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Finite, axially symmetric deformation of plastic fibre-reinforced materials

M. ARCISZ (WARSZAWA)

IN THIS paper we consider an axially symmetric deformation [defined by (1.2)] of an ideal rigid-plastic material reinforced by inextensible fibres parallel to the symmetry axis. We derive the system of five differential equations for five functions: three functions f, g, h, describing the motion of the medium, and two functions p, q, characterizing the mean pressure and the normal stress in the direction of the fibres. A particular case of motion (1.2) is considered by assuming that $\varphi \equiv \Phi$. This assumption enables us to determine the functions of motion independently of the equations of motion. The equations of motion (one of them is, in this case, identicaly satisfied) constitute a system of two linear equations for the functions p and q.

W pracy rozpatruje się osiowo-symetryczną deformację [zdefiniowaną wzorami (1.2)] materiału sztywno idealnie-plastycznego, wzmocnionego nierozciągliwymi włóknami równoległymi do osi symetrii. Wyprowadzono układ pięciu równań na pięć poszukiwanych funkcji: trzy funkcje f, g, h, opisujące ruch ośrodka, oraz dwie funkcje p i q charakteryzujące ciśnienie średnie i naprężenie normalne w kierunku ułożenia włókien. Zbadano szczególny przypadek ruchu (1.2) przy założeniu $\varphi = \Phi$. Założenie to pozwala na znalezienie funkcji opisujących ruch ośrodka niezależnie od równań ruchu. Równania ruchu, z których jedno jest w tym przypadku spełnione tożsamościowo, stanowią układ równań liniowych na funkcje p i q.

В работе исследована осесимметричная деформация жестко-идеально пластического материала, армированного нерастяжимыми нитями, расположенными параллельно оси симметрии. Выведена система пяти уравнений для пяти искомых функций, из которых три функции f, g, h описывают движение материала, а две остальные p и q характеризуют среднее давление и нормальные напряжения в направлении расположения нитей. Исследован частный случай движения (1.2), когда $\varphi \equiv \Phi$. Это предположение позволяет определить функции, описывающие движение материала независимо от уравнений движения, которые в этом случае образуют систему линейных уравнений относительно функций p и q, причем одно из них тождественно удовлетворяется.

J. F. MULHERN, T. G. ROGERS and A. J. M. SPENCER proposed a continuum model for fibre-reinforced materials [1]. Within the framework of this model, in the present paper we consider an axially symmetric deformation of a material reinforced by one family of fibres, parallel to the symmetry axis. Following the authors of paper [1], we assume that the material is incompressible, ideal rigid-plastic and locally transversely isotropic. It is further assumed that the direction of fibres is defined at *each* point of the medium (the continuum model of fibre-reinforced materials) and the material is inextensible in this direction.

We consider finite deformations using the material description.

1. Kinematics

Let us introduce two cylindrical coordinate systems: material X^K and spatial x^k. Hereafter we use the notation

$$X^1 = R, \quad X^2 = \Phi, \quad X^3 = Z,$$
$$x^1 = r, \quad x^2 = \varphi, \quad x^3 = z. \tag{1.1}$$

The coordinate reference system X^K is chosen in such a way that the Z-axis coincides with the symmetry axis. Therefore fibres are material lines Z.

A motion of the medium is described by the functions $x^i = x^i(X^K, t)$, and we assume that they satisfy the following conditions of axial symmetry:

$$\begin{aligned} r &= f(R, Z, t), \\ \varphi &= \Phi + g(R, Z, t), \\ z &= h(R, Z, t). \end{aligned} \tag{1.2}$$

Hence the deformation gradient has the form:

$$[x^i{}_{,K}] = \begin{bmatrix} f_{,R} & 0 & f_{,Z} \\ g_{,R} & 1 & g_{,Z} \\ h_{,R} & 0 & h_{,Z} \end{bmatrix}. \tag{1.3}$$

The functions (1.2) describing the motion must satisfy the incompressibility condition:

$$f_{,R}h_{,Z} - h_{,R}f_{,Z} = \frac{R}{f}, \tag{1.4}$$

the condition of inextensibility in the material, fibre direction — Z

$$(f_{,Z})^2 + (fg_{,Z})^2 + (h_{,Z})^2 = 1, \tag{1.5}$$

and the initial conditions:

$$f(R, Z, 0) = R, \quad g(R, Z, 0) = 0, \quad h(R, Z, 0) = Z. \tag{1.6}$$

The rate of deformation tensor d_{kl} is determined by the relation:

$$d_{kl} = X^K{}_{,k} X^L{}_{,l} \dot{E}_{KL}, \tag{1.7}$$

where $E_{KL} \equiv \frac{1}{2}(g_{ij} x^i{}_{,K} x^j{}_{,L} - G_{KL})$ is the Green strain tensor, G_{KL}, g_{kl} are metric tensors of the coordinate systems (X^K and x^k, respectively), and dot denotes material time derivative. From (1.7), taking into account (1.3), we obtain:

$$(1.8) \quad [d_{kl}] =$$
$$= \begin{bmatrix} \frac{f}{R}(h_{,Z}f_{,Rt} - h_{,R}f_{,Zt}) & \frac{f^3}{2R}(h_{,Z}g_{,Rt} - h_{,R}g_{,Zt}) & \frac{f}{2R}\left[(h_{,Z})^2\left(\frac{h_{,R}}{h_{,Z}}\right)_{,t} - (f_{,Z})^2\left(\frac{f_{,R}}{f_{,Z}}\right)_{,t}\right] \\ \frac{f^3}{2R}(h_{,Z}g_{,Rt} - h_{,R}g_{,Zt}) & ff_{,t} & \frac{f^3}{2R}(f_{,R}g_{,Zt} - f_{,Z}g_{,Rt}) \\ \frac{f}{2R}\left[(h_{,Z})^2\left(\frac{h_{,R}}{h_{,Z}}\right)_{,t} - (f_{,Z})^2\left(\frac{f_{,R}}{f_{,Z}}\right)_{,t}\right] & \frac{f^3}{2R}(f_{,R}g_{,Zt} - f_{,Z}g_{,Rt}) & \frac{f}{R}(f_{,R}h_{,Zt} - f_{,Z}h_{,Rt}) \end{bmatrix}$$

The rate of deformation tensor is then determined by three arbitrary functions $f(R, Z, t)$, $g(R, Z, t)$, $h(R, Z, t)$ which are connected by the relations (1.4), (1.5).

2. Constitutive relations

Following the authors of paper [1] we introduce the locally rectangular Cartesian coordinate system $\mathbf{e}_\alpha$, where the $\mathbf{e}_3$-axis is tangent to the material line Z (to the fibre direction) at each point x^k and at each instant of time t. The $\mathbf{e}_1$, $\mathbf{e}_2$-axes can be chosen arbitrarily; we choose them so that $\mathbf{e}_1$ is normal to the current surface $R = \text{const}$, and $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$. The decomposition of so chosen unit vectors in the cylindrical reference system has the form:

$$(2.1)\qquad [e^i_{\ \alpha}] = \begin{bmatrix} \frac{1}{\eta} h_{,Z} & -\frac{1}{\eta} fg_{,Z} f_{,Z} & f_{,Z} \\ 0 & \eta/f & g_{,Z} \\ -\frac{1}{\eta} f_{,Z} & -\frac{1}{\eta} fg_{,Z} h_{,Z} & h_{,Z} \end{bmatrix},$$

where

$$(2.2)\qquad \eta \equiv (f_{,Z})^2 + (h_{,Z})^2.$$

Hereinafter, the dash above a component of a tensor (e.g. $\bar{a}_{\alpha\beta}$) denotes a component of this tensor referred to the reference system $\mathbf{e}_\alpha$. Relations between components of an arbitrary tensor in the reference system $\mathbf{e}_\alpha$ ($\bar{a}_{\alpha\beta}$) and components of this tensor in a cylindrical system (a_{ij}) are:

$$(2.3)\qquad \bar{a}_{\alpha\beta} = e^i_{\ \alpha} e^j_{\ \beta} a_{ij}, \qquad a_{ij} = e^\alpha_{\ i} e^\beta_{\ j} \bar{a}_{\alpha\beta}.$$

The $\mathbf{e}_3$-direction defines at each point x^k and at each instant of time t, the direction of transvers isotropy of the material. Therefore the yield condition must be invariant with respect to a rotation about the $\mathbf{e}_3$-axis, and in general it is a function of five common invariants of the Cauchy stress tensor $\bar{\sigma}_{\alpha\beta}$ and $\mathbf{e}_3$-vector. In the paper [1], it was shown that bearing in mind the associated flow rule it is possible to reduce the number of invariants to three, taking into acount the incompressibility and the inextensibility in the Z-direction. Under these assumptions, the general form of the yield condition is

$$(2.4)\qquad F(I_3, I_4', I_5') = 0,$$

where

$$(2.5)\qquad \begin{gathered} I_3 = \bar{\sigma}_{13}^2 + \bar{\sigma}_{23}^2, \qquad I_4' = \frac{1}{2}(\bar{\sigma}_{11} - \bar{\sigma}_{22})^2 + 2\bar{\sigma}_{12}, \\ I_5' = \frac{1}{2}(\bar{\sigma}_{11} - \bar{\sigma}_{22})(\bar{\sigma}_{13}^2 - \bar{\sigma}_{23}^2) + 2\bar{\sigma}_{23}\bar{\sigma}_{13}\bar{\sigma}_{12}. \end{gathered}$$

Furthermore, we may require the yield condition to be a quadratic function of the stress tensor components and to be independent of I_5'. In particular, it may take the form proposed by HILL [2]

$$(2.6)\qquad F = \frac{I_3}{k_1^2} + \frac{1}{2}\frac{I_4'}{k_2^2} - 1 = 0,$$

where k_1, k_2 are material constants.

We assume the constitutive relations in the form of the associated flow rule:

$$\bar{d}_{\alpha\beta} = \lambda \frac{\partial F}{\partial \bar{\sigma}_{\alpha\beta}}, \tag{2.7}$$

and the yield condition in the form (2.6), and hence

$$\frac{1}{k_1^2}(\bar{\sigma}_{13}^2+\bar{\sigma}_{23}^2)+\frac{1}{k_2^2}\left[\frac{1}{4}(\bar{\sigma}_{11}-\bar{\sigma}_{22})^2+\bar{\sigma}_{12}^2\right]-1 = 0. \tag{2.8}$$

On the basis of (2.7) and (2.8), we obtain:

$$\begin{aligned} \frac{1}{2}(\bar{\sigma}_{11}-\bar{\sigma}_{22}) &= \Lambda \bar{d}_{11}, \quad \bar{\sigma}_{13} = \varkappa^2 \Lambda \bar{d}_{13}, \\ \bar{\sigma}_{12} &= \Lambda \bar{d}_{12}, \quad \bar{\sigma}_{23} = \varkappa^2 \Lambda \bar{d}_{23}, \end{aligned} \tag{2.9}$$

where

$$\Lambda \equiv \frac{k_2^2}{\lambda}, \qquad \varkappa \equiv \frac{k_1}{k_2}.$$

Substituting (2.9) into the yield condition (2.8), we are able to determine the function $\lambda = \lambda(\bar{d}_{\alpha\beta})$:

$$\lambda^2 = k_2^2[\varkappa^2(\bar{d}_{13}^2+\bar{d}_{23}^2)+(\bar{d}_{11}^2+\bar{d}_{12}^2)]. \tag{2.10}$$

Using (2.3), we now can write the above relations in the reference system (r, φ, z). Thus on the basis of $(2.3)_1$, substituting (1.8), we have:

$$\begin{aligned} \bar{d}_{11} = -\bar{d}_{22} &= \frac{(f\eta)_{,t}}{f\eta}, \\ \bar{d}_{33} &= 0, \\ \bar{d}_{12} &= \frac{f^2\eta^2}{2R}\left(g_{,R}-\frac{1}{\eta^2}\xi g_{,Z}\right)_{,t}, \\ \bar{d}_{13} &= \frac{\eta f}{2R}\left[(f^2 g_{,R} g_{,Z}+\xi)_{,t}-(f^2 g_{,Z})_{,t}\left(g_{,R}-\frac{1}{\eta^2}\xi g_{,Z}\right)\right], \\ \bar{d}_{23} &= \frac{1}{2f\eta}(f^2 g_{,Z})_{,t}, \end{aligned} \tag{2.11}$$

where

$$\xi \equiv f_{,R} f_{,Z} + h_{,R} h_{,Z}. \tag{2.12}$$

Furthermore, substituting (2.9) into $(2.3)_2$, we finally obtain:

$$\begin{aligned} \sigma_1^1 &= p+(f_{,Z})^2 q+\Lambda F_1^1, \\ \sigma_2^2 &= p+(fg_{,Z})^2 q+\Lambda F_2^2, \\ \sigma_3^3 &= p+(h_{,Z})^2 q+\Lambda F_3^3, \\ \sigma_2^1 &= f^2 g_{,Z} f_{,Z} q+\Lambda F_2^1, \\ \sigma_3^1 &= f_{,Z} h_{,Z}+\Lambda F_3^1, \\ \sigma_2^3 &= f^2 g_{,Z} h_{,Z} q+\Lambda F_2^3, \end{aligned} \tag{2.13}$$

where

$$F_1^1 \equiv \frac{1}{\eta^2}[(h_{,Z})^2-(f_{,Z}fg_{,Z})^2]\bar{d}_{11}-\frac{2}{\eta^2}fg_{,Z}f_{,Z}h_{,Z}\bar{d}_{12} -\varkappa^2\frac{2}{\eta}fg_{,Z}(f_{,Z})^2\bar{d}_{23}+\varkappa^2\frac{2}{\eta}f_{,Z}h_{,Z}\bar{d}_{13},$$

$$F_2^2 \equiv -\eta^2\bar{d}_{11}+\varkappa^2 2\eta fg_{,Z}\bar{d}_{23},$$

$$F_3^3 \equiv \frac{1}{\eta^2}[(f_{,Z})^2-(fg_{,Z}h_{,Z})^2]\bar{d}_{11}+\frac{2}{\eta^2}fg_{,Z}f_{,Z}h_{,Z}\bar{d}_{12} -\varkappa^2\frac{2}{\eta}fg_{,Z}(h_{,Z})^2\bar{d}_{23}-\varkappa^2\frac{2}{\eta}f_{,Z}h_{,Z}\bar{d}_{13}, \tag{2.14}$$

$$F_2^1 \equiv f^2g_{,Z}f_{,Z}\bar{d}_{11}+fh_{,Z}\bar{d}_{12}+\varkappa^2\frac{1}{\eta}ff_{,Z}[\eta^2-(fg_{,Z})^2]d_{23}+\varkappa^2\frac{1}{\eta}f^2g_{,Z}h_{,Z}\bar{d}_{13},$$

$$F_3^1 \equiv -\frac{1}{\eta_2}f_{,Z}h_{,Z}[1+(fg_{,Z})^2]\bar{d}_{11}+\frac{1}{\eta_2}fg_{,Z}[(f_{,Z})^2-(h_{,Z})^2]\bar{d}_{12} -\varkappa^2\frac{2}{\eta}fg_{,Z}f_{,Z}h_{,Z}\bar{d}_{23}-\varkappa^2\frac{1}{\eta}[(f_{,Z})^2-(h_{,Z})^2]\bar{d}_{13},$$

$$F_3^2 \equiv f^2g_{,Z}h_{,Z}\bar{d}_{11}-ff_{,Z}\bar{d}_{12}+\varkappa^2\frac{1}{\eta}fh_{,Z}[\eta^2-(fg_{,Z})^2]\bar{d}_{23}-\varkappa^2\frac{1}{\eta}f^2g_{,Z}f_{,Z}\bar{d}_{13},$$

and Λ, $\bar{d}_{\alpha\beta}$ are determined by the formulae (2.10) and (2.11), respectively. Thus we have expressed the state of stress by functions of motion and two arbitrary functions p, q, which have to be found from the equations of motion.

3. Equations of motion

The equations of motion written in the material, cylindrical system (R, Φ, Z), disregarding dynamical terms and mutual forces, take the form:

$$\frac{f}{R}[h_{,Z}\sigma^1_{1,R}+(h_{,R}g_{,Z}-g_{,R}h_{,Z})\sigma^1_{1,\Phi}-h_{,R}\sigma^1_{1,Z}-f_{,Z}\sigma^3_{1,R}+(f_{,Z}g_{,R}-f_{,R}g_{,Z})\sigma^3_{1,\Phi} +f_{,R}\sigma^3_{1,Z}]+\sigma^2_{1,\Phi}+\frac{1}{f}(\sigma^1_1-\sigma^2_2) = 0,$$

$$\frac{f}{R}[h_{,Z}\sigma^1_{2,R}+(h_{,R}g_{,Z}-g_{,R}h_{,Z})\sigma^1_{2,\Phi}-h_{,R}\sigma^1_{2,Z}-f_{,Z}\sigma^3_{2,R}+(f_{,Z}g_{,R}-f_{,R}g_{,Z})\sigma^3_{2,\Phi} +f_{,R}\sigma^3_{2,Z}]+\sigma^2_{2,\Phi}+\frac{1}{f}\sigma^1_2 = 0, \tag{3.1}$$

$$\frac{f}{R}[h_{,Z}\sigma^1_{3,R}+(h_{,R}g_{,Z}-g_{,R}h_{,Z})\sigma^1_{3,\Phi}-h_{,R}\sigma^1_{3,Z}-f_{,Z}\sigma^3_{3,R}+(f_{,Z}g_{,R}-f_{,R}g_{,Z})\sigma^3_{3,\Phi} +f_{,R}\sigma^3_{3,Z}]+\sigma^2_{3,\Phi}+\frac{1}{f}\sigma^1_3 = 0.$$

Substituting (2.13) into (3.1) and assuming that $p_{,\varphi} = p_{,\Phi} = 0$, we have:

$$
\begin{aligned}
&\frac{f}{R}(h_{,Z}p_{,R} - h_{,R}p_{,Z}) + f_{,Z}q_{,Z} + [f_{,ZZ} - f(g_{,Z})^2]q + A_1 = 0,\\
(3.2)\qquad &f^2 g_{,Z}q_{,Z} + (f^2 g_{,Z})_{,Z}q + A_2 = 0,\\
&\frac{f}{R}(-f_{,Z}p_{,R} + f_{,R}p_{,Z}) + h_{Z,}q_{,Z} + h_{,ZZ}q + A_3 = 0,
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &\equiv \Lambda\left[\frac{f}{R}(h_{,Z}F^1_{1,R} - h_{,R}F^1_{1,Z} - f_{,Z}F^3_{1,R} + f_{,R}F^3_{1,Z}) + \frac{1}{f}(F^1_1 - F^2_2)\right]\\
&\quad + \Lambda_{,R}\frac{f}{R}(h_{,Z}F^1_1 - f_{,Z}F^3_1) - \Lambda_{,Z}\frac{f}{R}(h_{,R}F^1_1 - f_{,R}F^3_1),\\
A_2 &\equiv \Lambda\left[\frac{f}{R}(h_{,Z}F^1_{2,R} - h_{,R}F^1_{2,Z} - f_{,Z}F^3_{2,R} + f_{,R}F^3_{2,Z}) + \frac{1}{f}F^1_2\right]\\
(3.3)\qquad &\quad + \Lambda_{,R}\frac{f}{R}(h_{,Z}F^1_2 - f_{,Z}F^3_2) - \Lambda_Z\frac{f}{R}(h_{,R}F^1_2 - f_{,R}F^3_2),\\
A_3 &\equiv \Lambda\left[\frac{f}{R}(h_{,Z}F^1_{3,R} - h_{,R}F^1_{3,Z} - f_{,Z}F^3_{3,R} + f_{,R}F^3_{3,Z}) + \frac{1}{f}F^1_3\right]\\
&\quad + \Lambda_{,R}\frac{f}{R}(h_{,Z}F^1_3 - f_{,Z}F^3_3) - \Lambda_{,Z}\frac{f}{R}(h_{,R}F^1_3 - f_{,R}F^3_3).
\end{aligned}
$$

The equations of motion (3.2) together with (1.4) and (1.5) constitute the set of five differential equations with five unknown functions

$$(3.4)\qquad f, g, h; p, g$$

of three variables R, Z, t.

It is possible to eliminate the functions p, q from the Eqs. (3.2) and hence to obtain the third equation (together with (1.4), (1.5)) for the functions of motion f, g, h. Multiplying the Eqs. (3.2) by $f_{,Z}, g_{,Z}, h_{,Z}$, respectively, and summing, subsequently multiplying by $f_{,R}, g_{,R}, h_{,R}$ and summing, we obtain the following system of two equations:

$$
\begin{aligned}
(3.5)\qquad &p_{,Z} + q_{,Z} + f_{,Z}A_1 + g_{Z,}A_2 + h_{,Z}A_3 = 0,\\
&p_{,R} + \nu q_{,Z} + \nu_{,Z}q + f_{,R}A_1 + g_{,R}A_2 + h_{,R}A_3 = 0(^1),
\end{aligned}
$$

where

$$(3.6)\qquad \nu \equiv f_{,R}f_{,Z} + f^2 g_{,R}g_{Z,} + h_{,R}h_{,Z}.$$

Further, using the Eq. $(3.2)_2$, we eliminate from the above equations $q_{,Z}$. Differentiating the first with respect to R, the second with respect to Z and substracting, we arrive at the equation containing the derivative $q_{,R}$ and q, only. This equation, together with $(3.2)_1$, constitute the system of two linear equations for q:

$$(3.7)\qquad q_{,R} + \alpha_2 q + \beta_2 = 0, \quad q_{,Z} + \alpha_1 q + \beta_1 = 0,$$

(1) This equation has been derived assuming $g_{,R} \neq 0$ (we multiplied the second equation by $g_{,R}$). For $g_{,R} = 0$ it is still valid and can be obtained directly from the first and third equations.

where we use the notation:

$$\alpha_1 \equiv \frac{(f^2 g_{,Z})_{,Z}}{f^2 g_{,Z}}, \quad \beta_1 \equiv \frac{A_2}{f^2 g_{,Z}},$$

(3.8)
$$\alpha_2 \equiv \frac{1}{\alpha_1}(v_{,ZZ} - 2v_{,Z}\alpha_1 + \alpha_{1,R} - v\alpha_{1,Z} + v\alpha_1^2),$$

$$\beta_2 \equiv \frac{1}{\alpha_1}(-2v_{,Z}\beta_1 + \beta_{1,R} - v\beta_{1,Z} + v\alpha_1\beta_1 + f_{,R}A_{1,Z} - f_{,Z}A_{1,R} + g_{,R}A_{2,Z} - g_{,Z}A_{2,R} + h_{,R}A_{3,Z} - h_{,Z}A_{3,R}),$$

and we have assumed that $\alpha_1 \neq 0$.

The integrability condition for the Eqs. (3.11) has the form:

(3.9)
$$(\alpha_{1,R} - \alpha_{2,Z})q + (\beta_{1,R} - \beta_{2,Z} - \alpha_1\beta_2 + \alpha_2\beta_1) = 0.$$

The requirement that (3.9) be identity for each q leads to two equations (the vanishing of the two parentheses) for functions f, g, h. Together with (1.4), (1.5), we would have four equations for three unknown functions, which in general cannot be satisfied. Therefore, we do not require the Eq. (3.9) to be an identity, and we find from it q. This value q is the necessary condition of integrability of the Eqs. (3.1). The sufficient condition is obtained by substituting q from (3.9) into the Eqs. (3.7) and equating the results:

(3.10)
$$\left[\frac{\beta_{1,R} - \beta_{2,Z} - \alpha_1\beta_2 + \alpha_2\beta_1}{\alpha_{1,R} - \alpha_{2,Z}}\right]_{,R} - \left[\frac{\beta_{1,R} - \beta_{2,Z} - \alpha_1\beta_2 + \alpha_2\beta_1}{\alpha_{1,R} - \alpha_{2,Z}}\right]_{,Z} + (\alpha_2 - \alpha_1)\left[\frac{\beta_{1,R} - \beta_{2,Z} - \alpha_1\beta_2 + \alpha_2\beta_1}{\alpha_{1,R} - \alpha_{2,Z}}\right] + (\beta_1 - \beta_2) = 0.$$

The Eq. (3.10), together with (1.4), (1.5), constitute the system of three equations for three functions of motion f, g, h. If these equations are solved, the function q is determined by the Eq. (3.9) and p can be found from (3.2) or (3.5). However, this procedure is rather cumbersome because of the extremely complicated form of the Eq. (3.10).

4. A particular case

Now, we consider a particular case of motion (2.1), assuming that

(4.1)
$$g \equiv 0 \quad \text{and hence} \quad \varphi \equiv \Phi.$$

4.1. Kinematics

A motion is now described by two functions $f(R, Z, t)$ and $h(R, Z, t)$, which can be found from the system of equations:

(4.2)
$$f_{,R}h_{,Z} - h_{,R}f_{,Z} = \frac{R}{f},$$

(4.3)
$$(f_{,Z})^2 + (h_{,Z})^2 = 1.$$

We are seecking a solution of this system with the initial conditions (1.6) and the following boundary conditions:

$$\text{on } R = R_0: \quad f = \alpha(Z, t), \; h = \beta(Z, t), \tag{4.4}$$

$$\text{on } Z = Z_0: \quad f = \gamma(R, t), \; h = \delta(R, t). \tag{4.5}$$

Concerning the functions α, β, γ, δ, we assume that they are sufficiently smooth, satisfy the initial conditions (1.6), and are compatible with the system (4.2), (4.3). The latter implies the condition

$$(\alpha')^2+(\beta')^2 = 1. \tag{4.6}$$

Hereinafter, "prime" denotes the derivative with respect to Z.

Let us transform the system of Eqs. (4.2), (4.3) by means of a change of dependent variables into independent variables. The derivatives of functions f, g are expressed by the derivatives of the functions R, Z as follows:

$$\begin{aligned} f_{,R} &= \frac{Z_{,h}}{\Delta}, \quad f_{,Z} = -\frac{R_{,h}}{\Delta}, \\ h_{,R} &= -\frac{Z_{,f}}{\Delta}, \quad h_{,Z} = \frac{R_{,f}}{\Delta}, \end{aligned} \tag{4.7}$$

where

$$\Delta = R_{,f}Z_{,h}-Z_{,f}R_{,h}. \tag{4.8}$$

After substituting (4.7), (4.8) into (4.2), (4.3), we obtain:

$$U_{,f}Z_{,h}-U_{,h}Z_{,f} = f, \tag{4.9}$$

$$(U_{,f})^2+(U_{,h})^2 = f^2, \tag{4.10}$$

where

$$U \equiv \frac{1}{2}R^2. \tag{4.11}$$

Instead of the system of non-linear equations, we now have two separate equations, each for one unknown function.

The Eq. (4.10) can be solved by the method of characteristics (see e.g. [3]). Introducing the notation

$$p \equiv U_{,f}, \quad q \equiv U_{,h}(^2), \tag{4.12}$$

it can be written in the form:

$$p^2+q^2-f^2 = 0. \tag{4.13}$$

The characteristic system for the Eq. (4.13) has the form:

$$\frac{df}{ds} = 2p, \quad \frac{dh}{ds} = 2q, \quad \frac{dU}{ds} = 2f^2, \quad \frac{dp}{ds} = 2f, \quad \frac{dq}{ds} = 0. \tag{4.14}$$

(²) We allow the double meaning of letters p and q in view of the customary notations.

Integrating, we obtain:

$$f = \frac{1}{2}(f_0+p_0)e^{2s}+\frac{1}{2}(f_0-p_0)e^{-2s}, \quad h = 2q_0 s+h_0,$$

(4.15)
$$p = \frac{1}{2}(f_0+p_0)e^{2s}-\frac{1}{2}(f_0-p_0)e^{-2s}, \quad q = q_0,$$

$$U = \frac{1}{8}(f_0+p_0)^2 e^{4s}-\frac{1}{8}(f_0-p_0)^2 e^{-4s}+q_0^2 s-\frac{1}{2}f_0 p_0+U_0,$$

where f_0, h_0, p_0, q_0, U_0 are the values of f, h, p, q, U on the line $s = 0$; they can be determined on the basis of the boundary conditions (4.4). Thus we have:

(4.16)
$$f_0 = \alpha(\zeta, t), \quad h_0 = \beta(\zeta, t), \quad U_0 = \frac{1}{2}R_0^2.$$

(We have replaced the latter Z by ζ in order to avoid identification of the unknown function Z with the parameter along the line $R = R_0$). The derivatives p_0, q_0 are determined from the system:

(4.17)
$$p_0^2+q_0^2 = \alpha^2, \quad U_0' = p_0\alpha'+q_0\beta',$$

whence

(4.18)
$$p_0 = -\varepsilon\alpha\beta', \quad q_0 = \varepsilon\alpha\alpha',$$

where $\varepsilon = \pm 1$.

Substituting (4.16) and (4.18) into (4.15), we obtain:

(4.19)
$$f = f(s, \zeta; t) \equiv \frac{1}{2}\alpha(1-\varepsilon\beta')e^{2s}+\frac{1}{2}\alpha(1+\varepsilon\beta')e^{-2s},$$

(4.20)
$$h = h(s, \zeta; t) \equiv 2\varepsilon\alpha\alpha' s+\beta,$$

(4.21)
$$p = p(s, \zeta; t) \equiv \frac{1}{2}\alpha(1-\varepsilon\beta')e^{2s}-\frac{1}{2}\alpha(1+\varepsilon\beta')e^{-2s},$$

(4.22)
$$q = q(\zeta; t) \equiv \varepsilon\alpha\alpha',$$

(4.23)
$$U = U(s, \zeta; t) \equiv \frac{1}{8}\alpha^2(1-\varepsilon\beta')^2 e^{4s}-\frac{1}{8}\alpha^2(1+\varepsilon\beta')^2 e^{-4s} +(\alpha\alpha')^2 s+\frac{1}{2}\varepsilon\alpha^2\beta'+\frac{1}{2}R_0^2.$$

Assuming that the determinant

(4.24)
$$\Delta = f_{,s}h_{,\zeta}-h_{,s}f_{,\zeta} = 2ph_{,\zeta}-2qf_{,\zeta} \neq 0,$$

we obtain, from (4.19) and (4.20), $s = s(f, h; t)$, $\zeta = \zeta(f, h; t)$, and substituting into (4.23), we arrive at the solution:

(4.25)
$$U = U(f, h, t).$$

Let us draw attention to some limitations of the validity of our solution. It follows from the Eq. (4.10) that on the line $f = 0$ we have $p = q = 0$. Therefore, the assumption (4.24) is not satisfied, and we have a singularity on $f = 0$. In particular, the boundary value problem formulated on this line leads to the solution $f = R$. In the case $\alpha \neq 0$, we should expect restrictions on the region of validity of the solution of the type $f \geqslant f^* \neq 0$, where f^* depends on the boundary conditions. Furthermore, boundary conditions have to be so chosen that

$$U \equiv \frac{1}{2} R^2 \geqslant 0. \tag{4.26}$$

Hereinafter, we assume

$$\alpha \neq 0 \quad \text{and} \quad \alpha' \neq 0 \tag{4.27}$$

(for $\alpha' = 0$, we obtain the solution $R^2 - R_0^2 = f^2 - \alpha^2$) and from the Eq. (4.19) and (4.20) we obtain a relation determining $\zeta = \zeta(f, h; t)$:

$$\frac{h-\beta}{\varepsilon\alpha\alpha'} = \ln\left|\frac{f - \varepsilon\sqrt{f^2-(\alpha\alpha')^2}}{\alpha(1-\varepsilon\beta')}\right|, \tag{4.28}$$

and

$$s = s(f, h, t) = \frac{1}{2}\ln\left|\frac{f - \varepsilon\sqrt{f^2-(\alpha\alpha')^2}}{\alpha(1-\varepsilon\beta')}\right|. \tag{4.29}$$

The mentioned above restriction has the form:

$$f \geqslant \alpha\alpha'. \tag{4.30}$$

It remains to solve the linear equation (4.9) with the boundary condition (4.5). Now, the derivatives $U_{,f}$, $U_{,h}$ are known functions of f, h and t, and they depend on the boundary condition for the Eq. (4.10).

To conclude the paper, we present an example with numerical results. Let us consider the domain $R_1 \leqslant R \leqslant R_0$, $Z \geqslant 0$, $0 \leqslant t \leqslant t_1$ and the following boundary conditions:

$$\begin{aligned} &\text{on } R = R_0: \quad f = \alpha(Z, t) \equiv \sqrt{2uZ + h^2}, \\ &\qquad h = \beta(Z, t) \equiv \frac{1}{2a}\Big(\alpha\sqrt{\alpha^2-a^2} - b\sqrt{b^2-a^2} \\ &\qquad\qquad - a^2\ln|\alpha + \sqrt{\alpha^2-a^2}| + a^2\ln|b + \sqrt{b^2-a^2}|\Big), \\ &\text{on } Z = 0: \quad h = \delta(R, t) = 0, \end{aligned} \tag{4.31}$$

where a and b are functions of time satisfying the initial conditions (1.6); hence

$$a(0) = 0, \quad b(0) = R_0, \quad a'(0) \neq 0, \quad b'(0) = 0. \tag{4.32}$$

In (4.31), we assumed the form of the function $\alpha(Z, t)$, $\beta(Z, t)$ was found from (4.6), and the integration constant from the condition on the line $Z = 0$ $(4.31)_3$.

The Eq. (4.25) has the form:

$$\begin{aligned} U = {} & \frac{1}{2} f\sqrt{f^2-a^2} - \frac{1}{2}a^2\ln|f^2 + \sqrt{f^2-a^2}| - ah - \frac{1}{2}b\sqrt{b^2-a^2} \\ & + \frac{1}{2}a^2\ln|b + \sqrt{b^2-a^2}| + \frac{1}{2}R_0^2. \end{aligned} \tag{4.33}$$

Substituting $p = \sqrt{f^2-a^2}$, $q = -a$ into (4.9), we obtain:

$$\sqrt{f^2-a^2}Z_{,h}+aZ_{,f} = f, \tag{4.34}$$

which, after introducing the new dependent variable

$$V \equiv aZ-\frac{1}{2}f^2, \tag{4.35}$$

takes the form:

$$\sqrt{f^2-a^2}\,V_{,h}+aV_{,f} = 0. \tag{4.36}$$

This is an equation of the type $U_{,f}V_{,h}-V_{,f}U_{,h} = 0$ which has solution:

$$V = G^*(U, t), \tag{4.37}$$

where G^* is an arbitrary function of its arguments; therefore, we have

$$aZ = G^*(U, t)+\frac{1}{2}f^2. \tag{4.38}$$

Solving (4.33) and (4.38) with respect to f and h, we finally obtain:

$$f = \sqrt{2aZ+G^2(R, t)},$$
(4.39)
$$h = \frac{1}{2a}\left(f\sqrt{f^2-a^2}-a^2\ln|f+\sqrt{f^2-a^2}|-R^2-b\sqrt{b^2-a^2}+a^2\ln|b+\sqrt{b^2-a^2}|+R_0^2\right),$$

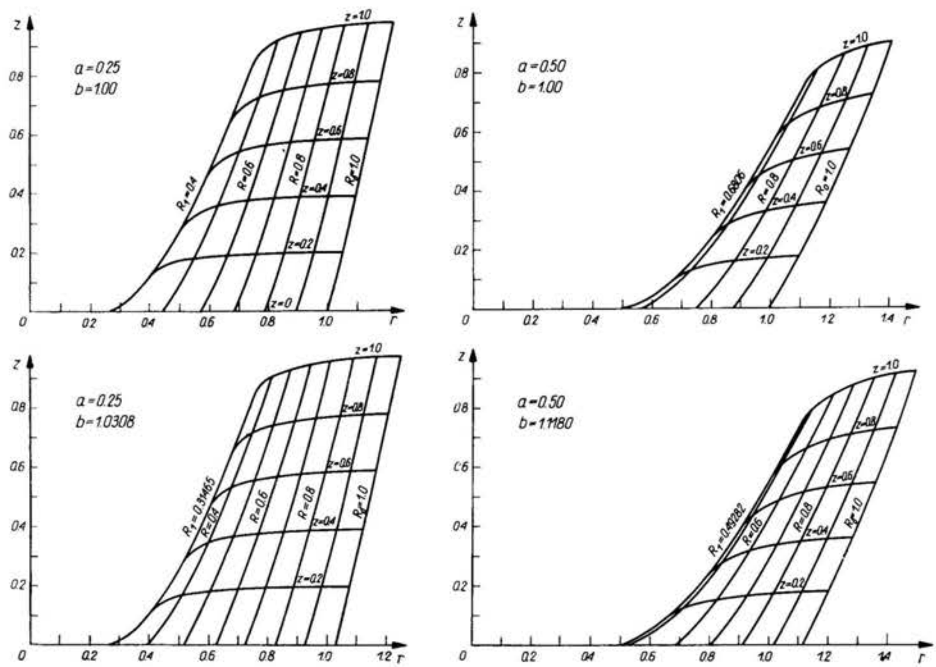

FIG. 1.

where the function $G(R, t)$ is determined from the condition $(4.31)_3$, which has the form:

$$G\sqrt{G^2-a^2}-a^2\ln|G+\sqrt{G^2-a^2}-R^2-b\sqrt{b^2-a^2}+a^2\ln|b+\sqrt{b^2-a^2}|+R_0^2 = 0. \tag{4.40}$$

It is evident that the function $f(R, Z, t)$ satisfies the initial condition $f(R, Z, 0)$: for $t = 0$ ($a = 0$, $b = R_0$), we have $G(R, 0) = R$, and the boundary condition $f(R_0, Z, t) = \sqrt{2aZ+b^2}$, because $G(R_0, t) = b$. The function $h(R, Z, t)$ for $t = 0$ has a singularity of the type 0/0, and it is easy to verify that $\lim\limits_{t\to 0} h(R, Z, t) = Z$.

The above solution is restricted by (4.26) and (4.30). These should be regarded as restrictions on R_1 and t_1; the geometric meaning of these restrictions is as follows: the lines $Z = \text{const}$ become tangent to the line $R = R_1$.

Numerical results are shown in Fig. 1. These are curves (4.39) calculated for the following values of the parameters:

$$R_0 = 1,$$
$$a = 0.25, \quad a = 0.5,$$
$$b = \text{const} = 1, \quad b = \sqrt{1+a^2}.$$

4.2. Statics

The relations (2.13) with (2.14) and (2.10), (2.11) determine $\sigma_{ij} = \sigma_{ij}(d_{kl})$ to within two arbitrary functions p and q. Taking into account the assumption (4.1), we obtain:

$$\sigma_2^1 = \sigma_2^3 = 0$$

and the remaining components by setting $g \equiv 0$.

The second equation of motion is, as expected, identicaly satisfied. The two remaining equations of motion constitute a linear system for the functions p and q. Using (3.5), we have:

$$(p+q)_{,Z}+f_{,Z}A_1+h_{,Z}A_3 = 0,$$
$$p_{,R}+\xi q_{,Z}+\xi_{,Z}q+f_{,R}A_1+h_{,R}A_3 = 0,$$

where ξ, A_1, A_3 are defined by (2.12) and (3.3).

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH

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