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On the optimal nonhomogeneity of an elastic bar in torsion; numerical examples

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THE PROBLEM of optimization of a linearly-elastic, nonhomogeneous bar subject to torsion is examined. Application of the algorithm formulated in [1] is studied on two examples of bars with square and elliptic cross-sections. It is shown that the maximum rigidity of the bar is achieved when its cross-section is composed of two materials with compliances u_{max} and u_{min}.

W pracy rozpatruje się zagadnienie optymalizacji niejednorodności liniowo-sprężystego pręta poddanego skręcaniu. Zbadano przydatność algorytmu sformułowanego w [1] na przykładzie pręta eliptycznego i kwadratowego. Okazało się, że maksymalną sztywność ma pręt złożony z dwu materiałów o podatliwościach u_{max} i u_{min}.

Рассматривается оптимальная задача о кручении линейно-упругого неоднородного стержня. Алгоритм численного решения, сформулированный в работе [1], исследован на примерах стержней эллиптического и квадратного сечений. Показано, что наибольшей жесткостью обладает составной стержень из материалов, характеризуемых податливостями u_{max} и u_{min}.

1. Statement of the problem

THE PROBLEM of determination of the optimum type of nonhomogeneity of a twisted bar was studied in [1]. In the present paper we shall discuss certain numerical solutions concerning bars with elliptical and square cross-sections.

Let us consider an elastic, prismatic and nonhomogeneous bar of a given cross-section D. The nonhomogeneity of the bar is described by the compliance function $1/G(x,y) = u(x,y) \in \Gamma$, Γ being a subset of $L^\infty(D)$ bounded according to the inequalities

$$0 < u_{min} \leqslant \operatorname{vrai\,sup} u(x,y) \leqslant u_{max} \tag{1.1}$$

and subject to one of the constraints

$$\int_D \frac{dD}{u} = \frac{D}{u_{av}} \quad \text{or} \quad \int_D u dD = u_{av} D, \tag{1.2}$$

u_{av} denoting the given average compliance.

In [4] the Prandtl function z was shown to minimize the functional

$$\int_D [u(\operatorname{grad} z)^2 - 2z] dD \tag{1.3}$$

under the condition $z|_{\delta D} = 0$; a different formulation of the problem may be found in [1].

If the function z was known, we would be able to determine the stress tensor components $\tau_{13} = -\partial z/\partial y$, $\tau_{23} = \partial z/\partial x$ and the torsional rigidity of the bar

$$I[u] = \int_D z dD. \tag{1.4}$$

Our purpose consists in prescribing a function $u(x, y) \in \Gamma$ in such a manner that the functional (1.4) attains its maximum value, and the functional (1.3) — its minimum value.

Let us consider the problem of maximization of $I[u]$ by taking into account the first of the constraints (1.2). From the theoretical analysis [1] it follows that three possibilities may be considered: the optimum compliance either takes the intermediate values (1.1), or is equal to $u_{\max}$ or $u_{\min}$, according to the local conditions

$$\begin{aligned} &u = u_{\max}, \quad w > u_{\max}, \\ &u = w, \quad u_{\min} < w < u_{\max}, \\ &u = u_{\min}, \quad w < u_{\min}, \end{aligned} \tag{1.5}$$

here $w = \gamma/(\text{grad } z)^2$, γ is a Lagrange multiplier for the integral constraint (1.2).

Let us now suppose that at the r-th step of iteration the function u_r is known. The value of u_{r+1} is calculated according to the following algorithm:

1. Calculate z_r from Eq. (1.3);
2. Construct a new control u_{r+1} locally different from u_r according to the conditions (1.5).

This algorithm was shown in [1] to be convergent.

2. Remarks on the numerical solution

According to the algorithm proposed, at the r-th step of iteration and at a given value of u_r we have to determine the corresponding Prandtl function z_r which is a solution of Eq. (1.3). A considerable difficulty consists in the fact that the line dividing the regions described by Eqs. (1.5) is still unknown.

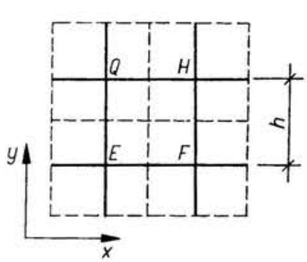

FIG. 1.

The solution will be sought for by means of a square net with the mesh size h, parallel respectively to the axes x, y. Let us assume the five-point approximation, the integrand (1.3) written for an interior mesh $GHEF$ has the form

$$Q = \frac{1}{4h^2}\,[(u_H+u_G)(z_H-z_G)^2+(u_E+u_F)(z_F-z_E)^2]+\frac{1}{4h^2}\,[(u_G+u_E)(z_G-z_E)^2 + (u_H+u_F)(z_H-z_F)^2]-\frac{1}{2}(z_G+z_H+z_E+z_F).$$

The condition necessary for the functional (1.3) to reach an extremum with respect to z at the point G is obtained by differentiating the sum of expressions (2.1) with respect to z_G. Conditions written for all meshes of the net form a system of linear equations $Az+b=0$; here z is unknown n-dimensional vector, where n is equal to the number of nodes of the net since b is an n-element vector depending on h; A is the $n\times n$ matrix. For an interior point G, for instance, the equation has the form

$$(2.2)\qquad (4u_G+u_A+u_B+u_E+u_F)z_G-(u_A+u_G)z_A-(u_H+u_G)z_H - (u_B+u_G)z_B-(u_E+u_G)z_E-2h^2 = 0.$$

STIEFEL and RUTISHAUSER [2] proposed several algorithms for minimizing the functionals of the type of Eq. (1.3), one of them being a combination of the methods of conjugate gradients and Chebysheff polynomials; this method makes it possible to solve great systems of linear equations, e.g. such as Eqs. (2.2).

In accordance with the results of Sec. 1, the parameters of the problem under consideration are the bounds $u_{\max}$ and $u_{\min}$ for the nonhomogeneity function and its mean value u_{av}. Once u_{av} is known, the Lagrange multiplier γ for the constraint (1.2) may be determined.

The procedure applied in numerical solutions is reversed: γ is treated as a parameter of the problem, while u_{av} has to be evaluated at every step r for the control function u_r.

In view of a limited computer store, the calculations are performed for a possibly large number of nodes: 126 nodes in the case of an elliptical section (mesh size $h = 0.1$) and 100 nodes for a bar with a square cross-section ($h = 0.05$).

The initial nonhomogeneity function $u_0(x, y)$ is assumed to be constant in D and equal to $u_{\max}$ or $u_{\min}$. In order to check the influence of u_0 on the final result and on the convergence of the algorithm applied, the values of the cost function I and of the average compliance u_{av} were calculated at every step r of the procedure.

On the basis of the results enabling to estimate the convergence of the functional I, the number of iteration steps is experimentally established for each case. The computations are terminated once the values calculated for two extreme initial values of $u_0(x, y)$ differ but at the third decimal place. It is observed that then the values of the control function in individual nodes of the net differ by magnitudes of order less than $O(h^2)$, and the solution obtained may be treated as an optimal one.

3. Elliptical cross-section

The region bounded by an ellipse of semi-axes $a = 1.5$, $b = 1.0$ is covered by a net with meshes $h = 0.1$ containing 126 interior nodes. Owing to the symmetry, only the region $D = \{0 \leqslant x \leqslant 1.5,\ 0 \leqslant y \leqslant \sqrt{1-2.25\,x^2}\}$ is considered. The dimensions of the cross-section are selected so as to make it possible to compare the results with the well known results concerning a circular cross-section [5]. The computations are performed for the parameters $u_{max} = 1.0$, $u_{min} = 0.5$, $\gamma = 0.3$. The results are shown in Fig. 2. Figure 2a presents the contour lines of the Prandtl function z. In accordance with the classical theory of torsion, the directions of tangents to the contour lines of z coincide with the directions of stress vectors, their absolute values being proportional to $|\text{grad } z|$ (Fig. 2b).

The state of stress in a cross-section may be analyzed on the basis of Figs. 2a and 2b. Figure 2c presents the ordinates of the nonhomogeneity function $u(x, y)$ and the lines

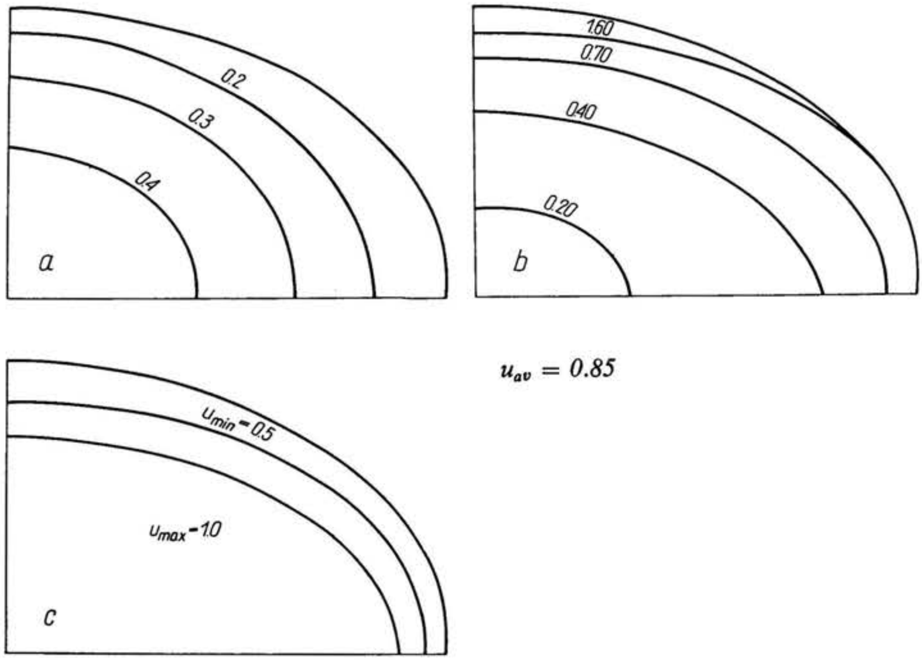

Fig. 2.

separating the regions in which $u(x, y)$ is equal to u_{max} or u_{min}. These lines represent the lines of constant gradients [cf. Eq. (1.5)] and are found from the analysis of the ordinates of the function $|\text{grad } z|$. The most rigid material should be located at the ends of the minor axis, where the stresses reach their maxima. Thickness of the reinforcement depends on the amuont of rigid and flexible material at our disposal, i.e. on the average compliance u_{av}.

In elliptical cross-sections the regions of rigid and flexible ($u = 1.0$) materials are separated by intermediate regime. From theoretical considerations it followed that with constraints of the type of $(1.2)_1$, the nonhomogeneity function $u(x, y)$ could have jumps only in the case of a circular cross-section; otherwise it was a continuous function.

4. Square cross-section

Let the cross-section be a square with sides $a = 1.0$. Owing to the symmetry properties of the problem, one quarter of the section is considered, i.e. the region $D = \{0 \leqslant x \leqslant 0.5,$

$0 \leqslant y \leqslant 0.5\}$. Assuming the meshes $h = 0.05$, a net with $n = 100$ nodes is constructed. Computations are performed for the following values of parameters: $u_{max} = 1.0$, $u_{min} = 0.5$, $\gamma = 0.006, 0.012, 0.015$. Figure 3 demonstrates the results for $\gamma = 0.012$. In Fig. 3a the contour lines of Prandtl's function are shown in Fig. 3b — absolute values of the gradient

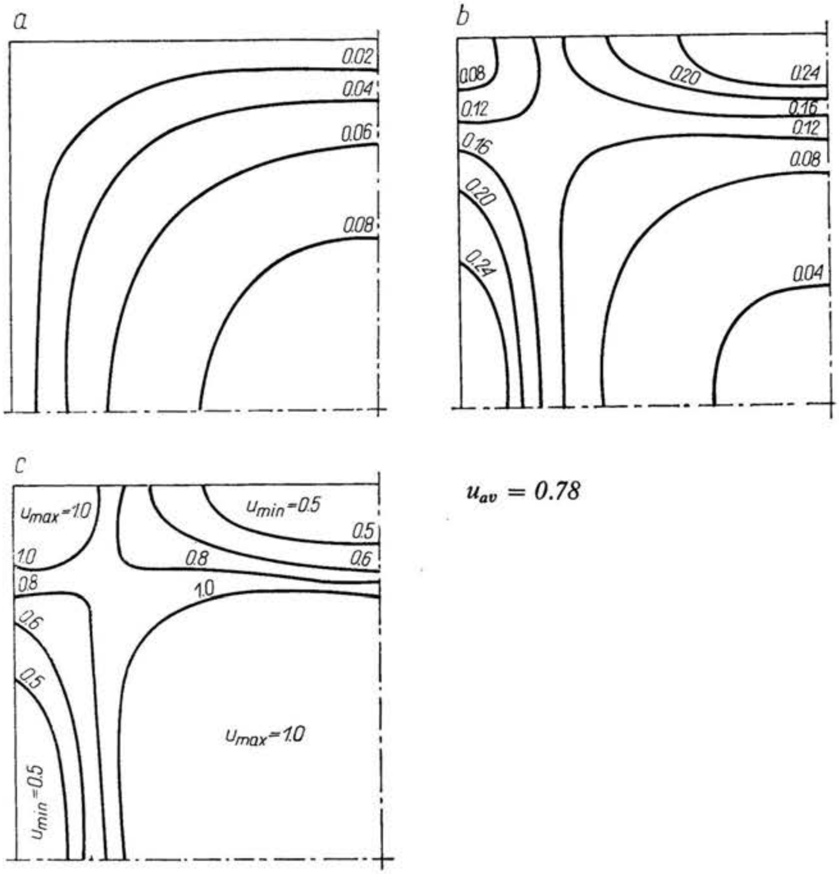

FIG. 3.

of z, $|\text{grad } z|$, and in Fig. 3c — the nonhomogeneities $u(x, y)$, the regions of constant values of $u(x, y)$ being marked.

Similarly to the case of a homogeneous square cross-section, the maximum shearing stresses occur at the centers of lateral surfaces what follows from the contour lines of $|\text{grad } z|$ (Fig. 3b). Depending on the proportions between the rigid and flexible materials (that is, on the average compliance u_{av}), the results are qualitatively different. When the rigid material prevails (u_{av} close to u_{min}), it forms a belt around the center of the cross-section. The center itself and the corners should be made of the flexible material. When the flexible material prevails (u_{av} close to u_{max}), the rigid reinforcements should by located at the centers of lateral surfaces (walls) of the bar. In [3] was considered the problem of optimization of the plastic nonhomogeneity of a prismatic bar, and the results obtained were qualitatively similar, what follows not only from engineering premises but also from the fact that the both problems may be reduced to a similar mathematical question belonging to the optimum control theory. In both cases we are seeking a maximum rigidity of the bar described by the functional (1.4), depending on the stress function z, and with identical constraints (1.1), $(1.2)_1$ imposed over the control function. In the elastic problem, the control function is the shear modulus, while in the plastic problem — the yield limit. The differential equations combining the stress function with the control function are in both cases different [Eq. (1.3)]. Considerable influence on the solution has the fact that the states of stress in both the elastic and plastic cases are described by similar functions. Rigid reinforcements placed at the points of greatest stresses are located in the same subregions of the cross-section.

5. Conclusions

The algorithm described is practically simple though it requires the knowledge of Weierstress conditions which must be formulated in each particular case thus causing serious difficulties. In addition, every iteration step requires a separate solution of the problem (1.3). Realization of that part of the program absorbs most of the computation time.

In [1] it was shown that the problem of optimal nonhomogeneity had a solution, though its uniqueness could not be proved. Consequently, the theorem of convergence of the algorithm does not secure that it converges to the solution of the problem stated. For the cross-sections and parameters considered in the paper, the algorithm converges rapidly and the results remain independent of the initial values assumed.

The numerical results obtained in the paper may serve as certain indications of theoretical nature: they enable us to draw conclusions concerning the uniqueness of solution and the convergence of the algorithm employed.

In view of a rather large mesh size and a simplified, five-point approximation, the results are of a qualitative character. However, from the point of view of engineering applications, the accuracy achieved seems to be satisfactory. Construction of intermediate regimes in practice is not possible; they are small, after all, and may be replaced by a homogeneous material. The maximum rigidity of the bar is attained by arranging two materials according to the conclusions drawn in Sec. 3 and 4.

All computations were performed at the Polish Academy of Sciences Computing Center, on ODRA-1204 digital computer.

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