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Dislocation lines in nonlocal elastic continua

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A NONLOCAL elastic medium containing dislocation lines is considered. The basic geometric and static equations are given in terms of distributions, and the corresponding general solution is found. A condition for the energy of dislocation line per unit length to be finite is derived.

Rozważono nielokalny ośrodek sprężysty zawierający linie dyslokacji. Podstawowe związki geometryczne i statyczne wyrażono za pomocą dystrybucji oraz znaleziono odpowiednie rozwiązanie ogólne. Wyprowadzono także warunek na to, by energia linii dyslokacji, przypadająca na jednostkę długości, była skończona.

В работе рассмотрено нелокальное упругое тело, содержащее дислокационные линии. В терминах теории дистрибуций сформулированы основные геометрические и статические соотношения и дан общий вид их решений. Выведено условие конечности удельной энергии единицы длины линии дислокаций.

1. Introduction

THE PRESENT paper is a continuation of [1] which will be referred to as I. It was shown there that equations of the form

$$\int \Phi_{ij}(\mathbf{x}, \mathbf{x}')u_j(\mathbf{x}')d^3x' = f_j(\mathbf{x}) \tag{1.1}$$

or, more generally,

$$L\mathbf{u} = \mathbf{f} \tag{1.2}$$

cannot be applied directly to media which contain dislocations. For, in that case, there is no uniquely defined displacement field $\mathbf{u}$, and the left-hand sides of those equations have no clear meaning. We shall examine this problem more thoroughly in the case of an elastic medium containing dislocation lines.

2. Geometric relations

Consider a dislocation line L with the Burgers vector $\mathbf{b}$. Such a dislocation can be regarded as created by cutting the medium along a surface G, the glide surface, and shifting the sides of the cut with respect to each other by the Burgers vector $\mathbf{b}$. In consequence, the resulting dislocation can be described by a displacement field defined everywhere except the line L and the surface G, and subjected to the condition:

$$u_i^+(\mathbf{x}) - u_i^-(\mathbf{x}) = b_i \quad \text{on } G, \tag{2.1}$$

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where the superscripts $\pm$refer to the corresponding limits from the opposite sides of G. This field, when continued across the surface G, results in a multivalued displacement field such that

$$u_i(\mathbf{x})|_n - u_i(\mathbf{x}|_m = (n-m)b_i, \quad \mathbf{x} \notin L, \tag{2.2}$$

where the integers n and m refer to the corresponding branches.

Since the differences (2.2) do not depend on $\mathbf{x}$, we have

$$u_{i,j}(\mathbf{x})|_n - u_{i,j}(\mathbf{x})|_m = 0, \quad \mathbf{x} \notin L, \tag{2.3}$$

i.e. the gradient $u_{i,j}$ ($\mathbf{x}$) does not depend on the branch. Thus we have

$$u_{i,j}(\mathbf{x})|_n = u_{i,j}(\mathbf{x}), \quad \mathbf{x} \notin L, \tag{2.4}$$

where $u_{i,j}(\mathbf{x})$ is single-valued and defined for $\mathbf{x} \notin L$. In place of the Eqs. (2.1) or (2.2), we can write:

$$\int\limits_B u_{i,j}(\mathbf{x})dx_j = b_i, \tag{2.5}$$

where B denotes an arbitrary closed contour which encircles the dislocation line L exactly once (orientation as in Fig. 1).

The displacement gradient (2.4) still has a great disadvantage: it is not defined on the dislocation line. Therefore, we assume that it can be extended to a distribution defined

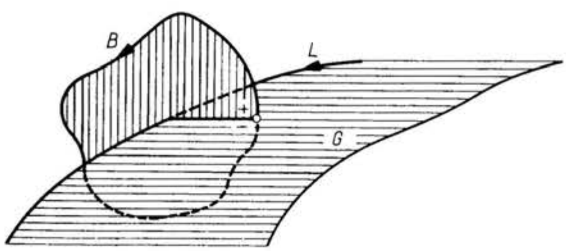

FIG. 1.

over the entire space, including the dislocation line. This distribution, denoted by $\beta_{ij}(\mathbf{x})$, will be called the distortion field. We shall regard it as a basic mathematical object describing the physical state of a dislocated medium. This is justified, for if we know the distortion field, we know the displacement gradient, since these two coincide in their common domain of definition — i.e.:

$$u_{i,j}(\mathbf{x}) = \beta_{ij}(\mathbf{x}), \quad \mathbf{x} \notin L. \tag{2.6}$$

The displacement gradient, in turn, determines the displacement field except for a constant term, which can be dispensed with as conveying no physical information.

On the other hand, two distortion fields which give the same displacement gradient do not necessarily coincide: they may differ by a distribution concentrated on the dislocation line. This opens the possibility of more detailed modelling of the dislocation core in continuum theory. This circumstance, useless in local theories, is of real significance in nonlocal ones.

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Now consider the curl of the distortion field, which will be denoted by α_{ik}:

$$\varepsilon_{klm}\beta_{im,l} = \alpha_{ik}. \tag{2.7}$$

It follows from the Eq. (2.6) that α_{ik} must be a distribution concentrated on the dislocation line L. Moreover, formal application of Stokes's theorem to the Eq. (2.5) yields the following condition on α_{ik}:

$$\int_S \alpha_{.k} ds_k = b_i, \tag{2.8}$$

where S is a surface bounded by contour B. This can be made rigorous, by considering regularizations of the distributions α_{ij} and β_{ij}. For, fixing contour B for a moment, we have

$$\int_S \alpha_{ij} * \varphi ds_k = \int_B \beta_{ij} * \varphi dx_j = b_i, \tag{2.9}$$

provided that φ is an infinitely differentiable function with a sufficiently small support and such that

$$\int \varphi d^3x = 1. \tag{2.10}$$

Thus, the integral in (2.8), defined as the limiting value of the left-hand side of the Eq. (2.9) when $\varphi(\mathbf{x}) \to \delta^{(3)}(\mathbf{x})$ and supp $\varphi \to \{0\}$, exists and equals the Burgers vector.

The Eq. (2.8) shows that α_{ik} has the form:

$$\alpha_{ik}(\mathbf{x}) = \int_L b_i\, \delta^{(3)}(\mathbf{x}-\mathbf{x}')dx'_k + \alpha'_{ik}(\mathbf{x}), \tag{2.11}$$

where the term α'_{ik} does not contribute to the integral in (2.8) and, in consequence, must be a combination of derivatives of Dirac delta distributed along the dislocation line. The simplest model of a dislocation line consists in assuming

$$\alpha_{ij}(\mathbf{x}) = \int_L b_i\, \delta^{(3)}(\mathbf{x}-\mathbf{x}')dx'_k, \tag{2.12}$$

i.e. $\alpha'_{ik} = 0$. More detailed models of the dislocation core can be created by choosing particular forms of $\alpha'_{ik} \neq 0$. Possible forms of α_{ik}(or α'_{ik}) are restricted by the condition:

$$\alpha_{ik,k} = 0 \tag{2.13}$$

or its equivalent;

$$\alpha'_{ik,k} = 0, \tag{2.14}$$

which is a direct consequence of the Eq. (2.7).

With α_{ik} given, the Eq. (2.7) can be considered the fundamental geometric equation for dislocated media. It has the same form as that given by KRÖNER [4] for continuous distribution of dislocations, and does not depend on the nature of interactions in the medium. When there are many dislocations, the resulting $\alpha_{ik}(\mathbf{x})$ equals the sum of the terms corresponding to single dislocations. Or, more generally, the line L can be regarded as consisting of all the dislocation segments and nodes of a dislocation net.

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3. Kinematic relations

Although basically we are interested here in problems of statics, a brief discussion of kinematics of dislocated media is necessary. We shall find it useful in calculating the work done in a deformation process, a problem which will be dealt with in the next section.

In a medium with no dislocations, the following relation between the velocity field $v_i(\mathbf{x})$ and the distortion field $\beta_{ik}(\mathbf{x})$ is valid:

$$v_{i,k} - \dot{\beta}_{ik} = 0. \tag{3.1}$$

This is a simple consequence of the displacement field $u_i(\mathbf{x})$ being defined in the whole medium. In a medium containing dislocations, $v_{i,k}$ and $\dot{\beta}_{ik}$ do not necessarily coincide on the dislocation lines. Therefore, we can write in general

$$v_{i,k} - \dot{\beta}_{ik} = J_{ik}, \tag{3.2}$$

where J_{ik} is another distribution concentrated on the dislocation lines. This quantity was introduced by KOSEVITCH [2] and, since it vanishes for stationary dislocations, called the dislocation current.

Combined, the Eqs. (3.2) and (2.7) yield:

$$\dot{\alpha}_{ik} + \varepsilon_{klm} J_{im,l} = 0. \tag{3.3}$$

This equation shows that whenever the dislocation lines move or the dislocation core changes its structure, the dislocation current cannot vanish.

4. The fundamental static equation for a dislocated medium

Now we need an equation of static equilibrium to be applicable to dislocated media with nonlocal interactions, and to replace the Eq. (1.1). As a generalization of the latter we assume an equation of the form:

$$M\beta = \mathbf{f}, \tag{4.1}$$

where β stands for the distortion field $\beta_{jl}(\mathbf{x})$ and M is again a certain linear operator. More specifically, we write this equation in the Fourier representation as

$$M_{ijl}(\mathbf{k})\hat{\beta}_{jl}(\mathbf{k}) = \hat{f}_i(\mathbf{k}), \tag{4.2}$$

$\beta_{jl}(\mathbf{k})$ being the Fourier transform of the distortion field and $M_{ijk}(\mathbf{k})$ a continuous tensor function of the wave vector $\mathbf{k}$. For a medium with no dislocations, this equation must be compatible with the Eq. (I.3.14). As

$$\hat{\beta}_{jl}(\mathbf{k}) = ik_l \hat{u}_j(\mathbf{k}), \tag{4.3}$$

when there are no dislocations, the corresponding compatibility condition is

$$ik_l M_{ijl}(\mathbf{k}) = \Lambda_{ij}(\mathbf{k}). \tag{4.4}$$

In the particular case, in which $M_{ijl}(\mathbf{k})$ is a tempered distribution, the Eqs. (4.1) and (4.4) can be written as

$$\Psi_{ijl} * \beta_{jl} = f_i \tag{4.5}$$

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and

$$\partial_l \Psi_{ijl} = \Phi_{ij} \tag{4.6}$$

respectively, where $\Psi_{ijl}(\mathbf{x})$ is a distribution whose Fourier transform equals M_{ijl}, i.e.:

$$\hat{\Psi}_{ijl}(\mathbf{k}) = M_{ijl}(\mathbf{k}). \tag{4.7}$$

From the form of the condition (4.4), it is clear that the operator M determines uniquely the operator L. In particular, this refers to the kernels Ψ_{ijl} and Φ_{ij}. The converse, however, is not true since any term $M'_{ijl}(\mathbf{k})$ such that

$$k_l M'_{ijl}(\mathbf{k}) = 0 \tag{4.8}$$

can be added to $M_{ijl}(\mathbf{k})$ with no resulting change in $\Lambda_{ij}(\mathbf{k})$.

5. The energy of a dislocated medium

Consider now the energy of a dislocated medium. Here, again, expressions of the form (I.4.1) or (I.4.2) are not applicable. The corresponding generalized expression is

$$W = \frac{1}{2}\frac{1}{(2\pi)^3}\int d^3k\, \hat{\beta}^*_{ij}(\mathbf{k}) K_{ijlm}(\mathbf{k}) \hat{\beta}_{lm}(\mathbf{k}), \tag{5.1}$$

where $K_{ijlm}(\mathbf{k})$ is another tensor function of the vector $\mathbf{k}$, such that

$$K_{ijlm}(\mathbf{k}) = K_{lmij}(\mathbf{k}) = K^*_{lmij}(\mathbf{k}) = K_{lmij}(-\mathbf{k}). \tag{5.2}$$

In order to establish the relation between $K_{ijlm}$and M_{ilm}, we shall consider the work done in the elementary process

$$\beta_{ij} \to \beta_{ij} + \delta\beta_{ij}, \qquad \delta\beta_{ij} = \dot{\beta}_{ij}\,\delta t. \tag{5.3}$$

From (5.1) and (5.2) we have

$$\delta W = \frac{1}{(2\pi)^3}\int d^3k\, \delta\hat{\beta}^*_{ij} K_{ijlm} \hat{\beta}_{lm} = \frac{1}{(2\pi)^3}\int d^3k\, \hat{\dot{\beta}}^*_{ij} K_{ijlm} \hat{\beta}_{lm}\,\delta t. \tag{5.4}$$

Taking into account the Eq. (3.2), we obtain:

$$\delta W = \frac{1}{(2\pi)^3}\int d^3k\, \hat{v}^*_i(-ik_j) K_{ijlm} \hat{\beta}_{lm}\,\delta t - \frac{1}{(2\pi)^3}\int d^3k\, \hat{J}^*_{ij} K_{ijlm} \hat{\beta}_{lm}\,\delta t. \tag{5.5}$$

In the particular case in which dislocations are stationary, the second term in this expresion vanishes. The first term must be compatible with the expression (I.4.2); thus, by the Eq.(4.2) we have

$$M_{ilm}(\mathbf{k}) = -ik_j K_{ijlm}(\mathbf{k}), \tag{5.6}$$

which is the relation we are seeking. Now, the expression (5.5) can be written as

$$\delta W = \frac{1}{(2\pi)^3}\int d^3k\, \hat{v}^*_i(\mathbf{k}) \hat{f}_i(\mathbf{k})\,\delta t - \frac{1}{(2\pi)^3}\int d^3k\, \hat{J}^*_{ij}(\mathbf{k}) \hat{\sigma}_{ij}(\mathbf{k})\,\delta t, \tag{5.7}$$

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where, by definition,

$$\hat{\sigma}_{ij}(\mathbf{k}) = LK_{ijlm}(\mathbf{k})\hat{\beta}_{lm}(\mathbf{k}). \tag{5.8}$$

The second term represents the work resulting from dislocation motion and determines the force on a dislocation line.

6. The general expression for the distortion field in a dislocated medium

The Eqs. (4.1) and (2.7) enable us to determine the distortion field created by an arbitrary distribution of dislocations and an arbitrary force field, provided that such a solution exists. Because those equations are linear, the distortion fields produced by forces and by dislocations can be considered separately.

Consider first the convolution case. By corresponding differentiation of the Eq. (4.5), we obtain [3]:

$$\Psi_{ijl} * \beta_{jl,m} = f_{i,m}. \tag{6.1}$$

From the Eq. (2.7) we have

$$\beta_{jm,l} = \beta_{jm,l} - \varepsilon_{lmk}\alpha_{jk}, \tag{6.2}$$

which after substitution into the Eq. (6.1) yields:

$$\Psi_{ijl} * \beta_{jm,l} = f_{i,m} + \Psi_{ijl} * \varepsilon_{lmk}\,\alpha_{jk}, \tag{6.3}$$

or, by the Eq. (4.6):

$$\Phi_{ij} * \beta_{jm} = f_{i,m} + \Psi_{ijl}\varepsilon_{lmk}\alpha_{jk}. \tag{6.4}$$

Provided that a fundamental solution of the Eq. (1.1) exists, the solution to the Eq. (6.4) can be represented as

$$\beta_{im} = G_{in} * f_{n,m} + G_{in} * \Psi_{njl} * \varepsilon_{lmk}\,\alpha_{jk}. \tag{6.5}$$

In general, the corresponding equation in the Fourier representation,

$$\hat{\beta}_{im} = \Lambda_{in}^{-1}\hat{f}_n ik_m + \Lambda_{in}^{-1} M_{njl}\varepsilon_{lmk}\hat{\alpha}_{jk}, \tag{6.6}$$

is valid. By inverse transformation it yields the solution whenever the result is a tempered distribution.

7. The energy of dislocation line per unit length

On substituting (6.6) into (5.1), we obtain an expression for the energy in terms of f_i and α_{ij}. For pure dislocation fields — i.e., when $f_i = 0$ — we have:

$$W = \frac{1}{2}\frac{1}{(2\pi)^3}\int d^3k\, K_{i'm'im}\Lambda_{i'n'}^{-1} M_{n'j'l'}\,\Lambda_{in}^{-1} M_{njl}\varepsilon_{l'm'k}\varepsilon_{lmk}\hat{\alpha}^{*}_{j'k'}\hat{\alpha}_{jk}. \tag{7.1}$$

In the case of a single straight dislocation line with $\hat{\alpha}_{ik}$ given by the Eq. (2.12), the corresponding expression for $\hat{\alpha}_{ik}$ is

$$\hat{\alpha}_{ik} = b_i\lambda_k\delta(\boldsymbol{\lambda}\cdot\mathbf{k}), \tag{7.2}$$

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where $\boldsymbol{\lambda}$ denotes the unit vector parallel to the dislocation line. The energy per unit length of the dislocation line equals:

$$(7.3) \qquad W_{lin} = \frac{1}{2}\frac{1}{(2\pi)^2} \int\limits_{\boldsymbol{\lambda}\cdot\mathbf{k}=0} d^2k\, K_{i'm'im} \Lambda^{-1}_{i'n'} M_{n'j'l'} \Lambda^{-1}_{in} M_{njl} \varepsilon_{l'm'k'} \varepsilon_{lmk} b_{j'} b_j \lambda_{k'} \lambda_k .$$

The integral in the Eq. (7.3) is divergent in the long wavelength limit. This divergence is entirely classical and can be removed by applying any classical procedure of cutting off the long range dislocation field. To estimate the convergence of the above integral in the short wavelength limit, let us consider K_{ijlm} such that

$$(7.4) \qquad K_{ijlm} = O(k^m) \quad \text{for} \quad k \to \infty.$$

Then, in general,

$$(7.5) \qquad M_{ijl} = O(k^{m+1}), \quad \Lambda_{ij} = O(k^{m+2})$$

in the same limit. It follows, according to the definition given in [1], that the singular order of the corresponding operator L equals

$$(7.6) \qquad s(L) = m+5.$$

The subintegral expression in (7.3) is $O(k^{m-2})$ and the corresponding integral is convergent when $m < 0$, i.e.

$$(7.7) \qquad s(L) < 5,$$

and divergent in the contrary case.

Thus, the inequality (7.7) is a necessary condition for the energy of the dislocation line per unit length to be finite. By arguments similar to those of [1], the same concerns the interaction energy of two parallel dislocations. By contrast with the corresponding inequality for point defects which, according to [1], reads

$$(7.8) \qquad s(L) > 8,$$

the inequality (7.7) shows that low values of $s(L)$ are preferred in the case of dislocations.

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