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Insensitivity of materials to the exchange of deformation paths

Part I. Insensitivity semigroups

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THE MAIN object of theory of materials (theory of constitutive equations) is to direct and to arrange the mechanical macroexperiment. In this paper a definition of the insensitivity semigroup of a material is given. It is a set of such all mappings of deformation paths, which do not influence the final value of stress. The algorithm for deriving the general constitutive operator insensitive with respect to prescribed semigroup is given.

Podstawowym celem teorii materiałów (teorii równań konstytutywnych) jest ukierunkowywanie i porządkowanie makroeksperymentu mechanicznego. W pracy podano definicję półgrupy niewrażliwości materiału. Jest to zbiór tych wszystkich przekształceń dróg odkształecnia, które nie wpływają na końcową wartość naprężenia. Podano algorytm znajdowania postaci ogólnej operatora konstytutywnego niezmienniczego względem z góry danej półgrupy.

Основной целью теории материалов (теории определяющих уравнений) является направливание и упорядочивание механического макроопыта. В работе дано определение полугруппы нечувствительности материала. Это множество всех тех преобразований путей деформирования, которые не влияют на конечное значение напряжений. Дан алгорифм построения определяющих операторов инвариантных относительно заданной полугруппы.

Contents. Part I

1. Introduction

THE NOTION of *a material* is a fundamental concept in the theory of continuum. In mechanics of continuous bodies, by a material we mean an operator called *a constitutive operator*, which assings a stress state to motion of a body. *The theory of materials* deals with properties and classification of such operators based on mechanical macroexperiments and general principles of physics. In recent decades, this theory has been developing intensively: we need mention but a few of numerous papers, ILYUSHIN [1, 2], NOLL [3, 4], RIVLIN and ERIKSEN [5], and SEDOV [6].

A demand for the theory of materials arises from a turbulent increase of various substances in contemporary technology as well as from the constantly increasing requirements concerning the more and more exact description of deformation and flow processes.

A fundamental aim of the theory of materials is the ordering and guiding a mechanical macroexperiment. This aim is attained by individualization of typical classes of materials and typical classes of processes. In the present paper, we wish to consider some further unexplored possibilities in this direction, relying on a more exact analysis of the invariance of constitutive operators.

In Part I of this paper a fundamental definition for the entire work, of *an insensitivity semigroup of material* is given.

That semigroup consists of all those mappings of deformation paths which preserve unchanged the final value of stresses. An algorithm is presented for deriving a general form of a constitutive operator, invariant with respect to a semigroup, given *a priori*.

One of the most important particular cases — namely that introduced by Noll, called isotropy groups of materials — is quoted herein.

Part II will propose a precise definition of a concept commonly used in mechanics and known as material *viscosity*.

2. Definition of an insensitivity semigroup of material

To focus our attention, we confine all considerations to purely mechanical theory of simple materials, which constitutes the nuckleus of the theory of materials and possesses most applications. We shall, herein, use a description based on a fixed reference configuration. All ideas of the present paper can be carried over to theories not purely mechanical, to non-simple materials, and even to non-local materials or to materials with microstructure. They can also be stated on the basis of a description without any reference configuration

Thus the starting point is a constitutive equation (cf. [7], Sec. 28):

$$\mathbf{T}(X, t) = \mathfrak{F}_{\mathbf{K}}[\mathbf{F}(X, t-s)]. \tag{2.1}$$

Here $\mathbf{T}(X, t)$ is the stress tensor evaluated at the particle X at the instant t, $\mathbf{F}(X, t-s)$ is the deformation gradient from a fixed local configuration $\mathbf{K}$ into a local configuration at the instant $t-s$, at the particle X under consideration, $s \geqslant 0$ is the time measured backwards from the instant $t \in (-\infty, +\infty)$. We assume that the stress unit E_0 and the time unit t_0 have been chosen, and we agree to understand by $\mathbf{T}$, t, s dimensionless quantities referred to them. Then we omit the explicit appearence of the symbol X for a fixed particle. We are also assuming, up to § 18, that the time t is fixed and, as a rule, we omit its appearence in all formulas.

Denote by $\mathscr{T}$ a set of tensors of the second-order, and by $\mathscr{N} \subset \mathscr{T}$ a subset of tensors whose determinant is positive. By R we mean the set of all non-negative real numbers, $\mathrm{R} \equiv [0, \infty)$. The set of non-singular tensor curves, parametrized by means of the dimensionless time s running over R, we denote by $\mathscr{A}$,

$$\mathscr{A} \equiv \{f\colon \mathrm{R} \to \mathscr{N} \mid \mathrm{Dom} f = \mathrm{R}\}. \tag{2.2}$$

Curves from $\mathscr{A}$ are called deformation processes or *deformation paths*. In accordance with the convention assumed, we write:

$$\mathbf{T} \equiv \mathbf{T}(X, t), \quad f(s) \equiv \mathbf{F}(X, t-s).$$

Thus, for a fixed time t, we consider a constitutive operator as the operator $\mathfrak{F}_{\mathbf{K}}: \mathscr{A} \to \mathscr{T}$, and we write:

$$\mathbf{T} = \mathfrak{F}_{\mathbf{K}}[f]. \tag{2.3}$$

The constitutive operator depends on the reference configuration according to the formula

$$\mathfrak{F}_{\mathbf{PK}}[f] = \mathfrak{F}_{\mathbf{K}}[f\mathbf{P}], \tag{2.4}$$

where $\mathbf{P} \in \mathscr{N}$ denotes local deformation from a local configuration $\mathbf{K}$ into the local configuration $\mathbf{PK}$, (cf. [7]). We know, moreover, that for every orthogonal tensor curve $\mathbf{Q} \in \mathscr{A}$ — i.e., one that satisfies the condition $\mathbf{Q}^T(s)\mathbf{Q}(s) = \mathbf{1}$ for each $s \in \mathrm{R}$ — the following formula

$$\mathfrak{F}_{\mathbf{K}}[\mathbf{Q}f] = \mathbf{Q}(0)\, \mathfrak{F}_{\mathbf{K}}[f]\mathbf{Q}^T(0) \tag{2.5}$$

holds true (principle of material objectivity, [7]). In the foregoing formulae, $f\mathbf{P}$, $\mathbf{Q}f$ denoted the product of tensors for each $s \in \mathrm{R}$.

In studying mechanics, insufficient attention is paid as a rule to defining a domain of operators with sufficient rigor, particularly the case with constitutive operators. In every concrete case, any information concerning the domain $\mathrm{Dom}\, \mathfrak{F}_{\mathbf{K}} \subset \mathscr{A}$ — i.e., the set of permissible deformation paths — is essential information about the material under consideration. In what follows, and particularly in Part II, assumptions on the domain of an operator will play an essential role. For the time being, let us cite the following important properties:

(i) for every $f \in \mathscr{A}$ and every number $\hat{s} \in \mathrm{R}$,

$$f \in \mathrm{Dom}\, \mathfrak{F}_{\mathbf{K}} \;\Rightarrow\; f_{\hat{s}} \in \mathrm{Dom}\, \mathfrak{F}_{\mathbf{K}}, \tag{2.6}$$

where $f_{\hat{s}}(s) \equiv f(\hat{s}+s)$, $s \in \mathrm{R}$;

(ii) for every $f \in \mathscr{A}$ and every $\mathbf{P} \in \mathscr{N}$,

$$f \in \mathrm{Dom}\, \mathfrak{F}_{\mathbf{K}} \;\Rightarrow\; f\mathbf{P} \in \mathrm{Dom}\, \mathfrak{F}_{\mathbf{K}}, \tag{2.7}$$

(iii) for every $f \in \mathscr{A}$ and every orthogonal curve $\mathbf{Q} \in \mathscr{A}$,

$$f \in \mathrm{Dom}\, \mathfrak{F}_{\mathbf{K}} \;\Rightarrow\; \mathbf{Q}f \in \mathrm{Dom}\, \mathfrak{F}_{\mathbf{K}}. \tag{2.8}$$

Property (i) follows from the manner in which we introduced $\mathfrak{F}_{\mathbf{K}}: \mathscr{A} \to \mathscr{T}$, since formula (2.1) holds for each $t \in (-\infty, +\infty)$; property (ii) is implicitly assumed in (2.4), and property (iii) in (2.5).

Conditions (2.4), (2.5) are, it seems, the only conditions which must, in purely mechanical theory, satisfy *a priori*, every constitutive operator. If we postulate further properties of the operator, then we are led to individualization of classes of simple materials. Studies concerning continuity of $\mathfrak{F}_{\mathbf{K}}$ by taking into account (2.5) and a suitable choice of topology in the set of deformation paths, lead, for instance, to materials with fading memory ([8, 9, 10, 11], and others). Considering the invariance of $\mathfrak{F}_{\mathbf{K}}$ with respect to exchanges of local reference configuration, NOLL [4, 7] achieved a fundamental classification of simple materials. The present paper is devoted to a broader conception of invariance of constitutive operators.

Consider all the following physical situations:

$$f \neq g \quad \text{and} \quad \mathfrak{F}_K[f] = \mathfrak{F}_K[g]. \tag{2.9}$$

We shall say that a material *does not feel any difference between two distinct deformation paths.* Let us introduce an equivalence relation $\mathscr{C} \subset \operatorname{Dom} \mathfrak{F}_K \times \operatorname{Dom} \mathfrak{F}_K$, defined by:

$$f \mathscr{C} g \quad \Leftrightarrow \quad \mathfrak{F}_K[f] = \mathfrak{F}_K[g]. \tag{2.10}$$

This equivalence relation decomposes a domain of an operator into a set of cosets $\operatorname{Dom} \mathfrak{F}_K/\mathscr{C}$, which are *level sets* for $\mathfrak{F}_K$, [12].

In elaborating and presenting this work, we were guided by the following conviction: *assumptions about level sets of a constitutive operator constitute the primitive and most fundamental information concerning a material.* The individualization of distinct classes of materials should begin with a description of level sets — i.e., sets of such, differing from each other, deformation paths which lead to the same final value of stresses.

Information about level sets is usually hidden in the information about invariance of an operator with respect to certain operations. For instance, formula (2.5) states that, if there exists an orthogonal curve $\mathbf{Q} \in \mathscr{A}$ such that $\mathbf{Q}(0) = \mathbf{1}$ and $f = \mathbf{Q}g$, then f, g belong to the same level set. This is a universal property of all constitutive operators; of course, we shall be interested in more special properties. With a view to discovering the nature of level sets in experiments, we proceed, in general, as follows. Choose a process f, say a one-dimensional tension test with constant velocity, or a class of processes $\mathscr{D}$, say cyclic tests. Further, we define a class Λ of operations of exchange processes — for instance, accelaration or retardation, the change of frequency and amplitude of cycles, superposition of prestrain, change of principal directions of deformations, and so forth. Taking a process f and an operation γ, we arrive at a new process; denote it by the symbol $f * \gamma$. We compare the response of the material on f as well as on $f * \gamma$.

To precise this idea, we introduce the following set of mappings:

$$\Gamma \equiv \{\gamma : \mathscr{A} \to \mathscr{A} \mid \operatorname{Dom} \gamma = \mathscr{A}\}. \tag{2.11}$$

A composition of mappings $\gamma, \delta \in \Gamma$ is a mapping $\gamma \circ \delta \in \Gamma$. Of course, $(\gamma \circ \delta) \circ \varepsilon = \gamma \circ (\delta \circ \varepsilon)$. Thus Γ is a semigroup [13]. The result of the operation $\gamma \in \Gamma$ on $f \in \mathscr{A}$ we agreed to denote by $f * \gamma \in \mathscr{A}$, $f * (\gamma \circ \delta) = (f * \gamma) * \delta$.

As a matter of fact, we are always dealing with subclasses of deformation paths. Firstly, the domain of the constitutive operator is always a subset of $\mathscr{A}$. In fact, the structure itself of $\mathfrak{F}_K$ can restrict the domain of its action — for instance, in the case of a viscous Newtonian fluid one may accept only continuous and differentiable deformation paths. However, even when the formal domain of the operator $\mathfrak{F}_K$ is the entire set $\mathscr{A}$ — as is the case for elastic materials — in reality it is reasonable to consider it on a certain class of deformation paths only. This class can be defined *a priori* by conditions of the type which restrict the magnitude of deformations, the magnitude of stretching, and so on, or, *a posteriori* by conditions of the type restricting the mean pressure (positive pressure in fluid), the stress deviator modulus (Huber-Mises yield condition), their combinations (models of soil media), and so forth. Secondly, in numerous situations we are interested only in a certain class of deformation paths — for instance, plane flow. Finally, we make experiments on subclasses $\mathscr{D} \subset \mathscr{A}$ only.

Let $\mathscr{D} \subset \mathscr{A}$ and $\mathscr{D}*\gamma \equiv \{f \in \mathscr{A} | f = g*\gamma$ for a certain $g \in \mathscr{D}\}$. In addition to Γ, we introduce

$$\Gamma_{\mathscr{D}} \equiv \{\gamma \in \Gamma | \ \mathscr{D}*\gamma \subset \mathscr{D}\}. \tag{2.12}$$

It is obvious that this is a subsemigroup in Γ.

We can now introduce the main object of this work.

DEFINITION 1. ***An insensitivity semigroup of a material*** *relative to the reference configuration* $\mathbf{K}$, *a class of deformation paths* $\mathscr{D} \subset \mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}$ *and a subsemigroup of the exchange of deformation paths* $\Lambda \subset \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}}$ *is a semigroup*:

$$\Omega(\mathbf{K}, \mathscr{D}, \Lambda) \equiv \{\gamma \in \Lambda | \mathfrak{F}_{\mathbf{K}}[f*\gamma] = \mathfrak{F}_{\mathbf{K}}[f] \quad \text{for every } f \in \mathscr{D}\}. \tag{2.13}$$

Then the condition $\Gamma_{\mathscr{D}} \subset \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}}$ *is satisfied by* $\mathscr{D}$ *a semigroup*

$$\Omega_{\mathbf{K},\mathscr{D}} \equiv \Omega(\mathbf{K}, \mathscr{D}, \Gamma_{\mathscr{D}}) \tag{2.14}$$

is called an insensitivity semigroup of a material relative to the configuration $\mathbf{K}$ *and the class* $\mathscr{D}$. A *semigroup*

$$\Omega_{\mathbf{K}} \equiv \Omega_{\mathbf{K},\,\mathrm{Dom}\,\mathfrak{F}_{K}} \tag{2.15}$$

is called a ***specific*** *insensitivity semigroup of a material relative to the configuration* $\mathbf{K}$.

The name introduced is correct, since for every $\gamma, \delta \in \Omega(\mathbf{K}, \mathscr{D}, \Lambda)$ we have $\gamma \circ \delta \in \in \Omega(\mathbf{K}, \mathscr{D}, \Lambda)$.

Every statement about insensitivity semigroups is a certain implicit statement concerning level sets. Notice that for every $f \in \mathrm{Dom}\ \mathfrak{F}_{\mathbf{K}}$

$$f*\Omega(\mathbf{K}, f, \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}}) \subset \mathrm{Dom}\ \mathfrak{F}_{\mathbf{K}} \tag{2.16}$$

is a level set represented by the deformation path f. In fact, the semigroup $\Gamma_{\mathrm{Dom}\,\mathfrak{F}_{K}}$, by definition, is transitive on $\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}$ — i.e., for every $f, g \in \mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}$ there exists $\gamma \in \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}}$ such that $g = f*\gamma$; if γ runs over $\Omega(\mathbf{K}, f, \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}})$, then $f*\gamma$ runs over the whole level set containing f.

In what follows, we shall be interested mainly in insensitivity semigroups of the type $\Omega_{\mathbf{K},\mathscr{D}}$. Every such semigroup contains identity on $\mathscr{D}$. The condition which defines $\Omega_{\mathbf{K},\mathscr{D}}$ can be stated somewhat differently. Denote by M the set of all constitutive operators with a common domain $\mathrm{Dom}\ \mathfrak{F}_{\mathbf{K}}$. Every mapping $\gamma \in \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}}$ generates a new mapping

$$\mathfrak{F}_{\mathbf{K}} \to \mathfrak{F}_{\mathbf{K}} \circ \gamma, \tag{2.17}$$

where, of course, $(\mathfrak{F}_{\mathbf{K}} \circ \gamma)[f] \equiv \mathfrak{F}_{\mathbf{K}}[f*\gamma]$. Thus a condition in (2.13) for $\Omega_{\mathbf{K},\mathscr{D}}$ can be stated in the form;

$$\mathfrak{F}_{\mathbf{K}} \circ \gamma = \mathfrak{F}_{\mathbf{K}} \quad \text{on} \quad \mathscr{D}. \tag{2.18}$$

We see that the operator $\mathfrak{F}_{\mathbf{K}}$ is to be a fixed point in M with respect to the semigroup $\Omega_{\mathbf{K},\mathscr{D}}$. In other words: $\mathfrak{F}_{\mathbf{K}}$ is to be *invariant* with respect to $\gamma \in \Omega_{\mathbf{K},\mathscr{D}}$.

The function Ω is defined on the triple Cartesian product of the family of subsemigroups of the semigroup Γ, of the family of all subsets of the domain of the operator $\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}$, and of the set of all local configurations. We may study properties of Ω from the view-point of algebraic structure of those sets. We confine our attention to one theorem only.

Let $\mathbf{P}$ be a non-singular tensor which transforms a local configuration $\mathbf{K}$ into a local configuration $\mathbf{PK}$. Let us introduce the following notation:

$\mathbf{P}\Lambda\mathbf{P}^{-1} \equiv \{\delta \in \Gamma |$ there exists a $\gamma \in \Lambda$, such that $f*\delta = [(f\mathbf{P})*\gamma]\mathbf{P}^{-1}$ for every $f \in \mathscr{A}\}$,

$\mathscr{D}\mathbf{P}^{-1} \equiv \{g \in \mathscr{A} |$ there exists a $f \in \mathscr{D}$, such that $g = f\mathbf{P}^{-1}\}$.

THEOREM 1

$$\Omega(\mathbf{PK}, \mathscr{D}\mathbf{P}^{-1}, \mathbf{P}\Lambda\mathbf{P}^{-1}) = \mathbf{P}\Omega(\mathbf{K}, \mathscr{D}, \Lambda)\mathbf{P}^{-1}, \tag{2.19}$$

in particular,

$$\Omega_{\mathbf{KP}} = \mathbf{P}\Omega_{\mathbf{K}}\mathbf{P}^{-1}. \tag{2.19'}$$

$$\Lambda_1 \subset \Lambda_2 \;\Rightarrow\; \Omega(\mathbf{K}, \mathscr{D}, \Lambda_1) \subset \Omega(\mathbf{K}, \mathscr{D}, \Lambda_2), \tag{2.20}$$

$$\mathscr{D}_1 \subset \mathscr{D}_2 \;\Rightarrow\; \Omega(\mathbf{K}, \mathscr{D}_1, \Lambda) \supset \Omega(\mathbf{K}, \mathscr{D}_2, \Lambda). \tag{2.21}$$

P r o o f. The first formula follows from the identity

$$\mathfrak{F}_{\mathbf{PK}}[f\mathbf{P}^{-1}] \equiv \mathfrak{F}_{\mathbf{K}}[f] = \mathfrak{F}_{\mathbf{K}}[f*\gamma] \equiv \mathfrak{F}_{\mathbf{PK}}[(f*\gamma)\mathbf{P}^{-1}] \equiv \mathfrak{F}_{\mathbf{PK}}[(((f\mathbf{P}^{-1}))*\gamma)\mathbf{P}^{-1}], \tag{2.22}$$

for every $f \in \mathscr{D}$ and every $\gamma \in \Omega(\mathbf{K}, \mathscr{D}, \Lambda)$. The second formula is a particular case of the first one, since due to (2.7)

$$(\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}})\mathbf{P}^{-1} = \mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}, \quad \mathbf{P}\Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}}\mathbf{P}^{-1} = \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}}.$$

The third and fourth formulae are obvious. Q.E.D.

If all operations of the semigroup Λ commute with the right-hand side tensor contraction, and if $\mathscr{D}$ is stable with respect to $\mathbf{P}$ — i.e., when

$$(f*\gamma)\mathbf{P} = (f\mathbf{P})*\gamma, \quad \mathscr{D}\mathbf{P}^{-1} = \mathscr{D} \tag{2.23}$$

— then the insensitivity semigroup relative to Λ is independent of the reference configuration

$$\Omega(\mathbf{PK}, \mathscr{D}, \Lambda) = \Omega(\mathbf{K}, \mathscr{D}, \Lambda). \tag{2.19''}$$

The specific insensitivity semigroup $\Omega_{\mathbf{K}}$ is always nonempty. What is more, it is even too rich, since it contains previously mentioned operations of the left-hand side contraction with orthogonal curves $\mathbf{Q} \in \mathscr{A}$, which give no information about a material. That can be avoided in the way described in paragraph 4. There arise two questions:

1. Do there exist materials with the trivial insensitivity specific semigroup (i.e., composed of operations of multiplication by $\mathbf{Q} \in \mathscr{A}$, $\mathbf{Q}(0) = \mathbf{1}$)?

2. Do there exist materials with the maximal specific insensitivity semigroup (i.e., $\Omega_{\mathbf{K}} = \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}}$)?

THEOREM 2. *If there exists a configuration* $\mathbf{K}$ *such that* $\Omega_{\mathbf{K}} = \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}}$, *then* $\Omega_{\mathbf{L}} = \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{L}}}$ *for every configuration* $\mathbf{L}$. *This takes place iff (if and only if).*

$$\mathbf{T} = \alpha\mathbf{1}, \quad \alpha = \alpha(X) \tag{2.24}$$

for all $\mathbf{K}$ *and all* $f \in \mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}$.

Proof. The first statement follows from (2.19) and (2.7). Necessity (2.24) is proved as follows. If $\Omega_{\mathbf{K}} = \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}}$, then the domain $\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}$ is the single level set, since $\Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}}$ acts inside it transitively. Thus $\mathfrak{F}_{\mathbf{K}}[f] = \mathbf{G} = \text{const}$. From the principle of material objectivity (2.5), we obtain $\mathbf{G} = \mathbf{R}\mathbf{G}\mathbf{R}^T$ for every orthogonal tensor $\mathbf{R}$, and this means that $\mathbf{G} = \alpha\mathbf{1}$. Q.E.D.

A hypothetic, completely insensitive material (2.24) would be a fluid which would not transfer shear stresses, and could exist only for strictly defined (at a particle X) pressure value. This material is valued by specialists dealing with closed crack, they prefer to fill up these cracks with this material rather than real fluids or gases.

Remark 1. On programming experiments and analysing their results, very often not the operator $\mathfrak{F}_{\mathbf{K}}$ itself is of interest to us but rather different operators generated by it — for instance,

$$\operatorname{tr}\mathfrak{F}_{\mathbf{K}},\ \operatorname{dev}\mathfrak{F}_{\mathbf{K}},\ \text{etc.} \tag{2.25}$$

Here, tr denotes the operation of taking the trace, and dev stands for the operation of taking the deviator of a symmetric tensor.

For anisotropic materials, we may be interested in operations of the type

$$K[f] \equiv \varphi(\mathfrak{F}_{\mathbf{K}}[f], \mathbf{L}_1, \ldots, \mathbf{L}_M), \tag{2.26}$$

where φ is the orthogonal invariant of the arguments indicated, and $\mathbf{L}_1, \ldots, \mathbf{L}_M$ is a system of tensors that define the group of material symmetry [14]. For instance, for transversally isotropic material $\mathbf{L}_1 = \mathbf{a} \otimes \mathbf{a}$, where $\mathbf{a}$ is the versor of the symmetry axis, and we may be interested in stress in the direction $\mathbf{a}$

$$\mathbf{a}\,\mathfrak{F}_{\mathbf{K}}[f]\mathbf{a}. \tag{2.27}$$

As an example of a more complicated operator may serve stress work done along the deformation path f. It is a functional W on $\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}$ defined by

$$\mathrm{W}[f] = \int_{-\infty}^{t} \operatorname{tr}\{(\mathfrak{F}_{\mathbf{K}}[f_\tau])\mathbf{D}(\tau)\}\,d\tau, \tag{2.28}$$

where $f_\tau(s) \equiv f(\tau+s)$, $\tau \in (-\infty, t]$, and $\mathbf{D}(\tau)$ is the stretching tensor at the instant τ, corresponding to the path f, in accordance with a known formula.

For all such operators the concept of an insensitivity semigroup — defined by the exchange of $\mathfrak{F}_{\mathbf{K}}$, in definition 1, into the operator under consideration — is resonable and useful. For operators (2.25), and, in general, for every operator of the type $a \circ \mathfrak{F}_{\mathbf{K}}$, where a is an arbitrary function defined on the set of symmetric tensors, we shall have

$$\Omega_a(\mathbf{K}, \mathscr{D}, \Lambda) \supset \Omega(\mathbf{K}, \mathscr{D}, \Lambda) \tag{2.29}$$

— i.e., performing an additional operation on the operator $\mathfrak{F}_{\mathbf{K}}$, we do not diminish the insensitivity semigroup. The first of the operators (2.25) provides numerous examples of situations in which Ω_a will be essentially a wider semigroup than Ω.

Remark 2. The concept of the insensitivity semigroup, for greater clarity stated above for a purely mechanical theory of simple materials, is particularly significant for the theory of not purely mechanical, and for materials which are not-simple. Since then, we deal with constitutive operators of several variables

$$\mathbf{T} = \mathfrak{F}_{\mathbf{K}}[f_1, \ldots, f_N], \tag{2.30}$$

where f_i are: deformation gradient, temperature and its gradient, other mechanical fields and their gradients, internal parameters describing a structure, etc. Thus there arises a possibility of an exchange operation of the type

$$(2.31) \qquad [f_1, \ldots, f_N] \to [\hat{f}_1(f_1, \ldots, f_N), \ldots, \hat{f}_N(f_1, \ldots, f_N)].$$

A description of insensitivity to exchanges of this type is the correct manner in which to state what are called temperature-deformation analogies, and others.

3. Representation theorem

Investigation of the insensitivity semigroup of a given constitutive operator is, in general, not too difficult, and sometimes even trivial. For instance, for an elastic material the insensitivity semigroup comprises all the exchanges of deformation paths which do not influence the final value of deformation — i.e., $\Omega_{\mathbf{K}} \supset \{\gamma \in \Gamma_{\mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}} \,|(f*\gamma)(0) = f(0)$ for every $f \in \mathrm{Dom}\,\mathfrak{F}_{\mathbf{K}}\}$. Of course, this property may serve as a definition of the elastic material (in the Cauchy sense). For isotropic, elastic material. referred to an unaltered configuration, it suffices to assume a weaker property $\mathbf{U}_{f*\gamma}(0) = \mathbf{U}_f(0)$, where $\mathbf{U}_f$ is the stretch tensor corresponding to f. For a Newtonian incompressible viscous fluid, it suffices to take $\mathbf{D}_{f*\gamma}(t) = \mathbf{D}_f(t)$. For more complicated known operators — for instance, those in the theory of "visco-elasticity" and "visco-plasticity" — the situation is, in general, much more involved.

The aim of the theory of materials consists, however, not so much in analysis of existing constitutive equations as in *elaborating a methodology to enable us to derive constitutive equations based on a "technological card" of the constitutive operator — a list of its fundamental properties found in a mechanical macroexperiment*. From this point of view, the following question possesses a fundamental meaning:

What conditions, in the form of a constitutive operator, do inclusion of a subsemigroup of mappings of the domain of an operator imposed, in advance, on its insensitivity semigroup? We shall give an answer to this question for the most important case, viz., for the semigroup $\Omega_{\mathbf{K},\mathscr{D}}$.

As frequently happens, it is simpler to pass to a more general language. Take, therefore, the operator

$$(3.1) \qquad \mathfrak{L} : \mathscr{P} \to T, \qquad \mathrm{Dom}\,\mathfrak{L} = \mathscr{P}.$$

We assume nothing special either about the operator $\mathfrak{L}$ or the sets $\mathscr{P}$, T; they should, however, posses sufficiently rich structure in order that the subsequent considerations may not become trivial ($\mathfrak{L} \neq \mathrm{const}$, $\mathscr{P}$ contains more than a single element, and so on). Denote

$$(3.2) \qquad \Pi \equiv \{\gamma : \mathscr{P} \to \mathscr{P} | \mathrm{Dom}\,\gamma = \mathscr{P}\}$$

and introduce the *insensitivity semigroup of the operator* $\mathfrak{L}$

$$(3.3) \qquad \Omega \equiv \{\gamma \in \Pi | \mathfrak{L} \circ \gamma = \mathfrak{L}\}.$$

We give to the question asked above greater precision as follows: determine all operators $\mathfrak{L}$ that satisfy the condition $\Omega \supset \Lambda$ — i.e., the following condition

$$(3.4) \qquad \mathfrak{L} \circ \gamma = \mathfrak{L} \quad \text{for every} \quad \gamma \in \Lambda,$$

where Λ is a subsemigroup in Π given in advance. This means that we are concerned with "general solutions" of some functional equations. We present an algorithm of finding such solutions based on the notions of Λ-orbit and Λ-separator. Similar, but weaker, algorithms have been used, explicitly or implicitly, in all problems of this kind (cf. [15, 16, 17]). An essential difficulty, in the present case, is implied by the fact that we are considering invariance with respect to a poor algebraic structure — semigroup.

Begin with a still more general situation. Let be given an equivalence relation $\mathrm{E} \subset \subset \mathscr{P} \times \mathscr{P}$. Consider the condition:

$$\mathfrak{L}[a] = \mathfrak{L}[b] \quad \text{if} \quad a\mathrm{E}b \tag{3.5}$$

for all $a, b \in \mathscr{P}$. Let us introduce the canonical mapping

$$\theta : \mathscr{P} \to \mathscr{P}/\mathrm{E} \tag{3.6}$$

which assigns to every element $a \in \mathscr{P}$ E-coset (an equivalence class with respect to E), to which it belongs, $a \in \theta(a)$.

Condition (3.5) simply means that each E-coset is contained in some level set of the operator $\mathfrak{L}$ — i.e., the partition into level sets is not finer than the partition into E-cosets. From this observation follows, at once, a "solution" of the functional equation (3.5):

operator $\mathfrak{L}$ satisfies the condition (3.5) iff there exists an operator $\mathfrak{S} : \mathscr{P}/\mathrm{E} \to \mathrm{T}$ such that

$$\mathfrak{L} = \mathfrak{S} \circ \theta. \tag{3.7}$$

Let us notice that E-cosets coincide with level sets — i.e., $\mathrm{E} = \mathrm{C}$ if $\mathfrak{S}$ is an injection.

It is not easy, in general, to pass to the operator $\mathfrak{S}$ since this passage requires a construction of some calculating apparatus on cosets. If one can find a suitable auxiliary set $\mathscr{M}$ such that E-cosets may be identified with elements of $\mathscr{M}$ — i.e., if there exists a bijection

$$t : \mathscr{P}/\mathrm{E} \to \mathscr{M} \tag{3.8}$$

— then, by introducing $\mu \equiv t \circ \theta$ and operator $\mathfrak{T} \equiv \mathfrak{S} \circ t^{-1}$, we write "solution" (3.7) of the Eq. (3.5) in an equivalent form:

$$\mathfrak{L} = \mathfrak{T} \circ \mu. \tag{3.9}$$

Finally, when neither (3.7) nor (3.9) is suitable, we proceed as follows.

We introduce an arbitrary mapping $\vartheta \in \Pi$ which satisfies the following conditions:

$$\begin{aligned} a * \vartheta &= b * \vartheta \quad \text{iff} \quad a\mathrm{E}b, \\ \vartheta \circ \vartheta &= \vartheta. \end{aligned} \tag{3.10}$$

Let us call every such mapping E-*separator*. The existence of E-separator is, like the existence of the set $\mathscr{M}$, based on the axiom of choice. In accordance with it, there exists a set of representatives for E-cosets, $\mathscr{E} \subset \mathscr{P}$. Let A be an arbitrary E-coset. Taking $e : \mathscr{P}/\mathrm{E} \to \mathscr{E}$ defined by the formula $e(\mathrm{A}) \equiv \mathrm{A} \cap \mathscr{E}$, we obtain E-separator $\vartheta \equiv e \circ \theta$. Every set of representatives $\mathscr{E}$ designates its own E-separator ϑ, and inversely. Now, if the formula (3.7) holds, then $\mathfrak{L} = \mathfrak{S} \circ \theta = \mathfrak{L} \circ \vartheta$. Inversely, if $\mathfrak{L} = \mathfrak{L} \circ \vartheta$ then $\mathfrak{L} = \mathfrak{S} \circ \theta$, where $\mathfrak{S} \equiv \mathfrak{L} \circ e$. Our result can be stated int he equivalent form:

Operator $\mathfrak{L}$ satisfies the condition (3.5) iff every E-separator ϑ belongs to its insensitivity semigroup — i.e., when

$$\mathfrak{L} = \mathfrak{L} \circ \vartheta. \tag{3.11}$$

Let us pass to the situation described by condition (3.4), where the information concerning level sets of the operator is hidden in its invariance with respect to some semigroup $\Lambda \subset \Pi$. We reduce it to the situation (3.5) by introducing a suitable equivalence relation. We now give a definition which is essential in all the subsequent results stated as theorems I–VIII of Part II.

First, let us take the following relation in $\mathscr{P} \times \mathscr{P}$:

$$\overleftarrow{\mathrm{E}}_\Lambda \equiv \{(a, b) \in \mathscr{P} \times \mathscr{P} | \quad \text{there exists a } \gamma \in \Pi, \text{ such that } a = b * \gamma\}, \tag{3.12}$$

$$\overrightarrow{\mathrm{E}}_\Lambda \equiv \{(a, b) \in \mathscr{P} \times \mathscr{P} | \quad \text{there exists a } \gamma \in \Pi, \text{ such that } a * \gamma = b\}. \tag{3.13}$$

Since, by hypothesis, the semigroup Λ possesses an identity element on $\mathscr{P}$, then both relations are reflexive. It is also obvious that both relations are transitive, but neither of them is symmetric in a general case.

We introduce the relation

$$\overleftrightarrow{\mathrm{E}}_\Lambda \equiv \overleftarrow{\mathrm{E}}_\Lambda \cup \overrightarrow{\mathrm{E}}_\Lambda. \tag{3.14}$$

This is a *tolerancy*(¹) — i.e., a reflexive and symmetric relation. In a general case, $\overleftrightarrow{\mathrm{E}}_\Lambda$ is not transitive. Let us introduce the minimal reflexive, symmetric and transitive relation containing $\overleftrightarrow{\mathrm{E}}_\Lambda$. That will be what is called, the *transitive closure of a tolerancy* [19],

$$\mathrm{E}_\Lambda \equiv \overleftrightarrow{\mathrm{E}}_\Lambda \cup \overleftrightarrow{\mathrm{E}}^2_\Lambda \cup \ldots \cup \overleftrightarrow{\mathrm{E}}^N_\Lambda \cup \ldots. \tag{3.15}$$

This relation we shall call Λ-equivalence. In other words, we are assuming the following definition:

DEFINITION 2. *Elements $a, b \in \mathscr{P}$ are equivalent with respect to the semigroup with identity $\Lambda \subset \Pi$, or briefly, Λ-equivalent, if*

$$a \mathrm{E}_\Lambda b. \tag{3.16}$$

Instead of $a\mathrm{E}_\Lambda b$, we shall write, in what follows, $a\Lambda b$.

*An explicit form of the definition reads: $a\Lambda b$; this means, according to (3.15), that for the pair (a, b) there exists a finite sequence $c_1, \ldots, c_N \in \mathscr{P}$, called Λ-**connecting sequence**, such that*

$$c_1 = a, \quad c_N = b, \tag{3.17}$$

$$\textit{for every} \quad i = 1, \ldots, N-1, \quad \textit{the pair } (c_i, c_{i+1}) \text{ is} \tag{3.18}$$

(¹) This concept was introduced by ZEEMAN [18]. Powers of tolerancy are defined as follows: $\mathrm{E}^2_\Lambda \equiv \{(a, b) | \text{there exists a } c \in \mathscr{P}, \text{ such that } a\overleftrightarrow{\mathrm{E}}_\Lambda c \text{ and } c\overleftrightarrow{\mathrm{E}}_\Lambda b\}, \ldots$

*Λ-**connected*** — i.e., *there exists $\gamma \in \Lambda$ such that*

$$c_i = c_{i+1} * \gamma \quad \text{or} \quad c_i * \gamma = c_{i+1}.$$

Cosets with respect to E_Λ — i.e., elements of $\mathscr{P}/\mathrm{E}_\Lambda$ — are called *Λ-**orbits***(²). Every E_Λ-separator is called Λ-separator. By using definition 2 and the foregoing statements, we obtain the following "solution" of the problem posed at the outset.

THEOREM 3. *A given subsemigroup $\Lambda \subset \Pi$ is a part of the insensitivity semigroup Ω of the operator $\mathfrak{L} : \mathscr{P} \to \mathrm{T}$ — i.e.,*

$$\mathfrak{L}[a * \gamma] = \mathfrak{L}[a] \tag{3.19}$$

for every $a \in \mathscr{P}$, $\gamma \in \Lambda$, iff there exists a set $\mathscr{M}$ and a mapping $\mu : \mathscr{P} \to \mathscr{M}$ with the property

$$\mu(a) = \mu(b) \quad \text{iff} \quad a\Lambda b, \tag{3.20}$$

and an operator $\mathfrak{T} : \mathscr{M} \to \mathrm{T}$ such that

$$\mathfrak{L} = \mathfrak{T} \circ \mu. \tag{3.21}$$

Another formulation: $\Lambda \subset \Omega$ iff every Λ-separator ϑ belongs to Ω — i.e.,

$$\mathfrak{L} = \mathfrak{L} \circ \vartheta. \tag{3.22}$$

P r o o f. The only statement that requires to be proved is the equivalence: $\Lambda \subset \Omega$, iff for every $a, b, \in \mathscr{P}$

$$\mathfrak{L}[a] = \mathfrak{L}[b] \quad \text{if} \quad a\Lambda b. \tag{3.23}$$

Suppose $\Lambda \subset \Omega$. The condition $a\Lambda b$ means, according to (3.15), that there exits a Λ-connecting sequence $c_1, \ldots, c_N \in \mathscr{P}$. It now follows from (3.17), (3.18), (3.19) that

$$\mathfrak{L}[a] \equiv \mathfrak{L}[c_1] = \mathfrak{L}[c_2] = \ldots = \mathfrak{L}[c_N] \equiv \mathfrak{L}[b]. \tag{3.24}$$

Inversely, suppose that the implication (3.23) holds. Then for arbitrary $a \in \mathscr{P}$ and $\gamma \in \Lambda$ we have $a\Lambda(a*\gamma)$; hence follows (3.19) — i.e., $\Lambda \subset \Omega$. Thus, we have reduced the condition of invariance with respect to Λ, (3.4), to the condition of constancy (3.5) on cosets of a certain equivalence relation. Therefore we may apply formulae (3.9), (3.11). Q.E.D.

"Solutions" (3.21), (3.22) will be called, in what follows, the *representation formulae.* They provide, as always in such situations, a manner of procedure only. In every concrete situation, the heart of the matter and the essence of the difficulty, for given $\mathscr{P}, \mathfrak{L}, \Lambda$, consists in the construction of the pair $(\mathscr{M}, \mu)$, or Λ-separator ϑ.

4. Exact insensitivity semigroups

An attempt to get rid of "dispensable" operations of a rotating deformation path in the past is quite troublesome and, therefore, it was not performed *before* the introduction of the definition of an insensitivity semigroup.

(²) A corresponding construction for groups of mappings is well known [28]. When Λ is a group, then $\overleftarrow{\mathrm{E}}_\Lambda = \overrightarrow{\mathrm{E}}_\Lambda = \overleftrightarrow{\mathrm{E}}_\Lambda = \mathrm{E}_\Lambda$.

Denote the set of those operations by $\mathcal{Q}$,

$$(4.1)\quad \mathcal{Q} \equiv \{\varrho \in \Gamma | \quad \text{there exists an orthogonal curve } \mathbf{Q} \in \mathscr{A},\ \mathbf{Q}(0) = \mathbf{1}, \quad \text{such that } f*\varrho = \mathbf{Q}f \text{ for every } f \in \mathscr{A}\}.$$

This is a group in the semigroup Γ. In agreement with the principle of material objectivity (2.5)

$$(4.2)\qquad \mathcal{Q} \subset \Omega_{\mathbf{K}},$$

for every material and every reference configuration.

Let us decompose every deformation path f into rotation $\mathbf{R}_f$ and stretch $\mathbf{U}_f$, $f \equiv \mathbf{R}_f \mathbf{U}_f$. Take the operation $\pi \in \Gamma$ defined by

$$(4.3)\qquad (f*\pi)(s) \equiv \mathbf{R}_f(0)\mathbf{U}_f(s),$$

where $\tilde{\mathscr{A}} \equiv \mathscr{A}*\pi$ is the set of deformation paths with constant rotation. It is obvious, that: (i) $f*\pi = g*\pi$ iff $f\mathcal{Q}g$ (i.e., $f = \mathbf{Q}g$ for some $\mathbf{Q}(s)$, $\mathbf{Q}(0) = \mathbf{1}$); (ii) $\pi \circ \pi = \pi$ — i.e., it is $\mathcal{Q}$-separator. According to the representation theorem (3.22), (4.2) is equivalent to

$$(4.4)\qquad \mathfrak{F}_{\mathbf{K}} = \tilde{\mathfrak{F}}_{\mathbf{K}} \circ \pi,$$

where $\operatorname{Dom} \tilde{\mathfrak{F}}_{\mathbf{K}} = \operatorname{Dom} \mathfrak{F}_{\mathbf{K}} \cap \tilde{\mathscr{A}}$.

In this way we have used that part of the information contained in the principle of material objectivity which concerns level sets of the constitutive operator(³).

It follows from the formula obtained, that in general, we might restrict our considerations of invariance to the cutoff operator $\tilde{\mathfrak{F}}_{\mathbf{K}}$, and to study the exchanges from $\Gamma_{\tilde{\mathscr{A}}}$ only — i.e., exchanges of deformation paths with constant rotation into paths with constant rotation.

DEFINITION 3. *By a **precise** insensitivity semigroup of a material we mean every semigroup* $\Omega(\mathbf{K}, \mathscr{D}, \Lambda)$ *if* $\mathscr{D} \subset \operatorname{Dom} \tilde{\mathfrak{F}}_{\mathbf{K}} \subset \tilde{\mathscr{A}}$. *By an **exact** insensitivity semigroup of a material we mean one which is isomorphic with a certain precise insensitivity semigroup of that material.*

Precise insensitivity semigroups are inconvenient in use. Numerous useful operations — for instance, the operations of multiplication of deformation path by a constant non-orthogonal tensor — do not enter into those semigroups since they lead out from $\tilde{\mathscr{A}}$.

A study of the exactness of insensitivity semigroups requires, in the general case, quite complicated algebraic constructions. None of the natural equivalence relations in Γ, connected with the projection π — for instance, the following relations $\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{E}_4$:

1. $\gamma \mathrm{E}_1 \delta$ iff there exists a $\varrho \in \mathcal{Q}$, such that $\gamma = \delta \circ \varrho$,
2. $\gamma \mathrm{E}_2 \delta$ iff there exists a $\varrho \in \mathcal{Q}$, such that $\gamma = \varrho \circ \delta$,
3. $\gamma \mathrm{E}_3 \delta$ iff $\gamma \circ \pi = \delta \circ \pi$,
4. $\gamma \mathrm{E}_4 \delta$ iff $\pi \circ \gamma \circ \pi = \pi \circ \delta \circ \pi$,

— are not two-sided stable relations [13], and none of them make it possible to construct a natural factor semigroup, dual to the partition of $\mathscr{A}$ into $\mathcal{Q}$-orbits.

(³) That principle contains, of course, additional information which does not concern level sets:

$$\mathfrak{F}_{\mathbf{K}}[\mathbf{R}_f(0)\mathbf{U}_f(s)] = \mathbf{R}_f(0)\mathfrak{F}_{\mathbf{K}}[\mathbf{U}_f(s)]\mathbf{R}_f^T(0)$$

which, for the present, will not be needed.

5. Configuration insensitivity (Noll's isotropy group)

A specially important type of insensitivity of material was under another name and in somewhat different context, introduced and studied by NOLL [4, 20]. His idea was to study equivalence relations in a set of local reference configurations of a material, introduced in the following way: local configurations $\hat{\mathbf{K}}$, $\mathbf{K}$ are equivalent if $\mathfrak{F}_{\hat{\mathbf{K}}} = \mathfrak{F}_{\mathbf{K}}$ and $\varrho_{\hat{\mathbf{K}}} = \varrho_{\mathbf{K}}$. One can say that $\hat{\mathbf{K}}$ and $\mathbf{K}$ are *indistinguishable in a mechanical macroexperiment*, since only stresses $\mathbf{T}$, deformations f, and density ϱ are the quantities measured in such experiments. Since $\hat{\mathbf{K}} = \mathbf{PK}$ for $\mathbf{P} \equiv \hat{\mathbf{K}}\mathbf{K}^{-1}$, then the central notion of Noll's theory is an *isotropy group of a material* $g_{\mathbf{K}}$ relative to the local configuration $\mathbf{K}$, introduced by the formula:

$$g_{\mathbf{K}} \equiv \{\mathbf{P} \in \mathscr{N} \mid \mathfrak{F}_{\mathbf{PK}} = \mathfrak{F}_{\mathbf{K}} \text{ and } \varrho_{\mathbf{PK}} = \varrho_{\mathbf{K}}\} = \{\mathbf{P} \in \mathscr{U} \mid \mathfrak{F}_{\mathbf{PK}} = \mathfrak{F}_{\mathbf{K}}\}, \tag{5.1}$$

where $\mathscr{U}$ is a group of unimodular tensors.

Let us consider this fundamental idea from our point of view. According to formula (2.4), the exchange of configurations $\mathbf{K} \to \mathbf{PK}$ is equivalent to the exchange $f \to f\mathbf{P}$, where $\det \mathbf{P} = 1$ — i.e., equivalent to the superposition of prestrain $\mathbf{P}$, which does not change the density (of shear prestrain) on any deformation path f.

Let us introduce a mapping of the group $\mathscr{U}$ into the semigroup of all exchanges of deformation paths Γ,

$$\chi : \mathscr{U} \to \Gamma, \quad f * \chi(\mathbf{P}) \equiv f\mathbf{P}. \tag{5.2}$$

Iff is obvious, that $\mathscr{U}$ is isomorphic to its image $\chi(\mathscr{U})$ (in fact, $\chi(\mathbf{P}_1\mathbf{P}_2) = \chi(\mathbf{P}_1) \circ \chi(\mathbf{P}_2)$; $\chi(\mathbf{P}_1) = \chi(\mathbf{P}_2)$ iff $\mathbf{P}_1 = \mathbf{P}_2$). The formula (5.1) may be written in the equivalent form:

$$\chi(g_{\mathbf{K}}) \equiv \{\mathbf{P} \in \mathscr{U} \mid \mathfrak{F}_{\mathbf{K}} \circ \chi(\mathbf{P}) = \mathfrak{F}_{\mathbf{K}}\}. \tag{5.3}$$

In other words,

$$\chi(g_{\mathbf{K}}) = \Omega(\mathbf{K}, \operatorname{Dom} \mathfrak{F}_{\mathbf{K}}, \chi(\mathscr{U})), \tag{5.4}$$

— i.e., Noll's isotropy group of a material is (isomorphic to) a special type of insensitivity semigroup. The group $\chi(g_{\mathbf{K}})$ can be called a *configuration insensitivity group of a material relative to* $\mathbf{K}$. Then the name of the "isotropy group" of a material relative to $\mathbf{K}$ will be suitable for the orthogonal part of $\chi(g_{\mathbf{K}})$.

A comprehensive description of groups $g_{\mathbf{K}}$ has been developed in the papers, [4, 20, 21, 22, 23]. Those results constitute an important part of the description of insensitivity semigroups of a material. Note that Noll's theorem: $g_{\mathbf{PK}} = \mathbf{P} g_{\mathbf{K}} \mathbf{P}^{-1}$ is but a particular case of (2.19).

For illustration of the representation theorem (3.21), we give a slightly modified proof of the basic Noll's theorem concerning simple fluids.

A simple fluid is, by definition, a simple material for which reference configurations with the same density are indistinguishable experimentally. In others words, a simple fluid is defined by the equality:

$$\Omega(\mathbf{K}, \operatorname{Dom} \mathfrak{F}_{\mathbf{K}}, \chi(\mathscr{U})) = \chi(\mathscr{U}). \tag{5.5}$$

It is obvious that two deformations paths f, g are $\chi(\mathscr{U})$-equivalent iff

$$f(s)[f(0)]^{-1} = g(s)[g(0)]^{-1}, \quad \det f(0) = \det g(0). \tag{5.6}$$

Hence $\chi(\mathscr{U})$-orbits in Dom $\mathfrak{F}_{\mathbf{K}}$ may be identified with the pairs $\mathbf{F}_{(t)}^{(t)}(s) \equiv f(s)[f(0)]^{-1}$, $\varrho(t)/\varrho_{\mathbf{K}} \equiv \det f(0)$. Applying the representation formula (3.21), we obtain:

$$\mathbf{T} = \mathfrak{T}_{\mathbf{K}}[\mathbf{F}_{(t)}^{(t)}(s), \varrho(t)/\varrho_{\mathbf{K}}]. \tag{5.7}$$

Applying the polar decomposition $\mathbf{F}_{(t)}^{(t)}(s) = \mathbf{R}_{(t)}^{(t)}(s)\mathbf{U}_{(t)}^{(t)}(s)$, $\mathbf{R}_{(t)}^{(t)}(0) = \mathbf{1}$ and the formula (2.5), we obtain *Noll's theorem*: A simple material is a simple fluid iff

$$\mathbf{T} = \mathfrak{T}_{\mathbf{K}}[\mathbf{U}_{(t)}^{(t)}(s), \varrho(t)/\varrho_{\mathbf{K}}], \tag{5.8}$$

where the operator is isotropic with respect to its tensor argument.

We shall consider arbitrary configuration insensitivity groups $\Omega(\mathbf{K}, \mathscr{D}, \chi(\mathscr{U}))$.

If $f \in \tilde{\mathscr{A}}$ — i.e., f is a deformation path with constant rotation, and $\mathbf{P} \in \mathscr{U}$ is not an orthogonal tensor — then, in general, $f\mathbf{P}$ will not be a path with constant rotation. We introduce, therefore, a mapping

$$p : \Gamma \to \Gamma_{\tilde{\mathscr{A}}}, \quad p(\gamma) \equiv \gamma \circ \pi \tag{5.9}$$

(cf. equivalence relation E_3 in § 4). Now $\chi(\mathbf{P}) \circ \pi : \tilde{\mathscr{A}} \to \tilde{\mathscr{A}}$.

LEMMA. *The groups $\chi(\mathscr{U})$, $\chi(\mathscr{U}) \circ \pi$ are isomorphic.*

Proof. First we show that

$$\chi(\mathbf{P}) \circ \pi = \pi \circ \chi(\mathbf{P}) \circ \pi \tag{5.10}$$

for every $\mathbf{P} \in \mathscr{U}$, (i.e., the relations E_3 and E_4 from § 4 coincide with each other on $\chi(\mathscr{U})$). In fact, let us write (5.2) in the equivalent form:

$$(f\mathbf{P}) * \pi = [(f * \pi)\mathbf{P}] * \pi \tag{5.11}$$

for every $f \in \mathscr{A}$. This means that for every $f \in \mathscr{A}$ we shall have

$$\mathbf{U}_{f\mathbf{P}} = \mathbf{U}_{(f*\pi)\mathbf{P}}, \quad \mathbf{R}_{f\mathbf{P}}(0) = \mathbf{R}_{(f*\pi)\mathbf{P}}(0). \tag{5.12}$$

Since $\mathbf{U}_{f*\pi} = \mathbf{U}_f$, then

$$\mathbf{U}_{f\mathbf{P}}^2 \equiv [f\mathbf{P}]^T[f\mathbf{P}] = \mathbf{P}^T f^T f\mathbf{P} = \mathbf{P}^T\mathbf{U}_f^2\mathbf{P} = \mathbf{P}^T\mathbf{U}_{f*\pi}^2\mathbf{P} = \mathbf{U}_{(f*\pi)\mathbf{P}}^2 \tag{5.13}$$

and

$$\mathbf{R}_{f\mathbf{P}}(0) = f(0)\mathbf{U}_{f\mathbf{P}}^{-1}(0) = (f*\pi)(0)\mathbf{U}_{(f*\pi)\mathbf{P}}^{-1}(0) = \mathbf{R}_{(f*\pi)\mathbf{P}}(0). \tag{5.14}$$

By now, the theorem is obvious since, firstly,

$$p[\chi(\mathbf{P}_1) \circ \chi(\mathbf{P}_2)] \equiv \chi(\mathbf{P}_1) \circ \chi(\mathbf{P}_2) \circ \pi = \chi(\mathbf{P}_1) \circ \pi \circ \chi(\mathbf{P}_2) \circ \pi = p[\chi(\mathbf{P}_1)] \circ p[\chi(\mathbf{P}_2)]; \tag{5.15}$$

thus p is a homomorphic embedding $\chi(\mathscr{U})$ in Γ. Secondly, if $p[\chi(\mathbf{P}_1)] = p[\chi(\mathbf{P}_2)]$ — i.e.,

$$(f\mathbf{P}_1) * \pi = (f\mathbf{P}_2) * \pi \quad \text{for every } f \tag{5.16}$$

— then, letting $f = \mathbf{1} = \text{const}$, we have $\mathbf{P}_1 = \mathbf{P}_2$. Q.E.D.

THEOREM 4. *Every configuration insensitivity group $\Omega(\mathbf{K}, \mathscr{D}, \chi(\mathscr{U}))$ is an exact one, for*

$$\Omega(\mathbf{K}, \mathscr{D}, \chi(\mathscr{U})) \circ \pi = \Omega(\mathbf{K}, \mathscr{D} * \pi, \chi(\mathscr{U}) \circ \pi). \tag{5.17}$$

P r o o f. It follows from the foregoing Lemma that the image $\Omega(\mathbf{K}, \mathscr{D}, \chi(\mathscr{U})) \circ \pi$ of the group $\Omega(\mathbf{K}, \mathscr{D}, \chi(\mathscr{U}))$ under the mapping p is a group. We now show that it is a precise group of configuration insensitivity written out on the right-hand side of the equality (5.9).

Let $\chi(\mathbf{P}) \in \Omega(\mathbf{K}, \mathscr{D}, \chi(\mathscr{U}))$. Then for every $f \in \mathscr{D}$,

$$\tilde{\mathfrak{F}}_{\mathbf{K}}[(f*\pi)*(\chi(\mathbf{P}) \circ \pi)] = \mathfrak{F}_{\mathbf{K}}[(f*\pi)*\chi(\mathbf{P})] = \mathfrak{F}_{\mathbf{K}}[f*\pi] = \tilde{\mathfrak{F}}_{\mathbf{K}}[f*\pi] \tag{5.18}$$

— i.e., $\chi(\mathbf{P}) \circ \pi \in \Omega(\mathbf{K}, \mathscr{D}*\pi, \chi(\mathscr{U}) \circ \pi)$. Inversely, if this inclusion is fulfilled then for every $f \in \mathscr{D}$

$$\begin{aligned} \mathfrak{F}_{\mathbf{K}}[f*\chi(\mathbf{P})] = \mathfrak{F}_{\mathbf{K}}[f*\chi(\mathbf{P})*\pi] = \tilde{\mathfrak{F}}_{\mathbf{K}}[f*(\chi(\mathbf{P}) \circ \pi)] = \tilde{\mathfrak{F}}_{\mathbf{K}}[f*(\pi \circ \chi(\mathbf{P}) \circ \pi)] \\ = \tilde{\mathfrak{F}}_{\mathbf{K}}[f*\pi] = \mathfrak{F}_{\mathbf{K}}[f], \end{aligned} \tag{5.19}$$

i.e.,

$$\chi(\mathbf{P}) \in \Omega(\mathbf{K}, \mathscr{D}, \chi(\mathscr{U})). \qquad \text{Q.E.D.}$$

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