\partial R} = \frac{\partial V}{\partial S}\cdot\frac{\partial S}{\partial R} = C^T s = r. \tag{6.10}$$

The condition of equilibrium for an arbitrary load P may now be written shortly as $R-P = 0$ or $\partial/\partial r[U(s)-P^T r] = 0$. An analogous form is also assumed by the condition of compatibility of displacements for an arbitrary p: $r-p = 0$ or $\partial/\partial R[V(S)-p^T R] = 0$. Two potentials, analogous and corresponding to each other, appear in these conditions:

(6.11)
(1) potential of forces $F_1 = U(s)-P^T r$,
(2) potential of displacements $F_2 = V(S) - p^T R$.

In the case of a system shown in Fig. 1a in which $r = s$ and $R = S$, functions F_1 and F_2 may be interpreted as the respective areas of figures indicated in Figs. 6a, 6b.

Using the analogy of potentials (6.11), the problem of determination of the compatibility state for given displacements p forced into the statically determinate system satisfying the conditions of equilibrium $(5.1)_2$, may be replaced by the problem of constrained minimization of the potential F_2 with the constraint $S = CR$. Function F_2 may be interpreted as the total complementary energy of the system. The necessary and sufficient condition of existence of min F_2,

$$\frac{\partial F_2}{\partial R} = \frac{\partial V}{\partial S}\cdot\frac{\partial S}{\partial R} - p = C^T s - p = 0 \tag{6.12}$$

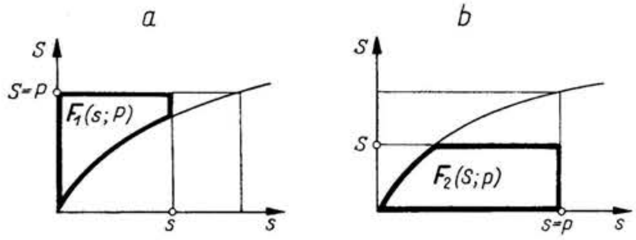

FIG. 6.

is the equation of compatibility of displacements having exactly one solution:

(6.13) $$\hat{s} = Ap, \quad \hat{S} = S(\hat{s}) = S(Ap) = \hat{S}(p), \quad \hat{R} = A^T\hat{S}(p) = \hat{R}(p),$$

uniquely determining the state of compatibility of displacements.

Using the method of Lagrange multipliers, we may replace the constrained minimization of F_2 with an equivalent problem of unconstrained determination of the stationary point of another displacement potential:

(6.14) $$F_2' = V(S) - p^T R + s^T(CR - S),$$

which may be interpreted as a suitably generalized complementary energy of the system, taking into account the complementary work done by non-equilibrated forces $CR-S$ on displacements s. Since the variables S and R are not subject to any constraint, they both are independent variables. This leads to two necessary and sufficient conditions of existence of a stationary point of F_2',

$$\frac{\partial F_2'}{\partial S} = \frac{\partial V}{\partial S} + \frac{\partial s^T}{\partial S}(CR-S) - s = H(V)\cdot(CR-S) = 0,$$

$$\frac{\partial F_2'}{\partial R} = -p + C^T s = 0,$$

or, in view of $\det H(V) \neq 0$, to

(6.15) $$CR - S = 0, \quad C^T s - p = 0.$$

From the Eqs. (6.15) it follows that the stationary point of F_2' satisfies simultaneously the conditions of compatibility and of equilibrium. Solution of the Eqs. (6.15) does not differ from (6.13).

The identity (4.1) yields, similarly to the Eq. (6.6), the result:

(6.16) $$F_2' = V(S) - p^T R + s^T(CR - S) = -U(s) + R^T(C^T s - p).$$

With $S = S(s)$ and using the variables s, R instead of S and R, the necessary and sufficient conditions of existence of a stationary point of F_2' assume the form:

$$\frac{\partial F_2'}{\partial s} = -\frac{\partial U}{\partial s} + CR = -S + CR = 0,$$

$$\frac{\partial F_2'}{\partial R} = C^T s - p = 0,$$

and the solution of the problem — the form (6.13).

Without going into further details we may also now establish the existence of a pair mutually dual problems

$$(6.17)\qquad \begin{aligned} &1)\quad \min_{D_3}[V(S)-p^T R] = \min_{D_3} F_2',\\ &2)\quad \min_{D_4} U(s) = \max_{D_4}[-U(s)] = \max_{D_4} F_2', \end{aligned}$$

where $D_2 = \{S, R\colon CR = S\}$, $D_4 = \{s\colon C^T s = p\}$.

6.1. Statically indeterminate systems

Generalization of the previous considerations to the case of statically indeterminate system consists in introducing certain constraints.

In the problem of seeking the states of equilibrium of forces under a prescribed loadings P, the set of compatibility conditions

$$(6.18)\qquad s = Ar = A_0 r_0 + A_* r_*, \quad r_* = p,$$

consists of the condition $(5.1)_1$ written by means of (5.4), and of the additional condition $r^* = p$ describing the kinematic constraints which make the system statically indeterminate. Condition $(6.18)_1$ presents the kinematic relations concerning the system released from redundant constraints.

In the case of constrained minimization of the potential $H_1 = U(s) - P^T r_0$ in the set of compatibility states determined by (6.18), r_0 is the independent variable. The necessary and sufficient condition of existence of $\min H_1$

$$(6.19)\qquad \frac{\partial H_1}{\partial r_0} = \frac{\partial U}{\partial s}\frac{\partial s}{\partial r_0} - P = A_0^T S - P = 0$$

is an equation of the displacement method having the solution

$$(6.20)\qquad \hat{r}_0 = \hat{r}_0(p, P), \quad \hat{s} = A_0 \hat{r}_0(p, P) + A_* p = \hat{s}(p, P), \quad \hat{S} = S(\hat{s}) = \hat{S}(p, P).$$

The concept of Lagrange multipliers leads to the construction of a new kineto-static potential L_1 which may be written [by means of (4.1)] in two equivalent forms:

$$(6.21)\qquad \begin{aligned} L_1 &= U(s) - P^T r_0 + S^T(Ar - s) + R_*^T(p - r_*) = \\ &= -V(S) + p^T R_* + r^T(A^T S - R) - r_0^T(P - R_0). \end{aligned}$$

From (6.21) it follows that potential L_1 in the set of compatibility states defined by (6.18) is transformed to the potential $H_1 = U(s) - P^T r_0$, and in the set of equilibrium states defined by conditions

$$(6.22)\qquad R = A^T S, \quad R_0 = P,$$

— to the potential $H_2 = -V(S) + p^T R_*$.

Determination of a constrained minimum of the potential $H_2' = -H_2 = V(S) - p^T R_*$ in the set of equilibrium states with respect to the independent variable R_* leads to the necessary and sufficient condition of existence of $\min H_2' = \max H_2$,

$$(6.23)\qquad \frac{\partial H_2'}{\partial R_*} = \frac{\partial V}{\partial S}\frac{\partial S}{\partial R_*} - p = C_*^T s - p = 0,$$

which is an equation of the force method. Its solution is

$$(6.24)\qquad \hat{R}_* = \hat{R}_*(P,p),\quad \hat{S} = C_0P+C_*\hat{R}_*(P,p) = \hat{S}(P,p),\quad \hat{s} = s(\hat{S}) = \hat{s}(P,p).$$

By analogy to the Eq. (6.11), we might also formulate the problem of determination of the displacements compatibility state at a prescribed displacements p, for which the set of equilibrium conditions

$$(6.25)\qquad S = CR = C_0R_0+C_*R_*,\quad R_0 = P,$$

would consist of the condition $(5.1)_2$ written by means of the Eq. (5.4), and of the additional condition $R_0 = P$ determining the given load of the system.

In a purely formal manner, by applying the analogy with previous considerations, a kineto-static potential may be introduced:

$$(6.26)\qquad L_2 = U(s)-P^Tr_0-R^T(C^Ts-r)+R_*^T(p-r_*) = \\ = -V(S)+p^TR_*-s^T(CR-S)-r_0^T(P-R_0),$$

differing only by the sign from its counterpart in (6.16). The potential in the set of compatibility states defined by the conditions

$$(6.27)\qquad r = C^Ts,\quad r_* = p,$$

is transformed to the potential H_1, and in the set of equilibrium states defined by (6.25) — into the potential H_2. It is seen that the two potentials L_1 and L_2 are equivalent in the sense that in the respective sets of states of the system they are transformed to identical energy potentials. Consequently, the necessary and sufficient conditions of existence of stationary points for L_1 and L_2 form equivalent systems of the equations

$$\begin{aligned}
&\partial L_1/\partial s = S-S = 0, && \partial L_2/\partial S = -s-s = 0,\\
&\partial L_1/\partial S = Ar-s = 0, && \partial L_2/\partial s = S-CR = 0,\\
(6.28)\quad &\partial L_1/\partial r_0 = A_0^TS-P = R_0-P = 0, && \partial L_2/\partial R_0 = r_0-C_0^Ts = 0,\\
(6.29)\quad &\partial L_1/\partial r_* = A_*^TS-R_* = 0, && \partial L_2/\partial R_* = p-r_* = p-C_*^Ts = 0,\\
&\partial L_1/\partial R_0 = r_0-r_0 = 0, && \partial L_2/\partial r_0 = R_0-P = 0,\\
&\partial L_1/\partial R_* = p-r_* = 0, && \partial L_2/\partial r_* = R_*-R_* = 0.
\end{aligned}$$

The identities $S-S = 0$, $-s+s = 0$, $r_0-r_0 = 0$, $R_*-R_* = 0$ occurring in the Eqs. (6.28), (6.29) are consequences of assuming the variables of S, s, r_0, R_* for Lagrange multipliers in the Eqs. $(6.21)_1$ and $(6.26)_2$. Arising from (5.4), the Eqs. $(6.28)_3$ and $(6.28)_4$ and also $(6.29)_3$, $(6.29)_4$ may be written jointly

$$(6.30)\qquad Ar-s = 0,\quad p-r_* = 0,\quad A^TS-R = 0,\quad R_0-P = 0;$$

$$(6.31)\qquad r-C^Ts = 0,\quad p-r_* = 0,\quad S-CR = 0,\quad R_0-P = 0.$$

The equations (6.30), (6.31) form two equivalent and complete sets of all equations of compatibility of displacements and equilibrium of forces, hence it follows that the two potentials L_1, L_2 have a common saddle point, being the solution of the problem and corresponding to the actual state of a statically indeterminate system subject to the loads P and p.

In conclusion, let us mention the mixed approach consisting in a two-stage solution of the problem. In the first stage, we are seeking $\overset{\min}{\underset{\max}{}} L'$ in the set D of states of the system partially satisfying the constraints, L' being uderstood as any of the potentials L_1, L_2 reduced to the set D. As a result, we determine a certain subset D' of extremal states of the system, and a new potential $L'' = \overset{\min}{\underset{\max}{}} L'$ defined in the subset D'. The second stage consists in finding $\overset{\max}{\underset{\min}{}} L''$ in the subset D', which leads to the determination of the actual state of the system satisfying all the required constraints.

Let us, for instance, consider a statically indeterminate system under the action of loads P and p. Assume that in the first stage of our procedure the potential

$$L'(s, r, R_*, P, p) = U(s) - P^T r_0 + R_*^T(p - r_*)$$

is minimized with the constraint $Ar = s$ defining the set D of states of the system. The necessary and sufficient conditions of existence of $\min\limits_{D} L'$

$$\frac{\partial L'}{\partial r_0} = A_0^T S - P = 0, \qquad \frac{\partial L'}{\partial r_*} = A_*^T S - R_* = 0,$$

have a solution in the form of a function of R_*:

$$\hat{r}_0 = \hat{r}_0(R_*, P), \quad \hat{r}_* = \hat{r}_*(R_*, P), \quad \hat{s} = A\hat{r}(R_*, P) = \hat{s}(R_*, P),$$

defining the set D' of states of the system. In view of the fact that in the first stage we have to deal with a statically determinate system released from redundant constraints, the sets D and D' are identical. The new potential

$$L'(\hat{s}, \hat{r}, R_*, P, p) = U(\hat{s}) - P^T \hat{r}_0 + R_*^T(p - \hat{r}_*) = L''(R_*, P, p)$$

is already a function of the single variable R_*.

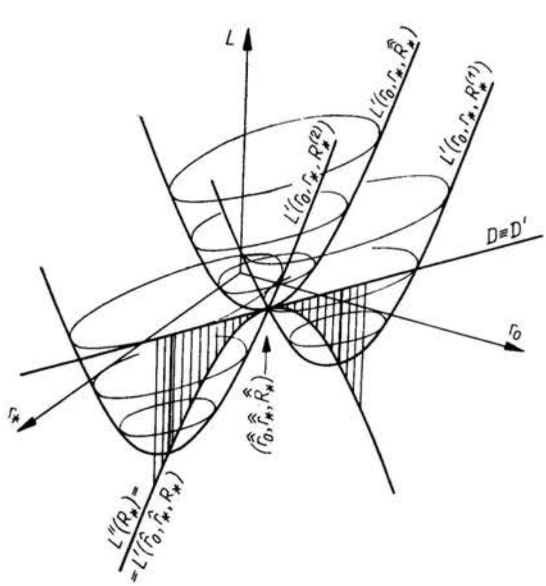

Fig. 7.

The second stage consists in calculating $\max_D L''$. The necessary and sufficient condition of existence of that maximum: $\partial L''/\partial R_* = 0$ is an equation enabling us to calculate $\hat{\hat{R}}_* = \hat{\hat{R}}_*(P, p)$ and, consequently, $\hat{\hat{r}} = \hat{r}[\hat{\hat{R}}_*(P, p)] = \hat{\hat{r}}(P, p)$, $\hat{\hat{s}} = A\hat{\hat{r}}(P, p) = \hat{\hat{s}}(P, p)$. The procedure is illustrated by Fig. 7.

7. Conclusions

A consistently dual approach to energy problems makes it possible to establish that the minimum theorems by Lagrange and Castigliano constitute two mutually dual aspects of a single, more general principle of stationary value of a certain Lagrange function. The function introduced in Sec. 6, L_1 or L_2, is the general kineto-static potential of mechanics of discrete, elastic, nodally loaded systems with bilateral constraints. The potential contains all the particular cases, what eliminates the necessity of using any other potentials (e.g. the set of potentials proposed by Goldenblat [11]). Identification of the kineto-static potential with the Lagrange saddle function explains why the mixed approach to the problem of calculation of statically indeterminate systems leads to the determination of a stationary point, and not of a minimum of that potential. Extremal theorems appear only in those particular cases when the potential is reduced to definite subsets of states of the system (compatibility of displacements or equilibrium of forces).

The approach to problems of mechanics of discrete systems presented in the paper has been purposely confined to nodal loads. That makes it possible to avoid discussing certain details inessential from the point of view of the general concept of dual approach which, on the other hand, may be generalized to the case of arbitrary loads. On the basis of mathematical programming methods, and of the Kuhn-Tucker theorem in particular, this approach enables to solve problems concerning systems with unilateral constraints. One of the principal aims of the paper consisted in proving the dual formulations to be purposeful and effective not in mathematical programming only.

References

1. F. Engesser, *Über statisch unbestimmte Träger bei beliebigem Formänderungsgesätze und über den Satz von der kleinsten Ergänzungsarbeit*, Zeitschrift des Architekten- u. Ingenieur-Vereins zu Hannover, **35**, 8, 733–744, 1889.
2. P. Funk, *Variationsrechnung und ihre Anwendung in Physik und Technik*, Springer-Verlag, Berlin 1962.
3. G. Hadley, *Non-linear and dynamic programming*, Addison-Wesley, London 1964.
4. P. G. Hodge, *Plastic analysis of structures*, McGraw-Hill, New York 1959.
5. T. Kozłowski, S. Piechnik, T. Stojek, *Application of variational methods to structural mechanics* [in Polish], „Inżynieria i Budownictwo" series, No. 13, Arkady, Warszawa 1967.
6. B. G. Neal, *The plastic methods of structural analysis*, Chapman and Hall, London 1965.
7. M. J. Sewell, *On dual approximation principles and optimization in continuum mechanics*, Phil. Trans. Roy. Soc. London, Ser. A, Math. Phys. Sci., **265**, 1162, 319–351, November 1969.
8. Westergaard H. M., *On the method of complementary energy*, Trans ASME, **107**, 765–803, 1942.
9. Н. П. Абовский, *Вариационные уравнения многоконтактных задач теории пологих оболочек и пластин* [N. P. Abovskii, Variational equations of multicontact problems of the theory of shallow

shells and plates, in Russian], Труды VII Всесоюзной конференции по теории оболочек и пластин, Москва, Изд. „Наука", 1970.

10. Ван Цзи-де, *Прикладная теория упругости* [WANG CHI-TEH, Applied elasticity, in Russian], Физматгиз, Москва 1959.
11. И. И. Гольденблат, *Экстремальные и вариационные принципы в теории сооружений* [I. I. GOLDENBLAT, Extremum and variational theorems in the theory of structures, in Russian], Строительная механика в СССР (1917–1957), Москва, Госстройиздат, 1957.
12. Р. Курант, Д. Гильберт, *Методы математической физики* [R. COURANT, D. HILBERT, Methods of mathematical physics, Russian translation], том 1, Гостехиздат, Москва 1951.
13. Л. С. Лейбензон, *Курс теории упругости* [L. S. LEIBENZON, Handbook of elasticity, in Russian], Огиз-гостехиздат, Москва 1947.
14. А. И. Лурье, *Аналитическая механика* [A. I. LURIE, Analytical mechanics, in Russian], Физматгиз, Москва 1961.
15. В. В. Новожилов, *Теория упругости* [V. V. NOVOZILOV, Theory of elasticity, in Russian], Судпромгиз, Ленинград 1958.
16. Я. А. Пратусевич, *Вариационные методы в строительной механике* [YA. A. PRATUSEVIC, Variational methods in structural mechanics, in Russian], Огиз-гостехиздат, Москва 1948.
17. И. М. Рабинович, *Курс строительной механики* [I. M. RABINOVIC, Handbook of structural mechanics, in Russian], том 2, Госстройиздат, Москва 1954.
18. Э. Рейсснер, *О некоторых вариационных теоремах теории упругости* [E. REISSNER, On certain variational theorems of the theory of elasticity, Russian translation], Сб. „Проблемы механики сплошной среды", Изд. АН СССР, Москва 1961.
19. *Современные методы расчета сложных статически неопределимых систем* [Modern methods of calculation of composite statically indeterminate systems, in Russian], Судпромгиз, Ленинград 1961.
20. А. А. Чирас, *Методы линейного программирования при расчете упруго-пластических систем* [A. A. ČIRAS, Methods of linear programming in calculating elastic-plastic systems, in Russian] Стройиздат, Ленинград 1969.

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