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On the existence of a magnetogasdynamic shock wave structure with negligible shear viscosity

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ON THE BASIS of properties of the generalized thermodynamic potential (defined by GERMAIN [2]), the existence and uniqueness of fast shock wave structure is proved, as well as the existence of slow shock waves. The proof is based on the assumptions that the gas is ideal and the first viscosity coefficient η is equal zero. The remaining three dissipation coefficients are assumed to be positive functions of class C^1 of the physical parameters.

W oparciu o własności uogólnionego potencjału termodynamicznego (zdefiniowanego przez GERMAINA [2]) udowodniono istnienie i jednoznaczność struktury szybkich fal uderzeniowych oraz istnienie struktury wolnych fal uderzeniowych. Dowód przeprowadzono przy założeniu, że gaz jest idealny, a współczynnik pierwszej lepkości η jest równy zeru. O pozostałych trzech współczynnikach dysypacji założono, że są dodatnimi funkcjami klasy C^1 parametrów fizycznych.

На основе свойств обобщенного термодинамического потенциала, введенного Жерменом в работе [2], доказаны существование и единственность структуры быстрых ударных волн, а также существование структуры медленных ударных волн. Доказательства основаны на предположении о том, что газ является идеальным, а коэффициент первой вязкости равен нулю ($\eta = 0$). На остальные коэффициенты диссипации, рассматриваемые как функции от физических параметров, наложены требования положительности и регулярности класса C^1.

1. Introduction

1.1

INVESTIGATIONS concerning the structure of magnetodynamic shock waves have been undertaken by MARSHALL [5], LUDFORD [4], GERMAIN [2], KULIKOVSKI and LIUBIMOV [3], ANDERSON [1] and others. The most important results were obtained by GERMAIN, who proved the fast shock waves to be stable and the intermediate ones to be unstable. But, in the case of slow shock wave — taking into account the serious difficulties in investigations of the existence of a shock waves structure — no satisfactory results have been obtained. The papers on this subject deal mainly with the limited problem of the influence of two dissipation coefficients on the existence of the shock wave structure. Particularly carefully discussed has been the case in which the first viscosity coefficient and heat conduction coefficient are equal to zero ($\eta = k = 0$) [4, 2, 3]. Connected with the latter problem is considerable misunderstanding cf. [1, 2, 3, 4, 6].

Investigated in the present paper is the problem of existence and uniqueness of the fast as well as the slow shock waves, with only one dissipation coefficient being disregarded (of the four occurring in the classical magnetodynamics of fluids).

It has been shown that, with the coefficient of shear viscosity being equal to zero ($\eta = 0$), the structure of the fast as well as the slow shock waves exists. The remainder of the positive dissipation coefficients ξ, k and $1/\mu^2\sigma$ are assumed to have continuous first derivatives with respect to the physical parameters. It seems that the results obtained and the methods applied may be a convenient starting point for proving the existence of the slow shock wave structure for all four positive dissipation coefficients.

1.2

On the basis of classical magnetodynamics of fluids, the structure of plane shock waves is described by the following system of ordinary differential equations (cf. [2]):

$$\frac{1}{M\sigma\mu^2}\frac{dB}{dx} = \left(\frac{B\tau}{\mu} - c_1 v + c_2\right), \qquad \frac{\eta}{M}\frac{dv}{dx} = (v - c_1 B),$$

$$\left(\zeta - \frac{4}{3}\eta\right) M \frac{d\tau}{dx} = \left(p + M^2\tau + \frac{B^2}{2\mu} - c_3\right), \tag{1.1}$$

$$\frac{k}{M}\frac{dT}{dx} = \left(e - \frac{M^2\tau^2}{2} - \frac{v^2}{2} - \frac{B^2\tau}{2\mu} - c_2 B + c_1 Bv + c_3\tau - c_4\right),$$

where τ denotes specific density, e — internal energy, T — temperature, $\mu = \text{const}$ — magnetic permeability, $M = u/\tau = \text{const}$, $[u, v, 0]$ — velocity vector, $\mathbf{E} = [0, 0, c_2\mu M]$ — electric field vector ($c_2 = \text{const}$), $\mathbf{B} = [c_1\mu M, B, 0]$ — magnetic induction vector ($c_1 = \text{const}$), c_3, c_4 — positive constants.

The system (1.1) has, in the most general case, four solutions $(B_i, v_i, \tau_i, T_i), (i = 1, 2, 3, 4)$ that can be treated as points $P_i(B_i, v_i, \tau_i, T_i)$ of the phase space (B, v, τ, T). Naturally, to every P_i there corresponds a certain entropy. Applying numeration in accordance with the growth of entropy — which is adopted from now on — the points P_1, P_2 determine the states of the fast shock wave, the points P_3, P_4 the states of the slow shock wave; the other pairs of points $P_i, P_j, i < j$, determine the states of intermediate shock waves. The above classification is adjusted by the following inequalities, holding for the point P_i:

$$\begin{aligned} &P_1: \quad u > c_f, &&P_2: \quad b_x < u < c_f, \\ &P_3: \quad c_s < u < b_x, &&P_4: \quad u < c_s. \end{aligned} \tag{1.2}$$

The normal component of the velocity u determined by τ from the equality $M = \dfrac{u}{\tau}$,

$b_x = \left(\dfrac{B_0^2\tau}{\mu}\right)^{\frac{1}{2}} = (M^2\mu c_1^2\tau)^{\frac{1}{2}}$ is the normal component of Alfven speed, c_f and c_s are speeds of fast and slow magnetoacoustic waves, respectively, being the roots of the biquadratic equation:

$$u^4 - u^2(a^2 + b_x^2 + b_y^2) + a^2 b_x^2 = 0, \tag{1.3}$$

$c_s < c_f$, $b_y \overset{\text{df}}{=} \left(\dfrac{B^2\tau}{\mu}\right)^{\frac{1}{2}}$ — tangent component of Alfven speed, $a^2 = -\dfrac{1}{\tau^2}\dfrac{\partial p}{\partial\tau}\Big|_{s=s_0}$, a — speed of sound.

The structure of a shock wave is described by the solution of the system (1.1) tending at $+\infty$ to the state behind and at $-\infty$ to the state in front of the shock wave.

Therefore, such a solution joins the singular points determining the states of the shock wave.

2. Equations describing shock wave structure for $\eta = 0$

When disregarding the shear viscosity ($\eta = 0$), the system (1.1) can be reduced to a system of three ordinary differential equations, which can be written in the form:

$$\begin{aligned} \varepsilon_1 \frac{dB}{dx} &= \frac{B\tau}{\mu} - c_1^2 B + c_2, \\ \varepsilon_2 \frac{d\tau}{dx} &= p + M^2\tau + \frac{B^2}{2\mu} - c_3, \\ \varepsilon_3 \frac{dT}{dx} &= e - \frac{M^2\tau^2}{2} + \frac{c_1^2 B^2}{2} - \frac{B^2\tau}{2\mu} - c_2 B + c_3\tau - c_4, \end{aligned} \tag{2.1}$$

where the following notations are adopted: $\varepsilon_1 = 1/M\sigma\mu^2$, $\varepsilon_2 = \zeta M$, $\varepsilon_3 = k/M$.

Moreover, between B and v the following relation holds:

$$v = c_1 B. \tag{2.2}$$

Therefore, the problem of existence of the shock wave structure for $\eta = 0$ can be reduced to investigation of the existence, in the three-dimensional space (B, τ, T), of the integral curves joining the respective singular points of the system (2.1). Taking into account the obvious unique relation between the singular points of the system (1.1) and the singular points of the system (2.1), the latter will also be denoted by P_i ($i = 1, 2, 3, 4$). We shall assume that the coefficients ε_i ($i = 1, 2, 3$), connected with the dissipation coefficients $1/\sigma\mu^2$, ζ and k, are positive functions of class C^1 defined in the region $0 = \{(B, \tau, T): :T > 0, \tau > 0\}$. Following the generalized dissipation and generalized thermodynamic of Germain potential, we shall define the functions F_1 and W_1 similarly.

These functions can be written in the form:

$$F_1 = \frac{1}{T}\left\{\frac{\varepsilon_1}{2}\left(\frac{dB}{dx}\right)^2 + \frac{\varepsilon_2}{2}\left(\frac{d\tau}{dx}\right)^2 + \frac{\varepsilon_3}{2T}\left(\frac{dT}{dx}\right)^2\right\}, \tag{2.3}$$

$$W_1 = \frac{1}{T}\left\{\frac{B^2\tau}{2\mu} + \frac{M^2\tau^2}{2} - \frac{c_1^2 B^2}{2} - f(\tau, T) + c_2 B - c_3\tau + c_4\right\}, \tag{2.4}$$

where f is a mass density of the free energy. It is easy to verify that if we denote by q_i ($i = 1, 2, 3$) B, τ, T, respectively, and by $\dot{q}_i (i = 1, 2, 3)$ their derivatives with respect to x, then the system (1.2) can be written in the form:

$$\frac{\partial W_1}{\partial q_i} = \frac{\partial F_1}{\partial \dot{q}_i}, \quad i = 1, 2, 3. \tag{2.5}$$

Because ε_i ($i = 1, 2, 3$) are positive functions of the variables B, τ, T and

$$(2.6) \qquad 2F = \sum_{i=1}^{3} \frac{\partial F_1}{\partial \dot{q}_i} \dot{q}_i,$$

then, taking into account (2.5), we have

$$2F = \sum_{i=1}^{3} \frac{\partial W_1}{\partial q_i} \frac{dq_i}{dx} = \frac{dW_1}{dx} > 0;$$

hence, along the integral curves of the system (2.1):

$$(2.7) \qquad W_1(x) - W_1(x_0) = 2 \int_{x_0}^{x} F dx > 0.$$

This means that W_1 increases along the integral curves of the system (2.1). Taking into consideration the well known thermodynamic relation:

$$(2.8) \qquad df = -SdT - pd\tau,$$

where S — entropy and p — pressure, we verify that

$$(2.9) \qquad S = W_1 + T \frac{\partial W_1}{\partial T}.$$

3. Investigation of the integral curves of the system (2.1) in the neighbourhood of the singular points

According to the definition, the coordinates (B_i, τ_i, T_i) of each singular point P_i satisfy the system of equations:

$$(3.1) \qquad \begin{gathered} \frac{B_i \tau_i}{\mu} - c_1^2 B_i + c_2 = 0, \\ p(T_i, \tau_i) + M^2 \tau_i + \frac{B_i^2}{2\mu} - c_3 = 0, \\ e(T_i, \tau_i) - \frac{M^2 \tau_i^2}{2} + \frac{c_1^2 B_i^2}{2} - \frac{B_i^2 \tau_i}{2\mu} - c_2 B_i + c_3 \tau_i - c_4 = 0. \end{gathered}$$

The linearized system (2.1) in the neighbourhood of P_i has the form:

$$(3.2) \qquad \begin{aligned} \varepsilon_{1i} \frac{d\bar{B}}{dx} &= \left(\frac{\tau_i}{\mu} - c_1^2\right)\bar{B} + \frac{B_i}{\mu}\bar{\tau}, \\ \varepsilon_{2i} \frac{d\bar{\tau}}{dx} &= \frac{B_i}{\mu}\bar{B} + \left[M^2 + \left(\frac{\partial p}{\partial \tau}\right)_i\right]\bar{\tau} + \left(\frac{\partial p}{\partial T}\right)_i \bar{T}, \\ \frac{\varepsilon_{3i}}{T_i} \frac{d\bar{T}}{dx} &= \left(\frac{\partial p}{\partial T}\right)_i \bar{\tau} + \frac{1}{T_i}\left(\frac{\partial e}{\delta T}\right)_i \bar{T}, \end{aligned}$$

where $\varepsilon_{ki} = \varepsilon_k(B_i, \tau_i, T_i)$, ($k = 1, 2, 3$; $i = 1, 2, 3, 4$), $B = B_i + \bar{B}$, $\tau = \tau_i + \bar{\tau}$, $T = T_i + \bar{T}$. The index i at the partial derivatives means that their value is taken at the point

(τ_i, T_i). During the linearization, we made use of the fact that the point P_i satisfies the system (2.1), and that p and e are connected by the thermodynamic relation:

$$\frac{\partial p}{\partial T} = \frac{1}{T}\left(\frac{\partial e}{\partial \tau} + p\right). \tag{3.3}$$

The form of the integral curves in the neighbourhood of the singular points can be obtained from the eigenvalues of the system (3.2). For our investigations suffices to know the signs of the eigenvalues. They can be found from certain facts known in the theory of quadratic forms.

Putting $\bar{B} = B_0 e^{\lambda x}$, $\bar{\tau} = \tau_0 e^{\lambda x}$, $\bar{T} = T_0 e^{\lambda x}$ and then dividing each equation by $e^{\lambda x}$, we obtain:

$$\lambda U X = A X, \tag{3.4}$$

where $X = [B_0, \tau_0, T_0]$.

$$U \stackrel{\mathrm{df}}{=} \begin{bmatrix} \varepsilon_{1i} & 0 & 0 \\ 0 & \varepsilon_{2i} & 0 \\ 0 & 0 & \dfrac{\varepsilon_{3i}}{T} \end{bmatrix}, \tag{3.5}$$

$$A \stackrel{\mathrm{df}}{=} \begin{bmatrix} \dfrac{\tau_i}{\mu} - c_1^2 & \dfrac{B_i}{\mu} & 0 \\ \dfrac{B_i}{\mu} & M^2 + \left(\dfrac{\partial p}{\partial \tau}\right)_i & \left(\dfrac{\partial p}{\partial T}\right)_i \\ 0 & \left(\dfrac{\partial p}{\partial T}\right)_i & \dfrac{1}{T_i}\left(\dfrac{\partial e}{\partial T}\right)_i \end{bmatrix}. \tag{3.6}$$

The symmetric matrices U and A can be treated as the matrices of certain quadratic forms, the matrix U corresponding to a positive form, since $\varepsilon_{1i}, \varepsilon_{2i}, \dfrac{\varepsilon_{3i}}{T_i} > 0$. But, from the theory of quadratic forms it is known that, for forms defined by matrices A and U, there exists a linear nonsingular transformation which transforms the first form into the form having the unitary matrix E, and the second form into canonical form. Let C be the matrix of such a transformation. Then:

$$C^T U C = E, \quad C^T A C = D, \tag{3.7}$$

where E — unitary matrix, D — diagonal matrix.

Applying in the system (3.4) the substitution $X = CY$, we obtain:

$$\lambda U C Y = A C Y, \tag{3.8}$$

and as a result of multiplying the left-hand side by C^T, we have:

$$\lambda C^T U C Y = C^T A C Y. \tag{3.9}$$

According to (3.7), we have finally:

$$\lambda E Y = D Y, \tag{3.10}$$

from which it follows that the number of positive and negative eigenvalues is equal to the number of positive and negative terms in the diagonal matrix D, respectively. From the inertia theorem of quadratic form it is seen that to determine the number of positive and negative eigenvalues suffices to transform, by means of an arbitrary nonsingular transformation, the quadratic form having the matrix A to the diagonal form. The number of positive (or negative) elements of such a diagonal matrix is equal to the number of positive (or negative) eigenvalues.

To the matrix A [see (3.6)] corresponds the quadratic form $g(XX)$ having the following form:

$$
\begin{aligned}
(3.11)\quad g(XX) &= \left(\frac{\tau_i}{\mu}-c_1^2\right)x_1^2+2\frac{B_i}{\mu}x_1x_2+\left[M^2+\left(\frac{\partial p}{\partial \tau}\right)_i x_2^2+2\left(\frac{\partial p}{\partial T}\right)_i x_2x_3\right.\\
&\quad + \frac{1}{T_i}\left(\frac{\partial e}{\partial T}\right)_i x_3^2 \equiv \left(\frac{\tau_i}{\mu}-c_1^2\right)\left(x_1+\frac{\dfrac{B_i}{\mu}}{\dfrac{\tau_i}{\mu}-c_1^2}x_2\right)^2\\
&\quad +\left[M^2+\left(\frac{\partial p}{\partial \tau}\right)_i-\frac{\left(\dfrac{\partial p}{\partial T}\right)_i^2}{\dfrac{1}{T_i}\left(\dfrac{\partial e}{\partial T}\right)_i}-\frac{\left(\dfrac{B_i}{\mu}\right)^2}{\dfrac{\tau_i}{\mu}-c_1^2}\right]x_2^2+\frac{1}{T_i}\left(\frac{\partial e}{\partial T}\right)_i\left(x_3+\frac{\left(\dfrac{\partial p}{\partial T}\right)_i}{\dfrac{1}{T_i}\left(\dfrac{\partial e}{\partial T}\right)_i}x_2\right)^2.
\end{aligned}
$$

It suffices to apply a nonsingular linear transformation with the matrix:

$$
(3.12)\qquad C=\begin{bmatrix} 1 & \dfrac{B_i}{\tau_i-\mu c_1^2} & 0\\ 0 & 1 & 0\\ 0 & \dfrac{\left(\dfrac{\partial p}{\partial T}\right)_i}{\dfrac{1}{T_i}\left(\dfrac{\partial e}{\partial T}\right)_i} & 1\end{bmatrix},\qquad C=1\neq 0,
$$

to reduce the quadratic form corresponding to the matrix A to the form:

$$
(3.13)\quad g(YY)=\left(\frac{\tau_i}{\mu}-c_1^2\right)y_1^2+\left[M^2+\left(\frac{\partial p}{\partial \tau}\right)_i-\frac{\left(\dfrac{B_i}{\mu}\right)^2}{\dfrac{\tau_i}{\mu}-c_1^2}-\frac{\left(\dfrac{\partial p}{\partial T}\right)_i^2}{\dfrac{1}{T_i}\left(\dfrac{\partial e}{\partial T}\right)_i}\right]y_2^2+\frac{1}{T_i}\left(\frac{\partial e}{\partial T}\right)_i y_3^2.
$$

It is easy to prove that the coefficients at y_1^2 and y_2^2, the quadratic form being defined by (3.13), are respectively equal to:

$$
\frac{\tau_i}{\mu}\left(1-\frac{b_{xi}^2}{u_i^2}\right),\qquad \frac{1}{\tau_i^2}\,\frac{(u_i^2-c_{fi}^2)(u_i^2-c_{si}^2)}{u_i^2-b_{xi}^2}.
$$

The coefficient at y_3^2 is always positive. Taking into consideration the above remark and the inequality (1.2), we can formulate the theorem:

THEOREM 1. *At the point P_1 all eigenvalues are positive, at the points P_2 and P_3 two eigenvalues are positive and one is negative, at P_4 two eigenvalues are negative and one is positive.*

Additional analysis is needed in the case in which $\tau_i = \mu c_1^2$, which takes place for $c_2 = 0$ and concerns "switch on" and "switch off" shock waves. Let us transform $g(XX)$ by the assumption that $\tau_i = \mu c_1^2$:

$$g(XX) = 2\frac{B_i}{\mu}x_1x_2 + \left[M^2 + \left(\frac{\partial p}{\partial \tau}\right)_i\right]x_2^2 + 2\left(\frac{\partial p}{\partial T}\right)_i x_2x_3 + \frac{1}{T_i}\left(\frac{\partial e}{\partial T}\right)_i x_3^2$$

$$\equiv \frac{1}{T_i}\left(\frac{\partial e}{\partial T}\right)_i\left[x_3 + \frac{\left(\frac{\partial p}{\partial T}\right)_i}{\frac{1}{T_i}\left(\frac{\partial e}{\partial T}\right)_i}x_2\right]^2 + \left[M^2 + \left(\frac{\partial p}{\partial T}\right)_i - \frac{\left(\frac{\partial p}{\partial T}\right)_i^2}{\frac{1}{T_i}\left(\frac{\partial e}{\partial T}\right)_i}\right]\times$$

$$\times\left(x_2 - \frac{\frac{B_i}{\mu}x_1}{M^2 + \left(\frac{\partial p}{\partial T}\right)_i - \frac{\left(\frac{\partial p}{\partial T}\right)_i^2}{\frac{1}{T_i}\left(\frac{\partial e}{\partial T}\right)_i}}\right)^2 - \frac{\left(\frac{B_i}{\mu}\right)^2}{M^2 + \left(\frac{\partial p}{\partial T}\right)_i - \frac{\left(\frac{\partial p}{\partial T}\right)_i^2}{\frac{1}{T_i}\left(\frac{\partial e}{\partial T}\right)_i}}x_1^2.$$

The coefficient of the first binomial of the second degree is always positive and the coefficient of the second has the opposite sign in comparison with the coefficient of x_1^2. At points P_2, P_3, as in the case in which $c_2 \neq 0$, two eigenvalues are positive and one negative. This completes the proof for $c_2 = 0$.

4. Qualitative analysis of the surface $W_1 =$ const in the neighbourhood of the singular points

We shall analyse the surface $W_1(B, \tau, T) = A$, where $W_1(B, \tau, T)$ is the function defined by (2.4) and A is a constant. It is evident that the domain in which the variable B, τ, T could change is limited by the inequalities $\tau > 0$, $T > 0$. The gradient of $W_1(B, \tau, T)$ is equal to zero only at the singular points of the system (2.1) [this results from the equivalence of the system (2.1) and (2.5)], hence the surfaces $W_1(B, \tau, T) =$ const, which do not pass through the points $P_i (i = 1, 2, 3, 4)$, are everywhere regular and the surfaces passing through the points P_i $(i = 1, 2, 3, 4)$ have singularities only at these points.

We shall start the investigations of the character of the surface $W_1 =$ const in the neighbourhood of the singular points of system (2.1). Let A_i denote the value of the function $W_1(B, \tau, T)$ at the point P_i, $A_i = W_1(P_i) = S(P_i)$. We develop the function $W_1(B, \tau, T)$ in the neighbourhood of P_i into Taylor series, preserving the terms up to the second order. Since the first derivatives are zero, and the differential of the second order of the function

$W_1(B, \tau, T)$ is defined at P_i by the form $g(XX)$ [see (3.11)], then the development into Taylor series can be written as

$$W_1(B, \tau, T) = A_i + \frac{1}{2T_i} g(XX) + (0)^3,$$

where $X = [B-B_i, \tau-\tau_i, T-T_i]$, $(0)^3$ — remainder of the third order. By taking into account a sufficiently small neighbourhood of P_i, the term $(0)^3$ can be made negligible in comparison with the other terms.

The surface $W_1(B, \tau, T) = A_i$ in the neighbourhood of P_i can be sufficiently well described by the equations

$$A_i + \frac{1}{2T_i} g(XX) = A_i, \quad g(XX) = 0. \tag{4.1}$$

Applying a nonsingular linear transformation of the system, defined by the matrix C [see (3.12)], we finally obtain:

$$g(YY) = 0, \tag{4.2}$$

where $g(YY)$ is defined by (3.13). Following Theorem 1, we obtain the following corollary

COROLLARY 1. *In the neighbourhood of the point P_1, the surface $W_1(B, \tau, T) = A_1$ is reduced to the point P_1, in the neighbourhood of the point P_i $(i = 2, 3, 4)$ the surface $W_1(B, \tau, T) = A_i$ is topologically equivalent to a cone.*

To investigate the character of the surface $W_1(B, \tau, T) = A$ in the neighbourhood of P_i but not passing through P_i $(i = 1, 2, 3, 4)$, it suffices in the right-hand side of (4.1) to substitute $A_i + \delta$ for A_i, δ being close to zero. After performing transformations defined by the matrix C, we obtain:

$$g(YY) = T_i \delta. \tag{4.3}$$

But the Eq. (4.3) describes a quadric. When analysing the signs of coefficients in the form $g(YY)$, we finally obtain the following result:

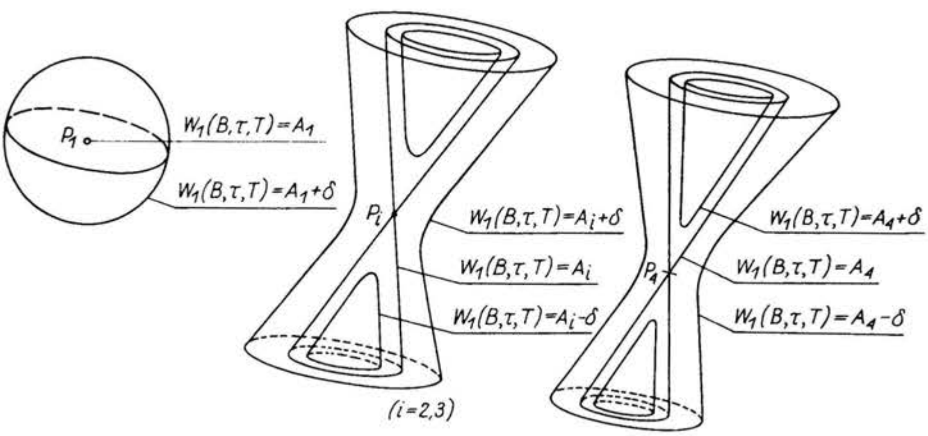

FIG. 1.

COROLLARY 2. *The surface* $W_1(B, \tau, T) = A_1+\delta$, $(\delta > 0)$ *is in the neighbourhood of* P_1 *topologically equivalent to a sphere, and the surface* $W_1(B, \tau, T) = A_i+\delta$, $(\delta > 0)$, *is in the neighbourhood of* P_i $(i = 2, 3)$ *topologically equivalent to a hyperboloid of one sheet and in the neighbourhood of* P_4 *it is topologically equivalent to hyperboloid of two sheets.*

Similarly, we obtain Corollary 3.

COROLLARY 3. *The surface* $W_1(B, \tau, T) = A_i-\delta$, $(\delta > 0)$, *is in the neighbourhood of* P_i *an empty set for* $i = 1$, *a set topologically equivalent to a hyperboloid of two sheets for* $i = 2, 3$ *and a set topologically equivalent to a hyperboloid of one sheet for* $i = 4$.

Corollaries 1, 2 and 3 are also immediate results of the Morse lemma. Figure 1 illustrates the above considerations.

5. Sections of the surface $W_1(B, \tau, T) = A$ by straight lines parallel to T axis

Our further considerations will be performed on the assumption that the equation of state for perfect gas holds —

$$p\tau = RT. \tag{5.1}$$

Then,

$$s-s_0 = c_v \ln RT\tau^{\gamma-1}, \quad e = c_v T. \tag{5.2}$$

To simplify the notation, we shall omit the constant S_0, which is not important in our considerations.

Taking into account in the formula defining W_1 [see (2.4)], the thermodynamical relation $f = e-TS$, and then applying the Eqs. (5.1) and (5.2), we obtain:

$$W_1(B, \tau, T) = \frac{1}{T}\left(\frac{B^2\tau}{2\mu}+\frac{M^2\tau^2}{2}-\frac{c_1^2 B^2}{2}-c_v T+c_v T\ln TR\tau^{\gamma-1} +c_2 B-c_3\tau+c_4\right). \tag{5.3}$$

Substituting in (5.3) $(B, \tau) = (B_0, \tau_0)$, we obtain a function of a single variable T. It is easy to see that for every (B_0, τ_0), where $\tau_0 > 0$,

$$\lim_{T\to\infty} W_1(B_0, \tau_0, T) = \infty \tag{5.4}$$

holds.

The limit of the function $W_1(B_0, \tau_0, T)$ at the point $T = 0$ can be expressed as follows:

$$\lim_{T\to\infty} W_1(B_0, \tau_0, T) = \begin{cases} +\infty & \text{for } (B_0, \tau_0)\in D \\ -\infty & \text{for } (B_0, \tau_0)\notin D \end{cases}, \tag{5.5}$$

where

$$D = \left\{(B_0, \tau_0): K(B_0, \tau_0) \overset{\mathrm{df}}{=} \frac{M^2\tau_0^2}{2}-\frac{c_1^2 B_0^2}{2}+\frac{B_0^2\tau_0}{2\mu}+c_2 B_0 -c_3\tau_0+c_4 > 0,\ \tau > 0\right\}. \tag{5.6}$$

Note that

$$(5.7)\qquad \frac{dW_1(B_0,\tau_0,T)}{dT} = \frac{1}{T^2}[c_v T - K(B_0,\tau_0)] = \frac{\partial W_1}{\partial T}\bigg|_{\substack{B=B_0\\ \tau=\tau_0}}.$$

From the above follows:

COROLLARY 4. *For every* $(B_0, \tau_0) \in D$, *the sign of* $\frac{dW_1(B_0,\tau_0,T)}{dT}$ *is changed one and only one point of the interval* $(0, \infty)$, *and for* $(B_0, \tau_0) \notin D$, *the function* $W_1(B_0, \tau_0, T)$ *incerases monotonically over the whole interval* $(0, \infty)$.

Denote by $\mathscr{L}_{(B_0,\tau_0)}$ a straight line parallel to the T axis and intersecting the plane (B, τ) at the point (B_0, τ_0).

From the corollary 4, the following corollary ensures:

COROLLARY 5. *For every* $(B_0, \tau_0) \notin D$ *and for every* A, *the surface* $W_1(B, \tau, T) = A$ *has one and only one point of intersection with the straight line* $\mathscr{L}_{(B_0,\tau_0)}$. *If, on the contrary,* $(B_0, \tau_0) \in D$, *then the number of intersection points depends on the constant* A. *Thus the straight line* $\mathscr{L}_{(B_0,\tau_0)}$ *may intersect* $W_1(B, \tau, T) = A$ *at two points, may be tangent to the surface and have no common points with the surface.*

On the basis of (5.7) and corollary 4, we can state that, for every point $(B_0, \tau_0) \in D$, there exists $T^*_{(B_0,\tau_0)} = \frac{1}{c_v} K(B_0, \tau_0)$ such that $\min\limits_{0<T<\infty} W_1(B_0, \tau_0, T) = W_1(B_0, \tau_0, T^*_{(B_0,\tau_0)}) \overset{\mathrm{df}}{=} W_1^*(B_0, \tau_0) = S(\tau_0, T^*_{(B_0,\tau_0)})$.

Hence we have:

COROLLARY 6. *The straight line* $\mathscr{L}_{(B_0,\tau_0)}$ *intersects the surface* $W_1(B, \tau, T) = A$ *at two points if* $(B_0, \tau_0) \in D$ *and* $W_1^*(B_0, \tau_0) < A$, *and it is tangent to the surface if* $(B_0, \tau_0) \in D$ *and* $W_1^*(B_0, \tau_0) = A$. *If* $(B_0, \tau_0) \in D$ and $W_1^*(B_0, \tau_0) > A$, *then* $\mathscr{L}_{(B_0,\tau_0)}$ *has no common points with the surface* $W_1(B, \tau, T) = A$.

In order to characterize the domain D, let us investigate its boundary $\mathscr{K}$ defined by the equation $K(B, \tau) = 0$, $\tau > 0$. Definition of the function $K(B, \tau)$ can be derived from (5.6). The equation of $\mathscr{K}$ can be written in the form:

$$(5.8)\qquad B^2\left(\frac{\tau}{2\mu} - \frac{c_1^2}{2}\right) + c_2 B + \frac{M^2\tau^2}{2} - c_3\tau + c_4 = 0,$$

or

$$(5.9)\qquad \frac{M^2\tau^2}{2} + \left(\frac{B^2}{2\mu} - c_3\right)\tau + c_2 B - \frac{c_1^2 B^2}{2} + c_4 = 0.$$

The discriminant $\Delta_B = c_2^2 - 2\left(\frac{\tau}{\mu} - c_1^2\right)\left(\frac{M^2\tau^2}{2} - c_3\tau + c_4\right)$ of the Eq. (5.8) is a polynomial of the third degree with respect to the variable τ, with negative coefficient by the highest order term. Hence a number N must exist such that for every $\tau > N$ the Eq. (5.8) has no roots. For $\tau = \mu c_1^2 \overset{\mathrm{df}}{=} \tau_*$ and $c_2 \neq 0$, we have $\Delta_B > 0$, and taking into account the continuity, we have $\Delta_B > 0$ in the neighbourhood of $\tau = \tau_*$. It is seen that for $\tau = \tau_*$ the Eq. (5.8) becomes linear with respect to B. Let us take $c_2 < 0$ (the opposite case $c_2 > 0$ can be obtained by changing the orientation of B axis), then with $\tau \to \tau_*^-$, we have $B \to -\infty$

and with $\tau \to \tau_*^+$ (for $\tau > \tau_*$), we have $B \to +\infty$. For $\tau = 0$, $\Delta_B > 0$, $(c_4 > 0)$, from (5.8) it is seen that the product of the roots B_1, B_2 is negative.

From analysis of the Eq. (5.9), it results that for $B = 0$, if there exist roots of this equation, then both of them are positive. The discriminant of the Eq. (5.9) is a polynomial of the fourth order with a positive coefficient of the highest order term; hence, for sufficiently great $|B|$, it has two roots and, as may easily be noticed, they are of different signs.

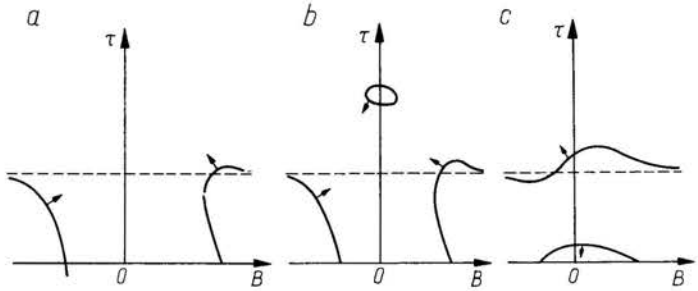

FIG. 2.

From the above it results that the qualitative character of the plot of the curve $\mathscr{K}$ can be shown — depending on the constants M, c_1, c_2, c_3, c_4 — as in Fig. 2.

The arrows in Fig. 2 indicate the domain D. For $c_2 = 0$ the plot of $\mathscr{K}$ can be simplified, being then symmetric with respect to the τ axis.

6. Connections between singular points of the curves Q_A and singular points of the surface $W_1(B, \tau, T) = A$. Analysis of the properties of the curves Q_A

As a result of the orthogonal projection of the surface $W_1(B, \tau, T) = A$ on the plane (B, τ), we obtain a set in the semiplane $\tau > 0$, the boundary of which consists of the B axis and the curve Q_A. In our considerations, the important role is played only by the curve Q_A and hence in discussing the boundary, we shall take into consideration only this curve. It is evident that points of the curve Q_A are the projections of the points of tangency of the surface $W_1(B, \tau, T) = A$ and the straight lines $\mathscr{L}_{(B,\tau)}$. Such points have to satisfy the following system of equations:

$$W_1(B, \tau, T) = A, \quad \frac{\partial W_1}{\partial T}(B, \tau, T) = 0, \tag{6.1}$$

or, on the basis of (2.9), the equivalent system:

$$S(\tau, T) = \text{A}, \quad \frac{\partial W_1(B, \tau, T)}{\partial T} = 0. \tag{6.2}$$

Making use of the formulae describing $S(\tau, T)$ [see (5.2)] and $\partial W_1/\partial t$ [see (5.7)] and then eliminating T from the system (6.2), we obtain the equation of Q_A:

$$\overline{K}(B, \tau) = \frac{1}{\gamma - 1} e^{\frac{A}{c_v}}, \tag{6.3}$$

where

$$\overline{K}(B,\tau) = \tau^{\gamma-1}K(B,\tau) = \tau^{\gamma-1}\left(\frac{M^2\tau^2}{2} - \frac{c_1^2B^2}{2} + \frac{B^2\tau}{2\mu} + c_2B - c_3\tau + c_4\right). \tag{6.4}$$

Note that to every point $(B, \tau) \in D$ [see (5.6)] there corresponds one and only one curve Q_A [the one that passes through (B, τ)], and to every $(B, \tau) \in D$ and to every curve Q_A there corresponds one and only one point (B, τ, T) of $W_1(B, \tau, T) = A$ — namely the point $(B, \tau, e^{\frac{A}{c_v}}\tau^{1-\gamma}/R)$. Therefore, we have established a one to one correspondence between the points of $(B, \tau) \in D$ and certain points of the (B, τ, T) space. Naturally, the latter fulfill the condition $\partial W_1/\partial T = 0$. In particular, to every singular point P_i $(i = 1, 2, 3, 4)$ there corresponds one and only one point of the (B, τ) plane. This will be denoted by $\overline{P}_i(B_i, \tau_i)$.

The gradient of the function $W_1(B, \tau, T)$ vanishes at the singular points of the system (2.1), and similarly the gradient $\overline{K}(B, \tau)$ vanishes at the point $\overline{P}_i(B_i, \tau_i)$. Indeed:

$$\frac{\partial\overline{K}}{\partial B} = \tau^{\gamma-1}\frac{\partial K}{\partial B} = \tau^{\gamma-1}\left(\frac{B\tau}{\mu} - c_1^2B + c_2\right)$$

and

$$\frac{\partial\overline{K}}{\partial\tau} = (\gamma-1)\tau^{\gamma-2}K(B,\tau) + \tau^{\gamma-1}\frac{\partial K}{\partial\tau} = \tau^{\gamma-1}\left(\frac{\gamma-1}{\tau}K(B,\tau) + M^2\tau + \frac{B^2}{2\mu} - c_3\right).$$

We must prove that if $P_i(B_i, \tau_i, T_i)$ satisfies the system (3.1) with $e = c_vT$ and $p = RT/\tau$, then $\overline{P}_i(B_i, \tau_i)$ satisfies the equations

$$\begin{gathered}\tau^{\gamma-1}\left(\frac{B\tau}{\mu} - c_1^2B + c_2\right) = 0,\\ \frac{\gamma-1}{\tau}\left[K(B,\tau) + M^2\tau + \frac{B^2}{2\mu} - c_3\right] = 0.\end{gathered} \tag{6.5}$$

It is not difficult to observe that taking into account the equation of state for an ideal gas in system (3.1) and eliminating T from this system, a system equivalent to (6.5) can be obtained.

On the other hand, if the point $\overline{P}_i(B_i, \tau_i)$ satisifes the Eqs. (6.5), then the corresponding point $P_i(B_i, \tau_i, e^{\frac{A}{c_v}}\tau_i^{1-\gamma}/R)$ satisfies the system (3.1). Moreover, note that by using the function $\overline{K}(B, \tau)$ we can establish the system describing the shock wave structure in the case in which $\eta = k = 0$. Indeed, it suffices to denote:

$$\bar{\varepsilon}_1 = \frac{\tau^{\gamma-1}}{2\sigma\mu M}, \quad \bar{\varepsilon}_2 = \frac{\zeta M\tau^{\gamma-1}}{2}, \quad \overline{F} = \bar{\varepsilon}_1\left(\frac{dB}{dx}\right)^2 + \bar{\varepsilon}_2\left(\frac{d\tau}{dx}\right)^2, \quad \overline{K} = \tau^{\gamma-1}\cdot K(B,\tau),$$

in order to obtain the final result:

$$\frac{\partial F}{\partial\dot{q}_j} = \frac{\partial\overline{K}}{\partial q_j} \quad j = 1, 2, \tag{6.6}$$

where $q_1 = B$, $q_2 = \tau$, $\dot{q}_j = dq_j/dx$.

It can easily be proved that along the integral curves of the system (6.6) the function $\overline{K}(B, \tau)$ increases. From the above consideration results Theorem 2.

THEOREM 2. *A point P_i of the surface $W_1(B, \tau, T) = A_i$ is singular if the corresponding point $\overline{P}_i$ of the curve Q_A: $K(B, \tau)\tau^{\gamma-1}$ = const is singular.*
Comparing the equation of the curve $\mathscr{K}$:

$$K(B, \tau) = 0, \tag{6.7}$$

with the equation of the family of curves Q_A:

$$K(B, \tau) = \frac{1}{\gamma-1} e^{\frac{A}{c_v}} \tau^{1-\gamma}, \quad \gamma > 1, \tag{6.8}$$

we can formulate the following theorem:

THEOREM 3. *For arbitrary $\delta > 0$ there exist such A_0 that in the domain $\{(B, \tau): \tau \geqslant \delta\}$ all the curves Q_A corresponding to $A < A_0$ lie sufficiently near the curve $\mathscr{K}$.*

P r o o f. Indeed, the fact that we consider only the domain $\{(B, \tau): \tau \geqslant \delta\}$ enables us to choose such A_0 that by $A < A_0$ the right-hand side of the Eq. (6.8) is sufficiently close to zero — i.e., from the right-hand side of the Eq. (6.7). By virtue of the continuity of the function $K(B, \tau)$, for an arbitrary bounded domain it is possible to choose such A_0 that for $A < A_0$ the parts of the curves Q_A belonging to this domain lie sufficiently near the curve $\mathscr{K}$. Since for $B \to \infty$ the curves Q_A and $\mathscr{K}$ have a common asymptote ($\tau = \tau_*$) and for sufficiently small A do not leave the domain $\{(B, \tau): \tau < L\}$, L being a constant, then the theorem holds for the whole domain $\{(B, \tau): \tau \geqslant \delta\}$.

Making use of the equation of the family of curves Q_A rewritten in the form

$$\left(\frac{M^2\tau^2}{2} - \frac{c_1^2 B^2}{2} + \frac{B^2\tau}{2\mu} + c_2 B - c_3\tau + c_4\right) = \frac{\tau^{1-\gamma}}{\gamma-1} e^{\frac{A}{c_v}}, \tag{6.9}$$

we can obviously state that every straight line $\tau = \tau_0$: crosses the curve Q_A at two points, is tangent to Q_A or has no common points with Q_A. The only exception is the straight line $\tau = \tau_*$, which crosses every curve Q_A at a single point (with $c_2 < 0$).

Substituting $B = B_0$ into the Eq. (6.9), we obtain the equation that must be satisfied by the coordinates τ of the intersection points of the straight line $B = B_0$ and the curve Q_A. The plot of the left-hand side of this equation (parabola) may intersect the plot of the right-hand side of this equation (generalized hyperbola) at one, two or three points. Hence it results that every straight line $B = B_0$ intersects the curve Q_A at one, two or

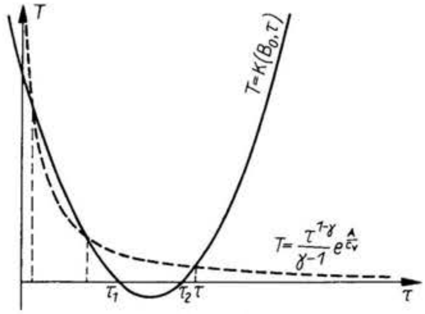

FIG. 3.

three points, respectively. On the basis of similar considerations, the following theorem can be proved (see Fig. 3).

THEOREM 4. *Let the straight line $B = B_0$ intersect the curve at the points (B_0, τ_1), (B_0, τ_2) [or at a single point (B_0, τ_0)], then one and only one point of intersection (B_0, τ) of the straight line and the curve satisfies the inequality $\tau > \max\ (\tau_1, \tau_2)$ (or $\tau > \tau_0$).*

Applying the Eq. (6.9) of the curve Q_A, we obtain:

THEOREM 5. *For every curve Q_A, there exists $d_A > 0$ such that the distance of this curve from B axis cannot be smaller than p_A.*

Indeed, taking an arbitrary but constant A, by virtue of the inequality $1-\gamma < 0$, for τ sufficiently small, the right-hand side of the Eq. (6.9) is great, while the left-hand side, for small τ, is upper bounded. Therefore, there exists $d_A > 0$ such that for $\tau < d_A$ the Eq. (6.9) cannot be satisfied.

7. Analysis of changes in the character of the curves Q_A

Proofs of existence of the shock wave structures will be based on the fact that, along the integral curves of the system of equations describing the structure, the function W_1 increases. To make use of this fact, we must analyse the surface $W_1(B, \tau, T) = A$, paying particular attention to changes which may — by means of continuity of the function $W_1(B, \tau, T)$ — occur only during the crossing of the parameter A through the values corresponding to singular points of the system (2.1). The properties proved in 3, 4, 5 and 6 enable us to reproduce with sufficient accuracy the shape of the surface $W_1(B, \tau, T) = A$, on the basis of its projection on the plane (B, τ). By virtue of Theorem 2, changes of a topological character in the surface $W_1(B, \tau, T) = A$ can be analysed on the basis of changes in the topological character of the curves Q_A. Analysis of the curves Q_A yields interesting information, which together with the proved properties of the surface $W_1(B, \tau, T) = A$ enable us to prove the existence of the shock waves structure. This analysis can conveniently be performed together with the analysis of the character of the curve $\mathscr{K}$. Hence we shall do it separately for each of the three cases considered in 5.

Let us begin from the set of constants M, c_1, c_2, c_3, c_4 to which corresponds a curve $\mathscr{K}$ shown in Fig. 2a. We may observe that for such a set of constants the discriminant of the Eq. (5.9)

$$\left(\frac{B^2}{2\mu} - c_3\right)^2 - M^2(2c_2 B - c_1^2 B^2 + 2c_4) = 0 \tag{6.10}$$

has only two real roots. Let us assume additionally that the system (2.1) has four singular points P_i $(i = 1, 2, 3, 4)$. The character of these points was determined in 3 and the character of the surface $W_1(B, \tau, T) = A$ in the neighbourhood of singular points P_i was analysed in 4. Taking into account a one to one correspondence of the singular points of the surface $W_1(B, \tau, T) = A$, and the singular points of the curve Q_A (see Theorem 2), the above assumption guarantees the existence of the points $\bar{P}_i$ $(i = 1, 2, 3, 4)$ in the plane (B, τ) (they are orthogonal projections of the points P_i on the surface (B, τ)). The points $\bar{P}_i$ are singular points of the system (6.6) and it is known (see [3]) that all the integral curves leave $\bar{P}_1$, two integral curves leave and two curves enter $\bar{P}_2$ and $\bar{P}_3$, and all the integral curves enter $\bar{P}_4$. From the character of the point $\bar{P}_i$ results the behaviour of the curves Q_A in their neighbourhood. The character of the curves Q can also be deduced from the character of the surface $W_1^{\bullet}(B, \tau, T) = A$.

We shall begin the analysis of the surface $W_1(B, \tau, T) = A$ for very small parameter A (A changes from $-\infty$ to $+\infty$). According to Theorem 3, the curves Q_A, being the boundary of the projection of the surfaces, have to pass in the neighbourhood of the curve $\mathscr{K}$. This concerns only the domain $\tau \geqslant \delta > 0$. The further behaviour of Q_A explains Theorem 4, and the fact that every straight line $\tau = \tau_0$ intersects Q_A at two points at most. Therefore, the curve Q_A, for sufficiently small A, consists of a single branch as shown in Fig. 4. In Figs. 5, 6, 8 and 9 replace P_i by $\bar{P}_i (i = 1, 2, 3, 4)$.

The domain bounded by the B axis and the curve Q_A forms an orthogonal projection of the surface $W_1(B, \tau, T) = A$ on the plane (B, τ). To every point in the shaded part

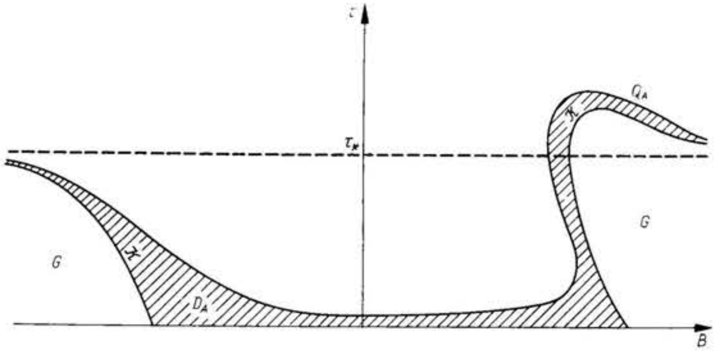

FIG. 4.

of the domain, defined by the condition $K(B, \tau) > 0$, there correspond two points on the surface $W_1(B, \tau, T) = A$. With increase of the constant A, the shaded domain $D_A = (B, \tau)$: $:0 < \bar{K}(B, \tau) < e^{\frac{A}{c_v}}/\gamma - 1, \tau > 0\}$ grows (if $A' > A'' \to D_{A'} \supset D_{A''}$) and the topological character of the curve Q_A changes for the first time when A exceeds the value $W_1(P_1) =$

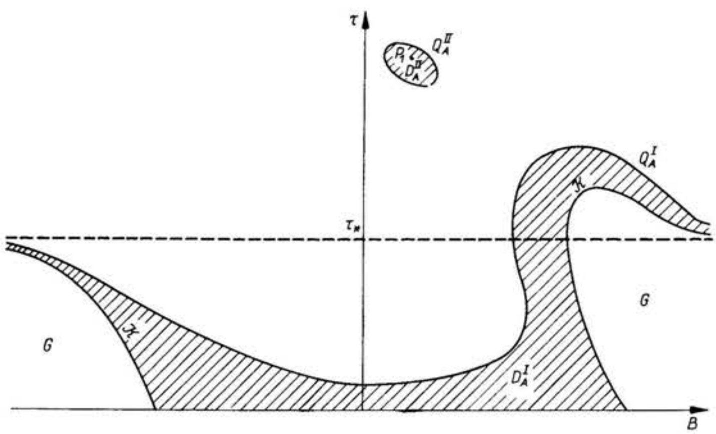

FIG. 5.

$= S(P_1)$. Taking into account the charakter of the point P_1 or the character of the point $\overline{P}_1$, we state that for $W_1(P_1) < A < W_1(P_2)$ the curve Q_A consists of two branches Q_A^{I} and Q_A^{II}, the second of which is closed. The form of the curve is presented in Fig. 5.

Now, D_A forms the union of two domains D_A^{I} and D_A^{II} (see Fig. 5). The further increase of A is accompanied by an increase in D_A^{I} and D_A^{II} and when A reaches the value $W_1(P_2)$, the branches Q_A^{I} and Q_A^{II} have one common point in the domain $\tau > \tau_*$. Indeed, if the singularities in P_2 did not correspond to the common boundary of Q_A^{I} and Q_A^{II}, then that would have to be realized at a further stage of increase of A and in the domain $\tau > \tau_*$, because the domains D_A and $\tau < \tau_*$ have to be disjoint. In the opposite case, Q_A could intersect the straight line $\tau = \tau_*$ at there points at least which is impossible. It is known, therefore, that in the domain $\tau > \tau^*$ there exist only two singular points; thus when A reaches the value $W_1(P_2)$, then Q_A^{I} must be in contact with Q_A^{II} (see Fig. 6).

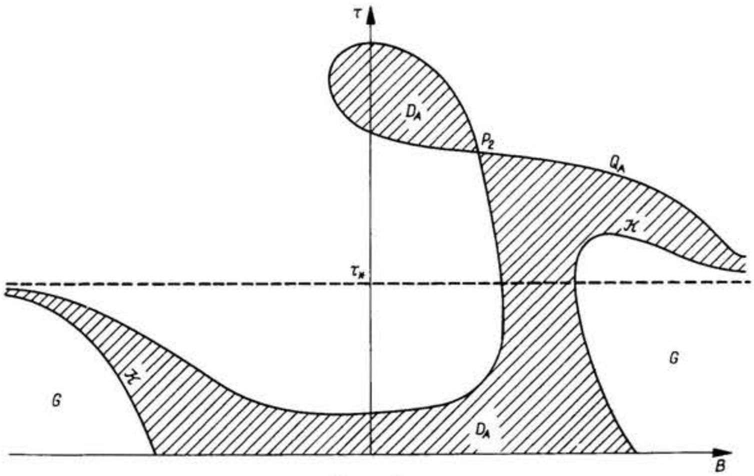

FIG. 6.

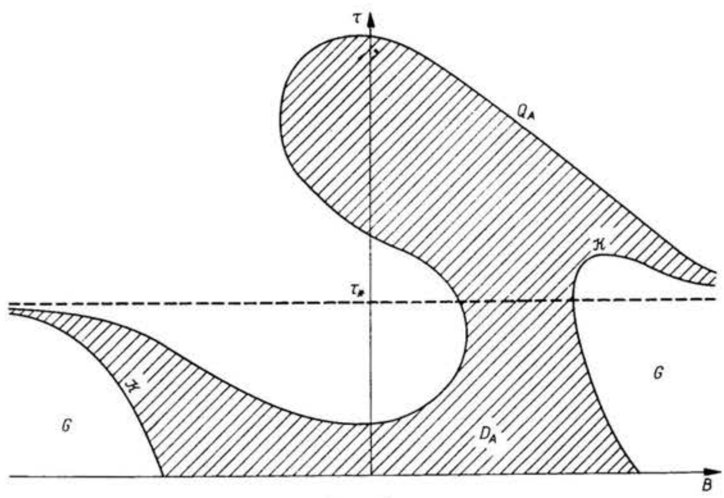

FIG. 7.

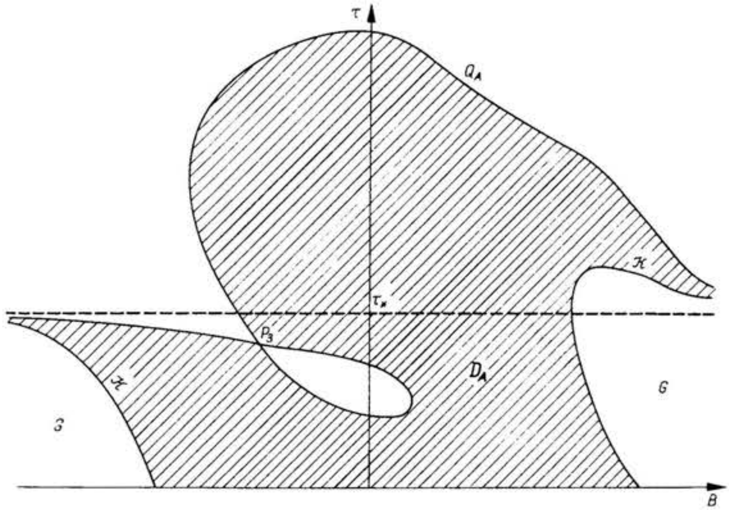

FIG. 8.

After exceeding the value of $S(P_2) = W_1(P_2) = A_2$ by A, the qualitative picture of the curve Q_A will be equivalent to the initial picture (corresponding to small A) for every $W(P_2) < A < W(P_3)$ (see Fig. 7).

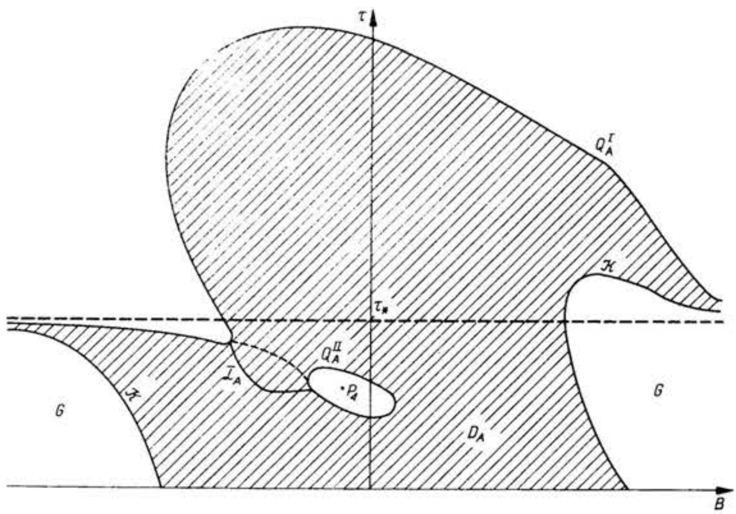

FIG. 9.

For $A = W_1(P_3) = A_3$ the curve Q_A must have the singularity at the point P_3. This point is situated in the domain $\tau < \tau_*$. From the character of the singular point P_3 it results that $\overline{P}_3$ is a saddle point. Hence the curve Q_A must form a loop in the domain $\tau < \tau_*$ (see Fig. 8).

With further increase of A, the curve Q_A is again divided into two branches Q_A^{I} and Q_A^{II} (see Fig. 9).

As A increases, the branch Q_A^{II} will include a smaller and smaller region and at the moment when A reaches the value $W_1(P_4)$, the curve Q_A will be reduced to a point, $\bar{P}_4$. With further increase of A, the curve Q_A will be characterized by the branch Q_A^{I} and its topological character will not be changed.

Note that the above considerations were made assuming that all the four singular points P_i do exist. From the discussion concerning the curves Q_A, it is easily seen that the singular point P_2 exists if the singular point P_1 exists, and the point P_4 exists if the point P_3 exists. Thus there remain two cases to analyse:

(i) there exist only P_1 and P_2,

(ii) there exist only P_3 and P_4.

In the first case, changes in the topological character of the curves Q_A will occur, as was shown above, up to the moment when A exceeds the value A_2. These changes are presented in Figs. 5, 6 and 7. For $A > A_2$, the topological character of the curves Q_A will be preserved.

In the second case, the first change of topological character of the curves Q_A will occur for $A = A_3$ (see Fig. 8). For $A_3 < A < A_4$, $(W_1(P_i) = A_i)$, the change of the curves Q_A will agree with that shown in Fig. 9. For $A = A_4$, the branch Q_A will be reduced to a point $\bar{P}_4$ and for $A > A_4$, the curve Q_A will consist of only one branch and its topological character will not change. The existence of fast shock wave corresponds only to the first case, the existence of slow shock wave only to the second.

For sets of constants M, c_1, c_2, c_3, c_4 such that the Eq. (6.10) has four real roots, the curve $\mathscr{K}$ consists of three branches, one of which is closed and is situated in the region $\tau > \tau_*$ (see Fig. 2b). From Theorems 3 and 5, and from obvious properties of the Eq. (6.10), it results that every curve Q_A, corresponding to a sufficiently small parameter A, consists of two branches Q_A^{I} and Q_A^{II} situated in the neighbourhood of the curve $\mathscr{K}$. The branch $Q_{A_i}^{II}$ forms a close curve and is situated in the region $\tau > \tau_*$ (see Fig. 10).

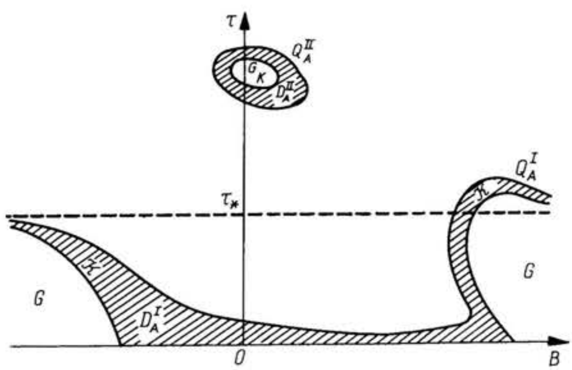

FIG. 10.

We shall prove that, in the situation now under consideration, existence of the point P_1 is not possible and existence of the point P_2 is necessary. Indeed, if the point P_1 exists, then for $A = A_1$ the curve Q_A would consist of the two branches Q_A^{I} and Q_A^{II} and the point P_1. For A, slightly greater than A_1, three branches of the curve Q_A ought to form. Two of them could be closed and they would be situated in the region $\tau > \tau_*$ (this results from the character of the point P_1). Because of the continuity of the function $\overline{K}(B, \tau)$, all the branches would have to be connected with each other to form a single curve. For reasons already indicated, the connection would have to occur in the region $\tau > \tau_*$, but this is linked with the necessity of existence of at least three singular points in the region $\tau > \tau_*$, and this is impossible. Note that the branches Q_A^{I} and Q_A^{II} for A increasing must join with each other (as a result of continuity of $\overline{K}(B, \tau)$) in the region $\tau > \tau_*$, and this proves the existence of P_2.

Thus, if the parameter A reaches the value A_2, the branches Q_A^{I} and Q_A^{II} join with each other and for $A > A_2$ the image of the curves Q_A will be in accordance with that presented in Fig. 7. In the next stage of increasing of A, a closed branch Q_A^{II} may occur in the case in which the point P_3 exists (see Fig. 8). The next change of topological character will be connected with degeneration of Q_A^{II} into a point $\overline{P}_4$ for $A = A_4$. For $A > A_4$, the curve Q_A consists of a single branch. It is possible that for the set of constants M, c_1, c_2, c_3, c_4, which is now under consideration the points P_3 and P_4 do not exist. Then, after the branches Q_A^{I} and Q_A^{II} join with each other for $A = A_2$, no other changes in the topological character of Q_A occur.

It remains to analyse the case corresponding to such a set of constants M, c_1, c_2, c_3, c_4 for which the Eq. (6.10) has no real roots. The curve $\mathscr{K}$ corresponding to that case is shown in Fig. 2c. For sufficiently small A, the curve Q_A consists of two branches, one of which forms a closed branch (see Fig. 11).

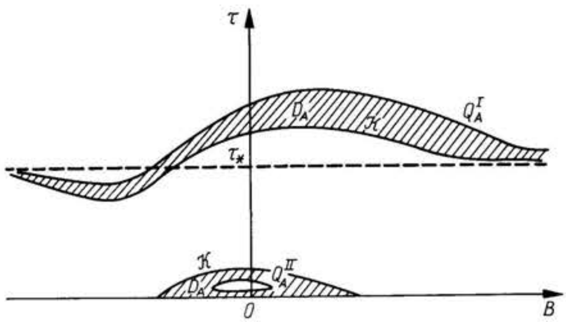

FIG. 11.

Note, however, that with a further increase of A, a closed branch of the curve Q_A cannot be formed, according to Theorem 4, in the region $\tau > \tau_*$; this excludes the possibility of existence of the point P_1 and hence the existence of the point P_2. Also in the region $\tau < \tau_*$ the branch Q_A^{II} cannot be separated into two parts because this would be connected with the necessity of existence of at least three singular points in the region $\tau < \tau_*$. Hence the point P_3 cannot exist. But with increase of A, the curve Q_A will bound a smaller and smaller region, up to the moment when for $A = A_4 = S(P_4)$ it will be degenerated into a point $\overline{P}_4$. Thus in the situation described, only one singular point exists, so that there are no shock waves here.

8. Proof of the existence of fast and slow shock wave structures

The results of the discussion in 7 enables us to reconstruct the shape of the surface $W_1(B, \tau, T) = A$. It is known, however, that the region $G_A = G \cup D_A$, where $G = \{(B, \tau): K(B, \tau) \leqslant 0, \tau > 0\}$, is the projection of the surface $W_1(B, \tau, T) = A$ on the plane (B, τ). This region is bounded by the curve Q_A and the straight line $\tau = 0$. Since $G \cap D = \phi$ (ϕ — empty set), then, according to Corollary 4, to every point $(B, \tau) \in G$ there corresponds one and only one point on the surface $W_1(B, \tau, T) = A$. But according to Corollary 6, every point $(B, \tau) \in D_A$ is an orthogonal projection of two and only two points situated on the surface $W_1(B, \tau, T) = A$. Straight lines parallel to T, passing through points of the curve Q_A are tangent to the surface $W_1(B, \tau, T) = A$, and every point (B, τ) situated on Q_A is an orthogonal projection of a single point of the surface $W_1(B, \tau, T) = A$.

A more careful analysis of the surface $W_1(B, \tau, T) = A$ enables us to state that projections of the points (B, τ, T) of the surface $W_1(B, \tau, T) = A$ for $T \to 0^+$ tend to the points of the curve $\mathscr{K}$, or to the points of the straight line $\tau = 0$.

Assuming that $\varepsilon_i (i = 1, 2, 3, 4)$ are positive functions of class C^1 of the variables (B, τ, T), we shall prove the existence of the fast and slow shock waves structure.

Existence of the fast shock waves structure is equivalent to existence of the integral curves of the system (2.1) connecting the singular point P_1 with the singular point P_2. A pair of such points belonging to the region $0 = \{(B, \tau, T): \tau > 0, T > 0\}$, as shown in 7, can exist only if the constants M, c_1, c_2, c_3, c_4 are so chosen that the Eq. (6.10) has two real roots only. To such a set of constants corresponds a curve $\mathscr{K}$ consisting of two branches (see Fig. 2a). Let us analyse changes of the surface $W_1(B, \tau, T) = A$ for $-\infty < A < \infty$. Let us begin from a very small A. Then, according to Theorems 3 and 5, the points of the curve Q_A have to be situated near the curve $\mathscr{K}$ or near B axis, and they belong obviously to the region D. The curves Q_A, for very small A, consist of one branch (see Fig. 4). On the basis of the interpretation of the region G_A bounded by the curve Q_A and the straight line $\tau = 0$, we can state that the surface $W_1(B, \tau, T) = A$ is, for small A, topologically equivalent to a plane. This situation cannot be changed with increase of A up to the value $A_1 = W(P_1)$. When this value is reached at an isolated point, $\overline{P}_1$ is adjoint to the curve Q_A. Hence the surface $W_1(B, \tau, T) = A_1$ consists of two disjoint parts — the part topologically equivalent to a plane, and the point P_1 being disjoint with the first part. For $A_1 < A < A_2$, the surface $W_1(B, \tau, T) = A$ consists of two parts, having no common points, since its projection consists of two disjoint regions (see Fig. 5). The part of the surface corresponding to the branch Q_A^{II} (see Fig. 5) forms a closed surface, the point P_1 being its internal point. (This results from 7 as well as from properties of the singular point P_1). As the parameter A increases, the parts of the surface $W_1(B, \tau, T) = A$ referred to approach — each other. They will be in contact when the parameter A reaches the value A_2 (this results from the behaviour of the curves Q_A and from a one to one correspondence between the points of the curve Q_A and the points of the surface $W_1(B, \tau, T) = A$.).

The analysis of the character of the singular points carried out in 4 shows that all the integral curves leave the point P_1. It is clear that for A sufficiently small, but greater than A_1, the intersection of the integral curves leaving the point P_1 with the surface $W_1(B, \tau, T) = A$ forms the closed part of the surface $W_1(B, \tau, T) = A$. With A increasing, the situation will be similar [W_1 increases along the integral curves of system (2.1)] up to the moment when A reaches the value A_2. Then, the two parts of the surface $W_1(B, \tau, T) = A$ will come into contact. Hence only one integral curve leaving the point P_1 must enter the point P_2. The other curve, of the integral curves considered as entering P_2, enters P_2 with the opposite sense, therefore it came out of the region bounded by closed surfaces. This proves the uniqueness of the fast shock wave structure.

The problem of existence of the slow shock wave structure for $\eta = 0$ is equivalent to the problem of existence of an integral curve of system (2.1) joining the singular point P_3 with the singular point P_4. The pair of points P_3, P_4 belonging to the region $0 = \{(B, \tau, T): \tau > 0,\ T > 0\}$ can exist in two cases only (see 7). The first case is connected with a set of constants M, c_1, c_2, c_3, c_4 for which the Eq. (6.10) has four real roots, the second for which the Eq. (6.10) has two real roots. Qualitalively different images of the curve $\mathscr{K}$ and, as a consequence, different regions G, correspond to those cases (see 2a and 2b). That cases a certain difference between the shapes of the surface $W_1(B, \tau, T) = A$ corresponding to the above cases. As was shown in 7, the qualitative image of the curves Q_A for $A > A_2$ is in both cases the same. The regions $G \cup D_A$ corresponding to those cases are also qualitatively the same.

Since along the integral curves of system (2.1) $W_1(B, \tau, T)$ increases, then every integral curve coming out of the point P_3 and coming into the point P_4 must be situated in the region $\{(B, \tau, T): A_3 \leqslant W_1(B, \tau, T) \leqslant A_4\}$. In connection with the above remarks, we shall carry out analysis of the cross-section of the two-dimensional manifold formed by the integral curves coming out of the point P_3 and the surface $W_1(B, \tau, T) = A$ for $A_3 < A < A_4$. For $A = A_3$, this manifold degenerates into a point. Since the point P_3 is an elementary singular point to which there correspond two positive eigenvalues and one negative eigenvalue, then, according to Hadamard-Peron's lemma, the manifold formed of the integral curves leaving the point P_3 is in the neighbourhood of this point diffeomorfic to a plane. By Corollary 1, the surface $W_1(B, \tau, T) = A_3$ is in the neighbourhood of the point P_3 topologically equivalent to a cone, and each of the surfaces $W_1(B, \tau, T) = A_3 + \delta$ ($\delta > 0$ — small) is in the neighbourhood of P_3 topologically equivalent to a hyperboloid of one sheet (see Fig. 1). The surface formed of the integral curves

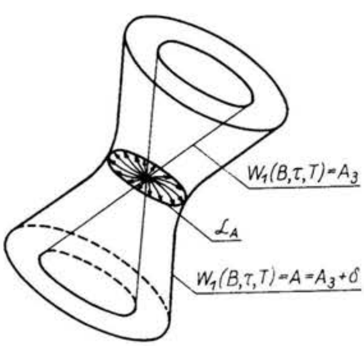

FIG. 12.

leaving the point P_3 cannot go into the interior of the cone $W_1(B, \tau, T) = A_3$ because the relation $W_1(B, \tau, T) < A_3$ holds there. This surface, being locally diffeomorphic to a plane, must intersect the surface $W_1(B, \tau, T) = A_3 + \delta$ (δ — sufficiently small) along the closed curve $\mathscr{L}_A$ (see Fig. 12). Let us observe that, $W_1(B, \tau, T) = A$ remaining on the surface, we cannot continuously transform the curve $\mathscr{L}_A$ into a point. Projection of the curve $\mathscr{L}_A$ into the plane (B, τ) must form a closed curve $\overline{\mathscr{L}}_A$, having common points with the branches Q_A^{I} and Q_A^{II} (see Fig. 9). With increase of A, the curve $\mathscr{L}_A$ cannot be split at a finite point of the region $0 = \{(B, \tau, T): T > 0,\ \tau > 0\}$ (that would contradict the continuity). Since the branch Q_A^{II} does not tend to infinity, and to every point situated on this branch there corresponds a finite point situated on the surface, then $\overline{\mathscr{L}}_A$ has to possess for all $A_3 < A < A_4$ a common point with the branch Q_A^{II}. But the branch Q_A^{II}, with increase of A, bounds a smaller region and for $A = A_4$ is degenerated to a point $\overline{P}_4$. Hence it results that the curve $\overline{\mathscr{L}}_A$ for $A = A_4$ passes through the point $\overline{P}_4$ and, in consequence, $\mathscr{L}_A$ for $A = A_4$ has to pass through P_4. This means that at least one integral curve leaving the point P_3 and entering P_4 exists. Thus the existence of the structure of slow shock waves is proved.

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