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BRIEF NOTES

The reduced stress intensity factor in the theory of cracks

Z. OLESIAK (WARSZAWA)

By means of the Huber-Mises-Hencky condition the reduced stress intensity factor has been introduced. It is shown that in the case of plane stress state the stress intensity factor has exactly the same value, while for the plane state of strain the reduced intensity factor depends on the Poisson ratio.

In an early paper M. T. Huber (1904) derived the expression for the elastic shear strain energy and proposed to assume it as the strength criterion. The appearence of the permanent, plastic deformations was then identified with the criterion of the crack destruction, i.e. the cohesion criterion. Since then the expression for the shear strain energy has been serving as the basis of a simple, still well justified, and sufficiently well supported by experiments in most of engineering cases, theory of material strength.

The same expression was used by R. Mises (1913) and by H. Hencky (1924) who applied it as the yield condition, or the strain energy plasticity condition. The approach towards the theories of strength has been changing, particularly in the last two decades, considerably. The initiation of a plastic zone in a solid body is, generally speaking, no longer identified with the crack generation. Besides, the existence of cracks does not necessarily mean the crack destruction. The scientists went deeper into the physical significance of the strength theories. Not only have been created numerous new theories and hypotheses, which take into account many effects and influences, not always justified by the experiments, but also new concepts have been introduced. However, the expression derived first by M. T. Huber in (1904) is still in the popular use in the theory of plasticity and is known as the Huber-Mises-Hencky (H-M-H) yield condition, in spite of a tremendous development of the theory of plasticity and the theoretical concepts. It is noteworthy that a great many of generalized theories take as the point of departure the equation derived by M. T. Huber.

According to the recent theories the existence of a crack in a stress field may cause the generation of a plastic deformation field in the neighbourhood of the crack tip. The determination of the plastic deformation zones is usually not an easy task, on the othe hand the calculated plastic zones sometimes do not agree with the experimental data. For example, in the case of the plane stress D. S. DUGDALE (1960) found, experimenting on steel sheets, that the plastic zone surrounding the crack occupies a very thin region. This fact was earlier taken into account by M. YA. LEONOV and V. V. PANASYUK (1959). Here, new problems arise, namely the dimension effect, the correspondence of the calculated plastic zone to that found from the experiment, etc.

In the development of the crack theory the notion of the stress intensity factor has been introduced. For each mode of the displacement there exists a corresponding stress intensity factor: for the plane state of stress and plane state of strain problems, antiplane and axi-symmetric problems we have six different coefficients (see I. N. SNEDDON, 1969).

Beside the stress intensity factors, at least in the problems in which the determination of the plastic zone is important, the reduced stress should play an important role and has a definite physical meaning. So far it was assumed that within the plastic zone (in Dugdale's approach) surrounding the crack of length $2a$ a one-dimensional, uniform state of stress existed, and for example for the mode I the yield condition was equated to a single stress component $\sigma_{yy}(x, 0)$.

The reason why authors were assuming that $\sigma_{yy}(x, 0)$ only was equal to the yield stress had probably the mathematical grounds and consisted in the simplest extension of the method used in the theory of elasticity approach to the elastic-plastic case. Namely using for example the Fourier transform technique, the purely elastic problem may be reduced to that of solving the dual integral equations for a single unknown function

$$\begin{aligned} \sigma_{yy}(x, 0) &= -p(x), \quad && 0 \leqslant x < a, \\ v(x, 0) &= 0, && x > a. \end{aligned} \tag{1.1}$$

Here

$$\begin{aligned} \sigma_{yy}(x, 0) &= -\mathscr{F}_c[\psi(\xi); \xi \to x], \\ 2Gv(x, 0) &= \mathscr{F}_c[\xi^{-1}(2-2\nu)\psi(\xi); \xi \to x], \end{aligned} \tag{1.2}$$

while

$$\begin{aligned} \sigma_{xx}(x, 0) &= -\mathscr{F}_c[\psi(\xi); \xi \to x], \\ \sigma_{xy}(x, 0) &\equiv 0; \end{aligned} \tag{1.3}$$

where $\mathscr{F}_c[\psi; \xi \to x] \equiv \sqrt{\frac{2}{\pi}} \int\limits_0^\infty \psi(\xi) \cos \xi x \, d\xi$, $\psi(\xi)$ denotes the required function, ν is the Poisson ratio. At this point the stress intensity factor is introduced and defined by the equation (for the mode I):

$$K_0 = \lim_{x \to a^+} \sqrt{2(x-a)}\, \sigma_{yy}(x, 0). \tag{1.4}$$

If we look at Eqs. (1.2) and (1.3) we see that both $\sigma_{yy}(x, y)$ and $\sigma_{xx}(x, y)$ are represented by exactly the same equation for $y \to 0$. This is so within the frames of the classical theory of elasticity. Moreover, the plane strain state differs from that of plane stress only by a constant.

Now, let us introduce the reduced stress intensity factor given by the equation resulting from the Huber-Mises-Hencky condition

$$\frac{1}{2} s_{ij} s_{ij} \equiv T^2 = Y^2, \tag{1.5}$$

where T is the strain energy, Y is the yield limit, and

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\delta_{ij}\sigma_{kk}.$$

For the two-dimensional case, and any mode, we define the reduced stress intensity factor by means of the equation

$$K_r = \lim_{x \to a^+} [\alpha(x-a)T(x, 0]^{1/2}; \tag{1.6}$$

where α is a constant. For the state of plane stress we have (mode I)

$$K_{r\sigma} = \lim_{x \to a^+} \sqrt{\alpha(x-a)}\sigma_{yy}(x, 0), \tag{1.7}$$

in this special case $\sqrt{2T} = \sigma_{yy}(x, 0)$, $\alpha = 2$, and $K_{r\sigma} = K_0$. For two-dimensional state of strain, however, there exists the stress component $\sigma_{zz}(x, 0) \neq 0$ which can be calculated from Hooke's relation

$$\sigma_{zz}(x, y) = \nu[\sigma_{xx}(x, y) + \sigma_{yy}(x, y)]; \tag{1.8}$$

now $\alpha = 2(1-2\nu)$ and the reduced stress intensity factor differs from the stress intensity factor by a constant, namely

$$K_{r\varepsilon} = \lim_{x \to a^+} \sqrt{\alpha(x-a)}\sigma_{yy}(x, 0) = \sqrt{1-2\nu}K_0. \tag{1.9}$$

If we take into account a thin plastic zone in the crack surface then the boundary conditions, from which the unknown function ψ is determined, are usually assumed to have the following form

$$\sigma_{yy}(x, 0) = \begin{cases} -p(x), & 0 < x < b, \\ Y(x) = Y_0, & b < x < a, \quad b > a, \end{cases} \tag{1.10}$$
$$v(x, 0) = 0, \quad x > a;$$

here $p(x)$ is the pressure on the crack surface, $2b$ denotes the crack length, $b-a$ denotes the width of the plastic zone. Since usually in both the cases, i.e., two-dimensional state of strain and two-dimensional state of stress, the discussion is based on the notion of the stress intensity factor, there is no difference in the computed width of crack. On the other hand, if we have assumed that the strain energy (H-M-H condition) is responsible for the attainment of the yield, we would obtain two different cases. For the two-dimensional state of stress the boundary condition remains unchanged, this is so because $K_{r\sigma} = K_0$. Though for the two-dimensional state of strain the plastic zone is determined from the same condition, the reduced stress intensity factor $K_{r\varepsilon}$ is no longer equal to K_0 and the first boundary condition (1.10) will take the form

$$\sigma_{yy}(x, 0) = \begin{cases} -p(x), & 0 < x < b, \\ (1-2\nu)^{-\frac{1}{2}} Y_0, & b < x < a. \end{cases} \tag{1.11}$$

This shows that the formal extension of Dugdale's approach, based on the experiments on thin sheets of steel, to the two-dimensional state of strain is not justified also theoretically.

Similar results can be obtained in the case of axial symmetry, then the reduced stress intensity factor differs from the intensity factor, besides the width of plastic zone, calculated from the reduced stress intensity factor varies from that found when it is assumed that $\sigma_{zz}(r, 0) = Y_0$ for the plastic zone $b < r < a$.

References

1. M. T. HUBER *Właściwa praca odkształcenia jako miara wytężenia*, Czasopismo Techniczne, Lwów 1904, *Zur Theorie der Berührung fester elastischer Körper*, Leipzig 1905, *Kryteria wytrzymałościowe w stereomechanice technicznej*, Inst. Wyd. SIMP, Warszawa 1948.
2. R. VON MISES, *Mechanik der fester Körper im plastisch deformablen Zustand*, Göttingen Nachrichten, 582–592, 1913.
3. H. HENCKY, *Zur Theorie plastischer Deformationen und der hierdurch im Material hervorgerufenen Nachspannungen*, Zeit. angew. Math. Mech., 4, 323–334, April 1924.
4. I. N. SNEDDON, *Crack problems in the classical theory of elasticity*, John Wiley & Sons, Inc., 1969.
5. Z. OLESIAK, M. WNUK, *Plastic energy dissipation due to a penny shaped crack*, Int. J. Fract. Mech, 4, 4, 1968.
6. D. S. DUGDALE, *Yielding of steel sheets containing slits*, J. Mech. Phys. Solids, May 1960.
7. M. YA. LEONOV, V. V. PANASYUK, Prikladna Mekhanika, 391–401, 1959.

INSTITUTE OF MECHANICS
UNIVERSITY OF WARSAW.

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