Archives of Mechanics • Archiwum Mechaniki Stosowanej • 25, 6, pp. 985-992, Warszawa 1973

On the fundamental singularity in the theory of shallow cylindrical shells

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THE PAPER presents a certain method of derivation of the fundamental singular solution for shallow cylindrical, circular shells. The solution is constructed by means of singular solutions of second order equation. The results derived are compared with solutions known from the literature.

W pracy podano pewną metodę otrzymania podstawowego rozwiązania osobliwego dla kołowych łupin walcowych o małej wyniosłości. Rozwiązanie konstruuje się za pomocą rozwiązań osobliwych równania drugiego rzędu. Otrzymane rezultaty porównano z rozwiązaniami znanymi w literaturze przedmiotu.

В работе предложен метод построения фундаментального сингулярного решения для пологих цилиндрических круговых оболочек. Решение построено при помощи особых решений уравнения второго порядка. Полученные результаты сравниваются с известными в литературе решениями.

1. Introduction

As THE fundamental singularity we shall understand the solution of the set of governing equations of the problem for an unbounded region with the loading term equal to a concentrated action. The fundamental singularity may be used as a starting point for the construction of particular singular solutions. For example, in the theory of elasticity the Somigliana tensor plays the role of fundamental singularity, while Green's, Neumann's or Robin's tensor fields are particular singular solutions satisfying appropriate boundary conditions [1]. The same appears in the theory of harmonic functions, where $u = 1/4\pi r$ is the fundamental singularity in three dimensions, while Green's, Neumann's and Robin's functions possess the same singularity and differ only in the boundary conditions.

In the technical applications, singular solutions form a basis for the construction of the surfaces of influence [2]. The singular solutions for shells may also be of some advantages in the stress concentration analysis in the vicinity of the action of concentrated loadings.

The problem of singular solutions for cylindrical shells, as also the problem of stress concentration due to a point load on the shallow shells, has been analysed by several authors. To list them all is almost impossible.

The fundamental singularity for shallow cylindrical shells was given for the first time by A. JAHANSHAHI in 1963 [3]. He presented his solution without a method of construction of the result. The correctness of the solution is proved by the fulfilment of the differential equation of the problem.

In 1966, also YU. SHEVLYAKOV and V. P. SHEVCHENKO [4] presented the fundamental singularity for shallow cylindrical shells. They obtained their solution by direct application of the Fourier integral transform to the set of equations of the problem, and next calculating somewhat complicated integrals of retransforms using the Residue Theorem. This last solution differs from that given by JAHANSHAHI. The difference lies in the regular part which does not disturb the singularity.

The aim of this paper is to give a simple method of construction of the fundamental singularity for shallow cylindrical shells. On the basis of the singular solution for a second-order equation, we deduce the fundamental singularity for the eighth-order equation of shallow cylindrical shells.

2. Governing equations

The point of departure for our considerations will be the set of well known equations of equilibrium for shallow cylindrical shells:

$$\begin{aligned} &u_{,\xi\xi}+\frac{1-\nu}{2}u_{,\eta\eta}+\frac{1+\nu}{2}v_{,\xi\eta}+\nu w_{,\xi} = -\frac{p_\xi(1-\nu^2)R}{Eh},\\ &\frac{1+\nu}{2}u_{,\xi\eta}+v_{,\eta\eta}+\frac{1-\nu}{2}v_{,\xi\xi}+w_{,\eta} = -\frac{p_\eta(1-\nu^2)R}{Eh},\\ &\nu u_{,\xi}+v_{,\eta}+w+\frac{h^2}{12R^2}\nabla^2\nabla^2 w = -\frac{p_r(1-\nu^2)R}{Eh}, \end{aligned} \tag{2.1}$$

where u, v, w are dimensionless displacements of the middle surface of the shell, R is the radius of curvature, $\nabla^2 = \dfrac{\partial^2}{\partial\xi^2}+\dfrac{\partial^2}{\partial\eta^2}$, and h the thickness of the shell.

The stress resultants in terms of displacements are (Fig. 1):

$$\begin{aligned} &N_\xi = D(u_{,\xi}+\nu v_{,\eta}+\nu w), && M_{\xi\eta} = -(1-\nu)\frac{K}{R}w_{,\xi\eta},\\ &N_\eta = D(v_{,\eta}+\nu u_{,\xi}+w), && Q_\xi = \frac{K}{R^2}\nabla^2 w_{,\xi},\\ &N_{\xi\eta} = \frac{1-\nu}{2}D(u_{,\eta}+v_{,\xi}), && Q_\eta = \frac{K}{R^2}\nabla^2 w_{,\eta},\\ &M_\xi = \frac{K}{R}(w_{,\xi\xi}+\nu w_{,\eta\eta}), && D = \frac{Eh}{1-\nu^2},\\ &M_\eta = \frac{K}{R}(w_{,\eta\eta}+\nu w_{,\xi\xi}), && K = \frac{Eh^3}{12(1-\nu^2)}. \end{aligned} \tag{2.2}$$

Making use of the well known Hilberts procedure [5, 6] and assuming $p_\xi = p_\eta = 0$, we transform the set of equations (2.1) into a single equation for the deflection function F:

$$F_{,\xi\xi\xi\xi}+\gamma^4\nabla^2\nabla^2\nabla^2\nabla^2 F = -\frac{2Rp_r}{Eh(1-\nu)}. \tag{2.3}$$

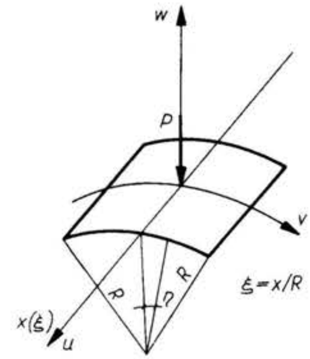

FIG. 1.

At the same time, the deformation components can be expressed in terms of F as follows:

$$u = \frac{1-\nu}{2}(F_{,\xi\eta\eta} - \nu F_{,\xi\xi\xi}),$$

(2.4) $$v = -\frac{1-\nu}{2}\left[(2+\nu)F_{,\xi\xi\eta} + F_{,\eta\eta\eta}\right],$$

$$w = \frac{1-\nu}{2}\nabla^2\nabla^2 F,$$

where $\gamma^4 = h^2/12R^2(1-\nu^2)$.

3. Fundamental singularity

In order to construct the fundamental singularity, we assume the loading function p_r equal to a concentrated force P acting at the origin of coordinates. Bearing in mind that ξ, η are the dimensionless coordinates, we take [7]:

(3.1) $$p_r = \frac{P}{R^2}\delta(\xi)\delta(\eta).$$

Substituting (3.1) in (2.3) and performing suitable transformations, we express the differential equation of the problem in the product form:

(3.2) $$\left[\prod_{i=1}^{i=4}\left(\nabla^2 + 2c_i\frac{\partial}{\partial\xi}\right)\right]F = -\frac{2P}{REh(1-\nu)\gamma^4}\delta(\xi)\delta(\eta),$$

where $c^4 = -1/16\gamma^4$, $i = 1, 2, 3, 4$.

The above Eq. (3.2) is homogeneous for all ϱ, ξ, except the origin of coordinates which is the point of application of the concentrated force P. Then we may expect the solution of (3.2) to be a sum of four functions. These four functions should be solutions of the second-order differential equations of the type of the operators appearing on the left side of (3.2) [8]. Moreover, we have to take into account the fact that each function F_i is singular at $\xi = \eta = 0$. Then we propose to assume following representation of the unknown displacement function:

$$F = \sum_{i=1}^{4} F_i, \quad \left(\nabla^2 + 2c_i \frac{\partial}{\partial \xi}\right) F_i = L_i \delta(\xi)\delta(\eta), \tag{3.3}$$

where L_i are unknown operators. Substituting (3.3) in (3.2), we obtain the following condition for L_i operators:

$$\nabla^2\nabla^2\nabla^2 \sum_{i=1}^{4} L_i - 2\frac{\partial}{\partial \xi}\nabla^2\nabla^2 \sum_{i=1}^{4} c_i L_i + 4\frac{\partial^2}{\partial \xi^2}\nabla^2 \sum_{i=1}^{4} c_i^2 L_i - 8\frac{\partial^3}{\partial \xi^3}\sum_{i=1}^{4} c_i^3 L_i = -\frac{2P}{REh(1-\nu)\gamma^4}. \tag{3.4}$$

To satisfy the above condition (3.4), we choose following four cases:

	1 st case	2 nd case	3 rd case	4 th case
$\nabla^2\nabla^2\nabla^2 \sum_{i=1}^{4} L_i =$	$-\frac{2P}{REh(1-\nu)\gamma^4}$	0	0	0,
$-2\frac{\partial}{\partial \xi}\nabla^2\nabla^2 \sum_{i=1}^{4} c_i L_i =$	0	$-\frac{2P}{REh(1-\nu)\gamma^4}$	0	0,
$4\frac{\partial^2}{\partial \xi^2}\nabla^2 \sum_{n=1}^{4} c_i^2 L_i =$	0	0	$-\frac{2P}{REh(1-\nu)\gamma^4}$	0,
$-8\frac{\partial^3}{\partial \xi^3}\sum_{i=1}^{4} c_i^3 L_i =$	0	0	0	$-\frac{2P}{REh(1-\nu)\gamma^4}$.

Solving the above equations, we obtain four expressions for the operators L_i.

$$\begin{aligned}
&\text{1 st case} \qquad \nabla^2\nabla^2\nabla^2 L_i = 8c_i^4 \frac{P}{REh(1-\nu)},\\
&\text{2 nd case} \qquad \frac{\partial}{\partial \xi}\nabla^2\nabla^2 L_i = -4c_i^3 \frac{P}{REh(1-\nu)},\\
&\text{3 rd case} \qquad \frac{\partial^2}{\partial \xi^2}\nabla^2 L_i = 2c_i^2 \frac{P}{REh(1-\nu)},\\
&\text{4 th case} \qquad \frac{\partial^3}{\partial \xi^3} L_i = -c_i \frac{P}{REh(1-\nu)}.
\end{aligned} \tag{3.5}$$

If we now differentiate successively the two sides of the Eq. $(3.3)_2$ according to (3.5), we obtain:

(3.6)
$$\left(\nabla^2+2c_i\frac{\partial}{\partial\xi}\right)\nabla^2\nabla^2\nabla^2 F_i = 8c_i^4\frac{P}{REh(1-\nu)}\delta(\xi)\delta(\eta),$$
$$\left(\nabla^2+2c_i\frac{\partial}{\partial\xi}\right)\frac{\partial}{\partial\xi}\nabla^2\nabla^2 F_i = -4c_i^3\frac{P}{REh(1-\nu)}\delta(\xi)\delta(\eta),$$
$$\left(\nabla^2+2c_i\frac{\partial}{\partial\xi}\right)\frac{\partial^2}{\partial\xi^2}\nabla^2 F_i = 2c_i^2\frac{P}{REh(1-\nu)}\delta(\xi)\delta(\eta),$$
$$\left(\nabla^2+2c_i\frac{\partial}{\partial\xi}\right)\frac{\partial^3}{\partial\xi^3} F_i = -c_i\frac{P}{REh(1-\nu)}\delta(\xi)\delta(\eta).$$

Now, if we take into account that the function [9]

$$\Phi_i = -\frac{e^{-c_i\xi}}{2\pi}K_0\left(\sqrt{c_i^2(\xi^2+\eta^2)}\right), \quad \mathrm{Re}\sqrt{c_i^2}>0,$$

is the singular solution for the following equation:

$$\left(\nabla^2+2c_i\frac{\partial}{\partial\xi}\right)\Phi_i = \delta(\xi)\delta(\eta),$$

we obtain from (3.6) the following closed-form solution for derivatives of the deflection function F:

(3.7)
$$\nabla^2\nabla^2\nabla^2 F = -\frac{4P}{\pi REh(1-\nu)}\sum_{i=1}^{4} c_i^4 e^{-c_i\xi}K_0\left(\sqrt{c_i^2(\xi^2+\eta^2)}\right),$$
$$\frac{\partial}{\partial\xi}\nabla^2\nabla^2 F = \frac{2P}{\pi REh(1-\nu)}\sum_{i=1}^{4} c_i^3 e^{-c_i\xi}K_0\left(\sqrt{c_i^2(\xi^2+\eta^2)}\right),$$
$$\frac{\partial^2}{\partial\xi^2}\nabla^2 F = -\frac{P}{\pi REh(1-\nu)}\sum_{i=1}^{4} c_i^2 e^{-c_i\xi}K_0\left(\sqrt{c_i^2(\xi^2+\eta^2)}\right),$$
$$\frac{\partial^3}{\partial\xi^3} F = \frac{P}{2\pi REh(1-\nu)}\sum_{i=1}^{4} c_i e^{-c_i\xi}K_0\left(\sqrt{c_i^2(\xi^2+\eta^2)}\right),$$

where $K_0(\,)$ — is the modified Bessel function of the second kind.

Knowledge of the singular solutions for the derivatives of the deflection function enables the construction of singular solutions for stress resultants. Using (2.4) and (2.2), we express all the internal forces in terms of the deflection function F;

$$N_\xi = \frac{Eh(1-\nu)}{2}F_{,\xi\xi\eta\eta}, \qquad M_\xi = \frac{1-\nu}{2}\frac{K}{R}\nabla^2\nabla^2(F_{,\xi\xi}+\nu F_{,\eta\eta}),$$

$$N_\eta = \frac{Eh(1-\nu)}{2} F_{,\xi\xi\xi\xi}, \qquad M_\eta = \frac{1-\nu}{2}\frac{K}{R}\nabla^2\nabla^2(F_{,\eta\eta}+\nu F_{,\xi\xi}),$$

$$N_{\xi\eta} = -\frac{Eh(1-\nu)}{2} F_{,\xi\xi\xi\eta}, \qquad M_{\xi\eta} = -\frac{(1-\nu)^2}{2}\frac{K}{R}\nabla^2\nabla^2 F_{,\xi\eta}.$$

The above derivatives of the displacement function F can easily be expressed using (3.7) by the formulae:

(3.8)
$$F_{,\xi\xi\xi\xi} = -\frac{P}{2\pi REh(1-\nu)}\sum_{i=1}^{4} c_i^2\left(K_0 + \frac{c_i\xi}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1\right)e^{-c_i\xi},$$

$$F_{,\xi\xi\eta\eta} = -\frac{P}{2\pi REh(1-\nu)}\sum_{i=1}^{4} c_i^2\left(K_0 - \frac{c_i\xi}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1\right)e^{-c_i\xi},$$

$$F_{,\xi\xi\xi\eta} = -\frac{P}{2\pi REh(1-\nu)}\sum_{i=1}^{4} c_i^3\frac{\eta}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1 e^{-c_i\xi},$$

$$\nabla^2\nabla^2 F_{,\xi\xi} = -\frac{2P}{\pi REh(1-\nu)}\sum_{i=1}^{4} c_i^4\left(K_0 + \frac{c_i\xi}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1\right)e^{-c_i\xi},$$

$$\nabla^2\nabla^2 F_{,\eta\eta} = -\frac{2P}{\pi REh(1-\nu)}\sum_{i=1}^{4} c_i^4\left(K_0 - \frac{c_i\xi}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1\right)e^{-c_i\xi},$$

$$\nabla^2\nabla^2 F_{,\xi\eta} = -\frac{2P}{\pi REh(1-\nu)}\sum_{i=1}^{4} c_i^5\frac{\eta}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1 e^{-c_i\xi}.$$

Thus we arrive at the singular solutions for internal forces:

(3.9)
$$N_\xi = -\frac{P}{4\pi R}\sum_{i=1}^{4} c_i^2\left(K_0 - \frac{c_i\xi}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1\right)e^{-c_i\xi},$$

$$N_\eta = -\frac{P}{4\pi R}\sum_{i=1}^{4} c_i^2\left(K_0 + \frac{c_i\xi}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1\right)e^{-c_i\xi},$$

$$N_{\xi\eta} = \frac{P}{4\pi R}\sum_{i=1}^{4}\frac{c_i^3\eta}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1 e^{-c_i\xi},$$

$$M_\xi = -\frac{P}{\pi R^2}\frac{K}{Eh}\sum_{i=1}^{4} c_i^4\left[(1+\nu)K_0 + (1-\nu)\frac{c_i\xi}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1\right]e^{-c_i\xi},$$

$$M_\eta = -\frac{P}{\pi R^2}\frac{K}{Eh}\sum_{i=1}^{4} c_i^4\left[(1+\nu)K_0-(1-\nu)\frac{c_i\xi}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1\right]e^{-c_i\xi},$$

$$M_{\xi\eta} = \frac{P}{\pi R^2}\frac{K(1-\nu)}{Eh}\sum_{i=1}^{4} c_i^5\frac{\eta}{\sqrt{c_i^2(\xi^2+\eta^2)}}K_1 e^{-c_i\xi}.$$

In the formulae presented above, (3.8) and (3.9), the arguments of the K_n functions were omitted in the interests of brevity; they read:

$$K_n = K_n\left(\sqrt{c_i^2(\xi^2+\eta^2)}\right), \qquad \mathrm{Re}\sqrt{c_i^2} > 0.$$

In this way, we have obtained the fundamental singularity for thin shallow cylindrical shells.

4. Final remarks

The system given above can be presented in terms of Kelvin functions. Taking into account [10] that:

$$e^{\mp\frac{1}{2}in\pi}K_n\left(ze^{\pm i\frac{\pi}{4}}\right) = \mathrm{ker}_n(z)\pm i\,\mathrm{kei}_n(z), \tag{4.1}$$

it is possible to express the stress resultants (3.9) as well certain deflection derivatives in terms of Kelvin functions. For example, we show below second derivatives of the deflection w. Using (2.4), (3.8) and (4.1) we obtain:

$$\begin{aligned} w_{,\xi\xi} = \frac{PR}{4\pi K}\Bigg\{&\mathrm{ker}\frac{1}{2\gamma}\sqrt{\xi^2+\eta^2}\,\mathrm{ch}\frac{\sqrt{2}}{4\gamma}\xi\cos\frac{\sqrt{2}}{4\gamma}\xi \\ &-\mathrm{kei}\frac{1}{2\gamma}\sqrt{\xi^2+\eta^2}\,\mathrm{sh}\frac{\sqrt{2}}{4\gamma}\xi\sin\frac{\sqrt{2}}{4\gamma}\xi+\frac{\xi}{\sqrt{\xi^2+\eta^2}}\left[\mathrm{kei}_1\frac{1}{2\gamma}\sqrt{\xi^2+\eta^2}\,\mathrm{sh}\frac{\sqrt{2}}{4\gamma}\xi\cos\frac{\sqrt{2}}{4\gamma}\xi\right. \\ &\left.\left.+\mathrm{ker}_1\frac{1}{2\gamma}\sqrt{\xi^2+\eta^2}\,\mathrm{ch}\frac{\sqrt{2}}{4\gamma}\xi\sin\frac{\sqrt{2}}{4\gamma}\xi\right]\right\}, \\ w_{,\xi\eta} = \frac{PR}{4\pi K}\frac{\eta}{\sqrt{\xi^2+\eta^2}}&\left[\mathrm{kei}_1\frac{1}{2\gamma}\sqrt{\xi^2+\eta^2}\,\mathrm{sh}\frac{\sqrt{2}}{4\gamma}\xi\cos\frac{\sqrt{2}}{4\gamma}\xi\right. \\ &\left.+\mathrm{ker}_1\frac{1}{2\gamma}\sqrt{\xi^2+\eta^2}\,\mathrm{ch}\frac{\sqrt{2}}{4\gamma}\xi\sin\frac{\sqrt{2}}{4\gamma}\xi\right]. \end{aligned} \tag{4.2}$$

This result together with (3.9) coincide exactly with that obtained by YU. SHEVLYAKOV and V. P. SHEVCHENKO [4] but differs from the A. JAHANSHAHI solutions [3]. After examination of this last solution it is easy to observe that the difference between the A. Jahanshahi result and that presented above is a regular one. Let us consider, for example, in details the first derivative of the deflection w. Using (2.4) and (3.7), we obtain:

$$\frac{\partial w}{\partial\xi} = \frac{P}{\pi REh}\sum_{i=1}^{4} c_i^3 e^{-c_i\xi}K_0\left(\sqrt{c_i^2(\xi^2+\eta^2)}\right). \tag{4.3}$$

On the other hand, performing certain suitable transformations on the A. JAHANSHAHI results [3], adjusting the notations and coordinate systems, we arrive at his proposal as follows:

$$(4.4)\quad \frac{\partial w}{\partial \xi} = \frac{P}{\pi REh} \sum_{i=1}^{4} c_i^3 e^{-c_i \xi} K_0\left(\sqrt{c_i^2(\xi^2+\eta^2)}\right)$$

$$+\frac{Pi\sqrt{3(1-\nu^2)}}{4Eh} \sum_{i=1}^{4} c_i e^{-ic_i \xi} J_0\left(\sqrt{c_i^2(\xi^2+\eta^2)}\right),$$

where $J_0(\,)$ is the Bessel function of zero order and the first kind.

The first term of (4.4), being singular, agrees with (4.3), while the second term of (4.4) is a regular one. Then we may conclude that there is no difference in singularity. The same concerns further derivatives of deflection w.

The author of the present paper is of the opinion that the method presented here can be applied in further problems of fundamental singularity for higher order differential equations.

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Received April 16, 1973.