Archives of Mechanics • Archiwum Mechaniki Stosowanej • 26, 1, pp. 105–118, Warszawa 1974

Non-linear mechanics of constrained material continua. I. Foundations of the theory

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THE AIM of the paper is to give as regards continuum mechanics a more general formulation, in which we deal with three-dimensional non-polar continua on the motion of which are imposed certain restrictions called constraints. Various engineering theories (such as theories of plates and shells, finite element approaches etc.), which are not consistent with classical continuum mechanics, can be treated as exact theories when based on the mechanics of constrained continua. Moreover, using the concept of constrained continuum, we can give criteria for the applicability of particular engineering approaches. In this part of the paper, the foundations of the theory are investigated.

W pracy przedstawiono pewne uogólnienie mechaniki kontinuum przyjmując, że na ruch trójwymiarowego niebiegunowego ośrodka ciągłego są narzucone więzy. Różne uproszczone teorie i podejścia (np. teorie płyt i powłok, metoda elementów skończonych itd.), które nie są konsystentne z klasyczną mechaniką ośrodka ciągłego, można formułować w sposób ścisły na podstawie mechaniki ośrodka ciągłego z więzami. Na tej drodze można także ustalić zakres stosowalności różnych teorii przybliżonych. W tej części pracy podaje się podstawowe założenia i twierdzenia mechaniki ośrodka ciągłego z więzami.

В работе представлено некоторо обобщение механики континуум принимая, что на движение трехмерной неполярной сплошной среды наложены связи. Разные упрощенные теории и подходы (например теории плит и оболочек, метод конечных элементов и т. д.), которые неконсистентны с классической механикой сплошной среды, можно формулировать точным образом на основе механики сплошной среды со связями. По этому пути можно также установить область применяемости разных приближенных теорий. В настоящей части работы приводятся основные постулаты и теоремы механики сплошной среды со связями.

Introduction

Two models of real bodies are used in mechanics; one is a finite or even countable set of material points and the other is a material continuum. In what follows we shall confine ourselves to such mechanical phenomena only, in which the latter model can be used. We assume that the material continuum is non-polar and three-dimensional. However, there are many problems which are too complicated to be solved or even investigated properly within the classical formulation of continuum mechanics. Dealing with such problems, we introduce certain approximations; examples of such approximations can be found in the known theories of plates and shells or in the finite element approach. Those approximations, represented by suitable formulas, are not consistent with the axioms of classical continuum mechanics. This follows from the fact that in engineering mechanics we do not usually deal with classical material continua but with certain constrained material continua — i.e., material continua on the motion of which are imposed restrictions called constraints. The aim of the paper is to elaborate the general theory of constrained continuous media. Various engineering approaches can be treated as exact theories when they are based on the mechanics of constrained continua; moreover, in this way we can also evaluate the criteria of applicability of particular engineering approaches. In the special case of bodies with internal simple constraints, we derive from the general theory given in the paper the known formulas. In the first part of the paper we shall establish the basic axioms and theorems.

1. Constrained body

Let $\mathscr{B}$ be a continuous body and let us denote by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ the deformation function of $\mathscr{B}$ from the reference configuration $\varkappa$, where $\mathbf{X} \in \varkappa(\mathscr{B})$, $t \in R$, and $\mathbf{x}$ is a position vector in the physical (three-dimensional and Euclidean) space([1]). The continuous body $\mathscr{B}$ is said to be constrained if on the deformation function $\boldsymbol{\chi}(\mathbf{X}, t)$ are imposed certain restrictions of a geometrical or kinematical nature, called constraints. In what follows we shall assume that the constraints are given by a system of partial differenctial equations:

$$(1.1) \qquad \gamma_{ia}(\mathbf{X}, t, \boldsymbol{\chi}, \nabla\boldsymbol{\chi}, \ldots, \nabla^k\boldsymbol{\chi}) = 0; \quad \mathbf{X} \in \varkappa(\mathscr{B}_a), t \in R; \quad i = 1, 2, \ldots, p_a,$$

in different parts $\mathscr{B}_a$, $a = 1, 2, \ldots, r$, of the body $\mathscr{B}$, where $\mathscr{B}_a \cap \mathscr{B}_b = \phi$ for each $a \neq b$. For an unconstrained body we have $\bigcup_{a=1}^{s} \mathscr{B}_a = \phi$; with another special case of a constrained body we shall deal when $r = 1$ and $\mathscr{B}_1 = \mathscr{B}$. Examples of constraints (1.1) and their physical interpretation will be given in later Sections of the present paper; for the time being, we shall assume that the equations (1.1) are given *a priori*.

The mass of the constrained body will be introduced in the same way as for the unconstrained body; we shall denote by $\varrho = \varrho(\mathbf{X}, t)$ the mass density of the body in an arbitrary configuration $\boldsymbol{\chi}_t$, assuming the equation of continuity in the known form $\dot{\varrho} + \varrho \operatorname{div} \boldsymbol{\chi} = 0$, $J \equiv \det \nabla \boldsymbol{\chi}$.

The system of forces for the constrained body in the motion given by the deformation function $\boldsymbol{\chi}(\mathbf{X}, t)$, satisfying (1.1), will be characterized by the following conditions:

(i) The field $\mathbf{b} = \mathbf{b}(\mathbf{X}, t)$, $\mathbf{X} \in \varkappa(\mathscr{B})$, $t \in R$, of the density per unit mass of the external body force is given.

(ii) The field $\mathbf{p} = \mathbf{p}(\mathbf{X}, t)$, $\mathbf{X} \in \varkappa(\mathscr{B})$, $t \in R$, of the density per unit area of the external surface loads is given.

(iii) The stress-tensor field $\mathbf{T}(\mathbf{X}, t)$ $\mathbf{X} \in \varkappa(\mathscr{B}_a)$, $a = 0, 1, \ldots, r$, $t \in R$, is given ($\mathscr{B}_0$ is a three-dimensional part of the body in which there are no constraints (1.1).)

The rules of interpretation of the primitive concept of forces introduced by (i), (ii), (iii) are the same as in classical continuum mechanics.

As the basic axiom of the dynamics of constrained bodies we shall take the principle of virtual work. To this end, we shall define the virtual displacement field $\delta\boldsymbol{\chi} = \delta\boldsymbol{\chi}(\mathbf{X}, t)$; $\mathbf{X} \in \overline{\varkappa(\mathscr{B})} \equiv \varkappa(\mathscr{B}) \cup \partial\varkappa(\mathscr{B})$, $t \in R$, which is continuous in $\overline{\varkappa(\mathscr{B})}$ and satisfies in $\varkappa(\mathscr{B}_a)$, $a = 1, 2, \ldots, r$, for each deformation function $\boldsymbol{\chi}$ admissible by (1.1), the following system

([1]) We shall use the same notations as in [1].

of linear differential equations;

$$\frac{\partial \gamma_{ia}}{\partial \boldsymbol{\chi}} \cdot \delta\boldsymbol{\chi} + \frac{\partial \gamma_{ia}}{\partial \nabla\boldsymbol{\chi}} \cdot \nabla(\delta\boldsymbol{\chi}) + \ldots + \frac{\partial \gamma_{ia}}{\partial \nabla^k\boldsymbol{\chi}} \cdot \nabla^k(\delta\boldsymbol{\chi}) = 0; \tag{1.2}$$

$$a = 1, \ldots, r,; \quad i = 1, \ldots, p_a,$$

where the dot between the symbols denotes contraction with respect to all tensor indices. We postulate that the principle of virtual work

$$\oint_{\partial\chi_t(\mathscr{B})} \mathbf{p} \cdot \delta\boldsymbol{\chi}\, ds + \int_{\chi_t(\mathscr{B})} \varrho\,(\mathbf{b} - \ddot{\boldsymbol{\chi}}) \cdot \delta\boldsymbol{\chi}\, dv = \sum_{a=0}^{r} \int_{\chi_t(\mathscr{B}_a)} [\mathbf{T}(\nabla\boldsymbol{\chi}^{-1})^T] \cdot \nabla(\delta\boldsymbol{\chi})\, dv; \tag{1.3}$$

$$\boldsymbol{\chi}_t(\mathscr{P}) \equiv \boldsymbol{\chi}(\varkappa(\mathscr{P}), t), \; \mathscr{P} \subset \mathscr{B},$$

holds for any virtual displacement field $\delta\boldsymbol{\chi}$.

We shall also assume that a material of a body is simple at each particle $X \in \mathscr{B}_a$, $a = 0, 1, \ldots, r$, postulating the following constitutive equation

$$\mathbf{T}(\mathbf{X}, t) = \mathop{\mathscr{F}}_{s=0}^{\infty} \big(\mathbf{X}, \nabla\boldsymbol{\chi}(\mathbf{X}, t-s)\big). \tag{1.4}$$

In continuum mechanics of unconstrained bodies, the response functional $\mathscr{F}$ is defined for each motion of the body, but in mechanics of constrained material continua, the domain of the response functional depends on the character of constraints. To investigate the character of constraints (1.1), let us observe that the constraints can be introduced either on the basis of certain physical properties of the material of the body (bodies made of incompressible materials, for example) or in order to simplyfy the mathematical structure of problems under consideration (the hypothesis of normal element in the shell theory, for example). In the former case the constraints will be termed real and in the latter case we shall call them imaginary constraints. Thus the constitutive functional $\mathscr{F}$ is defined for all motions which are admissible by the real constraints; because all rigid motions are always admissible by the real constraints, then we shall assume that the constitutive equation (1.4) has to satisfy the principle of material frame indifference. If there are no real constraints, then we have to assume that the manifold of all motions admissible by imaginary constraints is not an empty set.

From the formal point of view, mechanics of constrained continua is expressed in terms of primitive concepts $\mathscr{B}$, $\boldsymbol{\chi}$, ϱ, $\mathbf{b}$, $\mathbf{p}$, $\mathbf{T}$ and axioms given by (1.1), (1.3) and (1.4). All primitive concepts are interpreted in the same way as in classical continuum mechanics. However, from the point of view of application of bodies with imaginary constraints to the classical problems of continuum mechanics, we have also to establish criteria of approximation of classical continuum mechanics problems by using the suitable constrained continua. This problem we shall analyse in the next Section.

2. Reaction forces

Let us transform the Eq. (1.3) to the form:

$$\int_{\chi_t(\mathscr{P})} \mathbf{s} \cdot \delta\boldsymbol{\chi}\, ds + \int_{\chi_t(\mathscr{B})} \varrho\mathbf{r} \cdot \delta\boldsymbol{\chi}\, dv = 0, \tag{2.1}$$

where $\mathscr{P} = \bigcup_{a=0}^{r} \partial\mathscr{B}_a$ is a material surface, where

$$
(2.2)\qquad
\begin{aligned}
&\varrho\mathbf{r} = \varrho\ddot{\boldsymbol{\chi}} - \varrho\mathbf{b} - \operatorname{div}\mathbf{T} && \text{for} \quad (\mathbf{X}, t) \in \bigcup_{a=0}^{r} \varkappa(\mathscr{B}_a)\times R,\\
&\mathbf{s} = \mathbf{T}^a\mathbf{n}^a - \mathbf{p} && \text{for} \quad (\mathbf{X}, t) \in [\partial\varkappa(\mathscr{B}_a) \cap \partial\varkappa(\mathscr{B})]\times R,\\
&\mathbf{s} = \mathbf{T}^a\mathbf{n}^a + \mathbf{T}^b\mathbf{n}^b && \text{for} \quad (\mathbf{X}, t) \in [\partial\varkappa(\mathscr{B}_a) \cap \partial\varkappa(\mathscr{B}_b)]\times R, \quad a \neq b,
\end{aligned}
$$

and where $\mathbf{n}^a$ is a unit exterior normal to the surface $\partial\boldsymbol{\chi}_t(\mathscr{B}_a)$ and $\mathbf{T}^a$ is a limit value on $\partial\boldsymbol{\chi}_t(\mathscr{B}_a)$ of the stress tensor $\mathbf{T}$ in $\boldsymbol{\chi}_t(\mathscr{B}_a)$. The vector field $\mathbf{r} = \mathbf{r}(\mathbf{X}, t)$ will be called the body reaction force and the vector field $\mathbf{s} = \mathbf{s}(\mathbf{X}, t)$ is said to be the surface reaction traction (for $\mathbf{X} \in \cup\ \partial\varkappa(\mathscr{B}_a) \cap \partial\varkappa(\mathscr{B})$) or the contact reaction forces (for $\mathbf{X} \in \cup\ [\partial\varkappa(\mathscr{B}_a) \cap \partial\varkappa(\mathscr{B}_b)]$). The fields $\mathbf{r}(\mathbf{X}, t)$, $\mathbf{s}(\mathbf{X}, t)$ of reaction forces, the deformation function $\boldsymbol{\chi}(\mathbf{X}, t)$ and the field of Cauchy stress tensor $\mathbf{T}(\mathbf{X}, t)$ are the basic unknowns in the mechanics of constrained continua. These functions have to satisfy the equations of constraints (1.1), the constitutive equations (1.4), the expressions (2.2) for the reaction forces, and the integral condition (2.1). The Eqs. (2.2) may also be called the equations of motion, the boundary conditions and the contact conditions, respectively, for the constrained body. Note that the Eqs. (1.1), (1.4), (2.1) and (2.2) do not represent explicitly the system of equations for the unknown functions $\boldsymbol{\chi}$, $\mathbf{T}$, $\mathbf{r}$, $\mathbf{s}$ (Eq. (2.1) is not one equation but has to be satisfied by each $\delta\boldsymbol{\chi}$), but are the starting point for obtaining such a system in different special cases.

The Eqs. (1.1) represent the intrisic constraints; in many particular problems we also deal with the boundary constraints — for example, when the deformation function is prescribed on a certain part $\tilde{\partial}\varkappa(\mathscr{B})$ of the boundary. In the latter case we have $\boldsymbol{\chi} = \tilde{\boldsymbol{\chi}}(\mathbf{X}, t)$ for $\mathbf{X} \in \tilde{\partial}\varkappa(\mathscr{B})$; the function $\tilde{\boldsymbol{\chi}}$ is known and has to be admissible by (1.1). It follows that $\delta\boldsymbol{\chi} = \mathbf{0}$ on $\tilde{\partial}\varkappa(\mathscr{B})\times R$.

Now, we shall prove two theorems, which are valid for an arbitrary form of constraints (1.1).

THEOREM 1. *If $\mathscr{B}_0$ is an unconstrained part of the body $\mathscr{B}$, then $\mathbf{r} = \mathbf{0}$ for each $\mathbf{X} \in \varkappa(\mathscr{B}_0)$, $t \in R$*(²).

Putting $\delta\boldsymbol{\chi} = \mathbf{0}$ for each $\mathbf{X} \sim \in \varkappa(\mathscr{B}_0)$, we shall obtain from (2.1) the condition

$$\int_{\chi_t(\mathscr{B})} \varrho\mathbf{r}\cdot\delta\boldsymbol{\chi}\, dv = 0,$$

which has to be satisfied for any field $\delta\boldsymbol{\chi}(\mathbf{X}, t)$, $\mathbf{X} \in \varkappa(\mathscr{B}_0)$, taking zero values on $\partial\varkappa(\mathscr{B}_0)$. By virtue of the du Bois-Reymond lemma, we conclude that $\mathbf{r} = \mathbf{0}$ for each $\mathbf{X} \in \varkappa(\mathscr{B}_0)$.

THEOREM 2. *For a body without intrisic constraints and with boundary constraints on $\tilde{\partial}\varkappa(\mathscr{B})$, only reaction tractions on $\tilde{\partial}\varkappa(\mathscr{B})$ can be different from zero.*

For a body without intrisic constraints we have $\mathscr{B} = \mathscr{B}_0$, and by virtue of Theorem 1 we can reduce (2.1) to the form

$$\int_{\partial\chi_t(\mathscr{B})} \mathbf{s}\cdot\delta\boldsymbol{\chi}\, ds = \int_{\tilde{\partial}\chi_t(\mathscr{B})} \mathbf{s}\cdot\delta\boldsymbol{\chi}\, ds + \int_{\mathring{\partial}\chi_t(\mathscr{B})} \mathbf{s}\cdot\delta\boldsymbol{\chi}\, ds = 0,$$

(²) By a part of a body we always mean the three-dimensional manifold.

where on $\partial\mathring{\chi}_t(\mathscr{B})$ the virtual displacements $\delta\chi$ are arbitrary. Putting $\delta\chi = \mathbf{0}$ on $\mathring{\partial}\chi_t(\mathscr{B})$, we conclude that the integral over $\tilde{\partial}\chi(\mathscr{B})$ must disappear. It follows that

$$\int\limits_{\mathring{\partial}\chi_t(\mathscr{B})} \mathbf{s}\cdot\delta\chi ds = 0$$

holds for each $\delta\chi$. Using the du Bois-Reymond lemma, we shall obtain $\mathbf{s} = \mathbf{0}$ for each $\mathbf{X}\in\mathring{\partial}\varkappa(\mathscr{B})$. Thus we conclude that in the problem considered, only reaction forces on $\tilde{\partial}\chi_t(\mathscr{B})$ can be different from zero.

Theorem 2 implies that the classical continuum mechanics (i.e. mechanics of bodies without intrinsic constraints) constitutes a special case of constrained continuum mechanics.

Let us confine ouerselves, for the time being, to the case in which all constraints are intrinsic and imaginary. To such a constrained body corresponds an unconstrained body with the same mass distribution and the same field of response functional, provided that the global reference configurations of the two bodies coincide. Let us denote by $\{\mathbf{f},\mathbf{g}\}$, where $\mathbf{f} = \mathbf{f}(\mathbf{X}, t)$, $\mathbf{X}\in\bigcup\limits_{a=0}^{r}\varkappa(\mathscr{B}_a)$ and $\mathbf{g} = \mathbf{g}(\mathbf{X}, t)$, $\mathbf{X}\in\bigcup\limits_{a=0}^{r}\partial\varkappa(\mathscr{B}_a)$, the external load system, in which $\mathbf{f}$ is a vector field of body forces and $\mathbf{g}$ is a vector field of surface and contact tractions. Using the same notations as in [1], the pair $\{\chi, \mathbf{T}\}$ will be called the dynamical process.

Using (2.2), we observe that the dynamical process $\{\chi, \mathbf{T}\}$ in a constrained body subjected to the external load system $\{\mathbf{b},\mathbf{p}\}$ can at the same time be treated, as the dynamical process in the corresponding unconstrained body subjected to the external load system $\{\mathbf{b}+\mathbf{r}, \mathbf{p}+\mathbf{s}\}$(3). It follows that if the external load system $\{\mathbf{r},\mathbf{s}\}$ can be disregarded as sufficiently small with respect to the external load system $\{\mathbf{b},\mathbf{p}\}$, then the dynamical process $\{\mathbf{X},\mathbf{T}\}$ obtained from the solution of the constrained body problem can be treated as a good approximation of the dynamical process in the corresponding unconstrained body. To evaluate the approximation, we have to introduce a suitable norm in the linear space of external load systems $\{\mathbf{f},\mathbf{g}\}$. A simple example of such a norm in the static case is given by

$$||\{\mathbf{f},\mathbf{g}\}|| = \Big(\int\limits_{\chi_t(\mathscr{B})} \alpha\mathbf{f}\cdot\mathbf{f}dv + \int\limits_{\chi_t(\mathscr{P})} \beta\mathbf{g}\cdot\mathbf{g}ds\Big)^{\frac{1}{2}},$$

where $\alpha = \alpha(\mathbf{X})$, $\beta = \beta(\mathbf{X})$ are given positive real valued functions, and $\alpha\beta^{-1}$ has a dimension of length. Let us denote by $\delta < 1$ the positive number, assuming that in the problem under consideration each external load system $\{\mathbf{f},\mathbf{g}\}$ can be disregarded with respect to the given load system $\{\mathbf{b},\mathbf{p}\}$ only if $||\{\mathbf{f},\mathbf{g}\}|| < \delta||\{\mathbf{b},\mathbf{p}\}||$; the value of δ depends on the character of the problem and is based on experience. If the value of δ is established, then the dynamical process $\{\mathbf{X},\mathbf{T}\}$ obtained from the solution of the constrained body

(3) By virtue of (2.2) and (1.4) we also conclude that any theorem of classical continuum mechanics will hold for a continuum with imaginary constraints if the system of forces $\{\mathbf{b},\mathbf{p}\}$ is replaced by the system of forces $\{\mathbf{b}+\mathbf{r},\mathbf{p}+\mathbf{s}\}$.

problem can be treated as a good approximation of the dynamical process for the corresponding unconstrained body, provided that there holds the following condition

$$\frac{\|\{\mathbf{r}, \mathbf{s}\}\|}{\|\{\mathbf{b}, \mathbf{p}\}\|} < \delta, \tag{2.3}$$

in which the reaction forces are obtained from the solution of the constrained body problem analysed. In a more general case, the form of constraints (1.1) may be influenced by a certain n-th dimensional vector function which is called control; in this case, the norm $\|\{\mathbf{r}, \mathbf{s}\}\|$ is a functional which expresses the target function and it is required to determine the dynamical process and the control, so that the functional $\|\{\mathbf{r}, \mathbf{s}\}\|$ will attain its minimum. This is a somewhat complicated problem which will be studied separately.

The conclusions given above can easily be generalized to the case in which the boundary constraints on the part $\tilde{\partial}\boldsymbol{\chi}_t(\mathscr{B})$ of the surface $\partial\boldsymbol{\chi}_t(\mathscr{B})$ are prescribed for each t. Denoting by $\tilde{\mathbf{s}} = \mathbf{s}(\mathbf{X}, t)$, $\mathbf{X} \in \tilde{\partial}\boldsymbol{\varkappa}(\mathscr{B})$ the reaction forces in supports maintaining the boundary constraints and putting $\tilde{\mathbf{s}} \equiv \mathbf{0}$ on $\partial\boldsymbol{\varkappa}(\mathscr{P})/\tilde{\partial}\boldsymbol{\varkappa}(\mathscr{B})$, we have to replace the condition (2.3) by the following:

$$\frac{\|\{\mathbf{r}, \mathbf{s}-\tilde{\mathbf{s}}\}\|}{\|\{\mathbf{b}, \mathbf{p}+\mathbf{s}\}\|} < \delta.$$

To conclude this Section, we shall formulate the three theorems which describe the connection between the form of constraints (1.1) and the reaction forces (2.2). To this end we denote by $\mathscr{A}$ the manifold of all functions $\boldsymbol{\chi}(\mathbf{X}, t)$ which are continuous in $\overline{\boldsymbol{\varkappa}(\mathscr{B})} \times R$ and satisfy in each $\boldsymbol{\varkappa}(\mathscr{B}_a) \times R$ the suitable system of equations (1.1).

THEOREM 3. *If for each deformation function $\boldsymbol{\chi} \in \mathscr{A}$, the relation $\boldsymbol{\chi}+\mathbf{c} \in \mathscr{A}$ holds for an arbitrary constant vector* $\mathbf{c}$ *in the physical (reference) space, then the resultant force of all reaction forces acting at the constrained body equals zero.*

If for each $\boldsymbol{\chi} \in \mathscr{A}$ and arbitrary $\mathbf{c}$ we have $\boldsymbol{\chi}+\mathbf{c} \in \mathscr{A}$, then the Eqs. (1.1) are invariant under arbitrary translation of the physical space; it follows that the functions γ_{ia} are independent of argument $\boldsymbol{\chi}$:

$$\frac{\partial \gamma_{ia}}{\partial \boldsymbol{\chi}} = \mathbf{0}, \quad i = 1, \ldots, p_a; \quad a = 1, \ldots, r. \tag{2.4}$$

From (2.4) and (1.2), we shall obtain the following equations for virtual displacements:

$$\frac{\partial \gamma_{ia}}{\partial \nabla \boldsymbol{\chi}} \cdot \nabla(\delta\boldsymbol{\chi}) + \ldots + \frac{\partial \gamma_{ia}}{\partial \nabla^k \boldsymbol{\chi}} \cdot \nabla^k(\delta\boldsymbol{\chi}) = 0. \tag{2.5}$$

The Eqs. (2.5) are satisfied for each $\delta\boldsymbol{\chi} = \mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector in the physical space. Substituting $\delta\boldsymbol{\chi} = \mathbf{c}$ into the Eq. (2.1), we shall satisfy the relation obtained only if

$$\int_{\chi_t(\mathscr{P})} \mathbf{s}\, ds + \int_{\chi_t(\mathscr{B})} \varrho \mathbf{r}\, dv = \mathbf{0} \tag{2.6}$$

which ends the proof.

THEOREM 4. *If for each deformation function $\chi \in \mathscr{A}$, the relation $\mathbf{Q}\chi \in \mathscr{A}$ holds for an arbitrary orthogonal tensor* $\mathbf{Q}$ *in the physical space, then the resultant moment of all reaction forces acting at the constrained body equals zero.*

If for each $\mathbf{X} \in \mathscr{A}$ and for an arbitrary orthogonal tensor $\mathbf{Q}$, we can write $\mathbf{Q}\chi \in \mathscr{A}$, then the Eqs. (1.1) are invariant under all orthogonal transformation of the physical space. The functions γ_{ia} are then hemitropic in the physical space — i.e., they satisfy the conditions

$$\frac{\partial \gamma_{ia}}{\partial \chi^{[n}}\chi_{n]} + \frac{\partial \gamma_{ia}}{\partial \chi^{[n}_{,a}}\chi_{n],a} + \ldots = 0, \tag{2.7}$$

where χ^m are components of χ, and where the index preceded by a comma denotes partial differentiation with respect to material coordinates (all tensor indices run over the sequence 1, 2, 3, the summation convention holds). From (2.7) and (1,2) it follows that $\delta\chi = \mathbf{E}\chi$, where $\mathbf{E}$ is an arbitrary skew-symmetric matrix and χ is an arbitrary deformations function, $\chi \in \mathscr{A}$. Substituting $\delta\chi = \mathbf{E}\chi$ into the Eq. (2.1), we shall satisfy the resulting equation for each $\mathbf{E}$ only if

$$\int\limits_{\chi_t(\mathscr{P})} \mathbf{s}\times\chi ds + \int\limits_{\chi_t(\mathscr{B})} \varrho\mathbf{r}\times\chi dv = 0, \tag{2.8}$$

which ends the proof of Theorem 4.

THEOREM 5. *If for each deformation function $\chi(\mathbf{X}, t) \in \mathscr{A}$, the relation $\chi(\mathbf{X}, t+c) \in \mathscr{A}$ holds for an arbitrary constant c, then the work done by all reaction forces acting at the constrained body equals zero.*

If for each $\chi(\mathbf{X}, t) \in \mathscr{A}$ we also have $\chi(\mathbf{X}, t+c) \in \mathscr{A}$, then the functions γ_{ia} are independent of argument t:

$$\frac{\partial \gamma_{ia}}{\partial t} = 0. \tag{2.9}$$

From (2.9) and (1.1) it follows that

$$\frac{\partial \gamma_{ia}}{\partial \chi}\cdot\dot{\chi} + \frac{\partial \gamma_{ia}}{\partial \nabla\chi}\cdot\dot{\overline{\nabla\chi}} + \ldots = \frac{d\gamma_{ia}}{dt} = 0. \tag{2.10}$$

Comparing (2.10) and (1.2), we conclude that if (2.9) holds then we can put $\delta\chi = \dot{\chi}$ for any admissible motion $\chi \in \mathscr{A}$. Thus we obtain the following relation:

$$\int\limits_{\chi_t(\mathscr{S})} \mathbf{s}\cdot\dot{\chi} ds + \int\limits_{\chi_t(\mathscr{B})} \varrho\mathbf{r}\cdot\dot{\chi} dv = 0, \tag{2.11}$$

which ends the proof of Theorem 5.

3. Principles of conservation

If the mechanics of constrained continua is based on the dynamical principle of virtual work (1.3), then the principle of conservation of momentum and that of moment of momentum are not axioms of the theory, but they may be proved, assuming that suitable conditions hold.

3.1. Principle of conservation of momentum.

For each part $\mathscr{P} \subset \mathscr{B}_a$, and an arbitrary $\mathscr{B}_a$, $a = 0, 1, \dots, r$, the following principle of conservation of momentum holds:

$$(3.1) \qquad \frac{d}{dt} \int_{\chi_t(\mathscr{P})} \varrho \dot{\boldsymbol{\chi}} dv = \int_{\partial\chi_t(\mathscr{P})} \mathbf{t}_{(\mathbf{n})} ds + \int_{\chi_t(\mathscr{P})} \varrho(\mathbf{b}+\mathbf{r}) dv, \qquad \mathbf{t}_{(\mathbf{n})} \equiv \mathbf{Tn}.$$

Equation (3.1) can be obtained directly from $(2.2)_1$, using the divergence theorem and taking into account that ϱ is a scalar density in the physical space (i.e., that the equation of continuity holds).

THEOREM 6. *If the conditions* (2.4) *hold, then the principle of conservation of momentum for the whole constrained continuum has the form*

$$(3.2) \qquad \frac{d}{dt} \int_{\chi_t(\mathscr{B})} \varrho \dot{\boldsymbol{\chi}} dv = \int_{\partial\chi_t(\mathscr{B})} \mathbf{p} ds + \int_{\chi_t(\mathscr{B})} \varrho \mathbf{b} dv,$$

being independent of the reaction forces.

To prove the theorem, we have to observe that an arbitrary constant vector $\mathbf{c}$ in the physical space constitutes now the virtual displacement (cf. Theorem 3). Substituting $\delta\boldsymbol{\chi} = \mathbf{c}$ into the Eq. (1.3) and after simple transformations, we arrive at the condition (3.2).

Using (3.1) and $(2.2)_{2,3}$ we can prove that if the condition (2.4) is not satisfied, then on the right-hand side of the Eq. (3.2) there will be an additive term expressing the resultant of all reaction forces acting on the body (cf. also Theorem 3).

3.2. Principle of conservation of moment of momentum.

For each part $\mathscr{P} \subset \mathscr{B}_a$ and an arbitrary $\mathscr{B}_a$, $a = 0, 1, \dots, r$, the following principle of conservation of moment of momentum holds:

$$(3.3) \qquad \frac{d}{dt} \int_{\chi_t(\mathscr{P})} \varrho \dot{\boldsymbol{\chi}} \times \boldsymbol{\chi} dv = \int_{\partial\chi_t(\mathscr{P})} \mathbf{t}_{(\mathbf{n})} \times \boldsymbol{\chi} ds + \int_{\chi_t(\mathscr{P})} \varrho(\mathbf{b}+\mathbf{r}) \times \boldsymbol{\chi} dv,$$

provided that the Cauchy stress tensor is symmetric: $\mathbf{T} = \mathbf{T}^T$. The principle of conservation (3.3) can be obtained by constituting the vector product of the Eq. $(2.2)_1$ with the position vector $\boldsymbol{\chi}$, and by integrating the equation obtained over the region $\boldsymbol{\chi}_t(\mathscr{P})$; the divergence theorem, the continuity equation and the condition $\mathbf{T} = \mathbf{T}^T$ have also to be taken into account.

THEOREM 7. *If the equations of constraints* (1.1) *are invariant under arbitrary orthogonal transformations of the physical space, and if the Cauchy stress tensor is symmetric, then the principle of conservation of momentum for the whole constrained continuum has the form:*

$$(3.4) \qquad \frac{d}{dt} \int_{\chi_t(\mathscr{B})} \varrho \dot{\boldsymbol{\chi}} \times \boldsymbol{\chi} dv = \int_{\partial\chi_t(\mathscr{B})} \varrho \times \boldsymbol{\chi} ds + \int_{\chi_t)\mathscr{B})} \varrho \mathbf{b} \times \boldsymbol{\chi} dv,$$

being independent of the reaction forces.

The Eqs. (1.1) are invariant under arbitrary orthogonal transformation of the physical space if the relation (2.7) holds; it follows that $\delta\chi = \mathbf{E}\chi$, for each $\chi \in \mathscr{A}$ and each antisymmetric tensor $\mathbf{E}$. Substituting $\delta\chi = \mathbf{E}\chi$ into (1.3), we shall observe that the right-hand side of the resulting equation vanishes for the symmetric stress tensor $\mathbf{T}$. The left-hand side of this equation also being equal to zero, simple manipulations lead to (3.4).

Using (3.3) and $(2.2)_{2,3}$ we can also prove that if the Cauchy stress tensor is symmetric but the equations of constraints (1.1) do not satisfy the condition (2.7), then on the right-hand side of the Eq. (3.4) the moments of all reaction forces will also be present (cf. Theorem 5).

Let us denote by P the rate at which the stresses do work (the stress power) per unit volume of the body in the actual configuration:

$$P = \mathbf{T}\cdot\mathbf{D}; \quad \mathbf{D} \equiv \frac{1}{2}[\operatorname{grad}\dot{\chi} + (\operatorname{grad}\dot{\chi})^T], \quad \operatorname{grad}\dot{\chi} \equiv \nabla\dot{\chi}(\nabla\chi)^{-1}. \tag{3.5}$$

Constituting the scalar product of the Eqs. $(2.2)_1$ with the vector $\dot{\chi}$, after simple calculations we arrive at the relation:

$$\frac{d}{dt}\int_{\chi_t(\mathscr{P})} \frac{1}{2}\varrho|\dot{\chi}|^2 dv + \int_{\chi_t(\mathscr{P})} P\,dv = \oint_{\partial\chi_t(\mathscr{P})} \mathbf{t}_{(\mathbf{n})}\cdot\dot{\chi}\,ds + \int_{\chi_t(\mathscr{P})} \varrho(\mathbf{b}+\mathbf{r})\cdot\dot{\chi}\,dv, \tag{3.6}$$

which holds for an arbitrary $\mathscr{P}$, $\mathscr{P} \subset \mathscr{B}_a$, $a = 0, 1, \dots, r$. The Eq. (3.6) can be called the principle of conservation of the kinetic energy, represented by the first term on the left-hand side of (3.6).

THEOREM 8. *If the conditions* (2.9) *hold, then the principle of conservation of the kinetic energy for the whole constrained continuum has the form*:

$$\frac{d}{dt}\int_{\chi_t(\mathscr{B})} \frac{1}{2}\varrho|\dot{\chi}|^2 dv + \int_{\chi_t(\mathscr{B})} P\,dv = \oint_{\partial\chi_t(\mathscr{B})} \mathbf{p}\cdot\dot{\chi}\,ds + \int_{\chi_t(\mathscr{B})} \varrho\mathbf{b}\cdot\dot{\chi}\,dv, \tag{3.7}$$

being independent of the reaction forces.

By virtue of (2.9) we have $\delta\chi = \dot{\chi}$ for any admissible motion χ, $\chi \in \mathscr{A}$ (cf. the proof of Theorem 5). Substituting $\delta\chi = \dot{\chi}$ into (1.3), we obtain finally the condition (3.7).

If the condition (2.9) does not hold, then the rate at which the reaction forces perform work has to be added to the right-hand side of the Eq. (3.7); this result can be obtained using (3.6) and $(2.2)_{2,3}$.

The connection of the theorems 6, 7, 8 with the theorems 3, 4, 5, respectively, can be seen.

4. Lagrange's equations of the first kind

The axioms of the constrained continuum mechanics, which were given in Sec. 1, do not represent, in explicit form, the system of equations for unknown functions. In this section, we shall obtain such equations in the form which corresponds to the form of Lagrange equations of the first kind, well known in analytical mechanics. Thus the equations we are to obtain will also be called Lagrange equations of the first kind for the constrained continuum. To simplify the calculations, we shall confine ourselves to the constraints of the form.

$$\gamma_i(\mathbf{X}, t, \boldsymbol{\chi}, \nabla\boldsymbol{\chi}, \ldots, \nabla^k\boldsymbol{\chi}) = 0; \quad i = 1, 2, \ldots, p; \quad \mathbf{X} \in \varkappa(\mathscr{B}), t \in R, \tag{4.1}$$

which are given for the whole constrained body. The special case of (4.1), in which $k = 1$ (simple constraints), has received a great deal of attention in many papers on incompressible bodies or bodies reinforced with inextensible cords (cf. [1, 2]).

We shall use the index notation of the tensor calculus; indices $\alpha, \alpha_1, \beta, \gamma, \gamma_1, \ldots$, will refer to material coordinates (components of the vector $\mathbf{X} \in \varkappa(\mathscr{B})$) and "$m$" will refer to ortonormal Carthesian coordinates in the physical space; all tensor indices run over the sequence $1, 2, 3$. Moreover.

$$a^{\beta^L} \equiv a^{\beta_1\beta_2\ldots\beta_L}, \; f_{\alpha^L} \equiv f_{,\alpha_1\alpha_2\ldots\alpha_L} \quad \text{for} \quad L = 1, 2, \ldots, k,$$

and $f_{,\alpha^L} \equiv f$, $a^{\beta^L} \equiv a$ for $L = 0$.

Let us denote by $\lambda^i = \lambda^i(\mathbf{X}, t)$, $\mathbf{X} \in \varkappa(\mathscr{B})$, $t \in R$, $i = 1, 2, \ldots, p$, Lagrange multipliers not determined for the time being. Multiplying the Eqs. (1.2) termwise by scalar multipliers λ^i and taking into account the form (4.1) of constraints, we shall write the sum:

$$\sum_{i=1}^{p} \sum_{L=0}^{k} \lambda_i \frac{\partial \gamma_i}{\partial \chi^m{}_{,\alpha^L}} (\delta\chi^m)_{,\alpha^L} = 0. \tag{4.2}$$

After many manipulations, we shall transform (4.2) to the form

$$(a_m + N_m{}^{\alpha}{}_{,\alpha})\,\delta\chi^m - \Big(\sum_{L=0}^{k-1} N_m{}^{\alpha\beta^L}\,\delta\chi_{m,\beta^L}\Big)_{,\alpha} = 0, \tag{4.3}$$

where we have denoted:

$$a_m \equiv \sum_{i=1}^{p} \lambda^i \frac{\partial \gamma_i}{\partial \chi^m}, \qquad N_m{}^{\alpha\beta^L} \equiv -\sum_{i=1}^{p} \sum_{M=0}^{k-1-L} (-1)^M \left(\lambda^i \frac{\partial \gamma_i}{\partial \chi^m{}_{,\alpha\beta^L\gamma^M}}\right)_{,\gamma^M}, \tag{4.4}$$

$$0 \leqslant L \leqslant k-1.$$

Let us integrate (4.3) over the region $\varkappa(\mathscr{B})$ occupied by the body in the reference configuration. Adding termwise the resulting equation to the Eq. (2.1) (we have to remember that $r = 1$, $\mathscr{B}_1 = \mathscr{B}$ and $\mathscr{B}_0 = \phi$), we obtain:

$$\int_{\varkappa(\mathscr{B})} (\varrho_R r_m - a_m - N_m{}^{\alpha}{}_{,\alpha})\,\delta\chi^m\, dv_R + \int_{\partial\varkappa(\mathscr{B})} \Big(s_m\,\delta\chi^m + \sum_{L=0}^{k-1} N_m{}^{\alpha\beta^L} \delta\chi^m{}_{,\beta^L} n_\alpha\Big) ds_R = 0, \tag{4.5}$$

where ϱ_R, $\mathbf{s}_R \equiv (s_m)$, $\mathbf{n}_R \equiv (n_\alpha)$ are, respectively, the mass density in $\varkappa$, the surface reaction traction related to $\varkappa$ and the unit vector normal to $\partial\varkappa(\mathscr{B})$. Since in the Eq. (4.5) there are p undetermined Lagrange multipliers, then we can treat the virtual displacements $\delta\boldsymbol{\chi}$ in (4.5) as, from the formal point of view, as arbitrary differentiablle functions independent of each other. It follows that the Eq. (4.5) holds for all functions $\delta\boldsymbol{\chi}$ that satisfy on the boundary $\partial\varkappa(\mathscr{B})$ the conditions $\delta\chi^m{}_{,\alpha^L} = 0$, $L = 0,1,, \ldots, k-1$. Thus the Eq. (4.5) will be satisfied only if the volume integral in (4.5) equals zero:

$$\int_{\varkappa(\mathscr{B})} (\varrho_R r_m - a_m - N_m{}^\alpha{}_{,\alpha}) \delta\chi^m dv_R = 0. \tag{4.6}$$

Using the du Bois-Reymond lemma, we obtain: $\varrho_R r_m = a_m + N_m{}^\alpha{}_{,\alpha}$ or

$$\varrho_R r_m = \sum_{i=1}^{p} \sum_{M=0}^{k} (-1)^M \left(\lambda^i \frac{\partial\gamma_i}{\partial\chi^m{}_{,\alpha^M}} \right)_{,\alpha^M}. \tag{4.7}$$

Because of $\varrho = \det(\nabla\chi)\varrho_R$, we also have

$$\varrho\mathbf{r} = \det(\nabla\chi) \sum_{i=1}^{p} \sum_{M=0}^{k} (-1)^M \nabla^M \left(\lambda^i \frac{\partial\gamma_i}{\partial\nabla^M\chi} \right). \tag{4.8}$$

Thus the basic system of the field equations for the constrained body under consideration is given by the equations of constraints (4.1) and by the equations of motion (cf. Eqs. $(2.2)_1$):

$$\operatorname{div}\mathbf{T} + \varrho\mathbf{b} + \varrho\mathbf{r} = \varrho\ddot{\chi}, \tag{4.9}$$

where the stress tensor $\mathbf{T}$ is related to the deformation $\chi(\mathbf{X}, t)$ by means of the constitutive equations (1.4), and the body reaction force $\mathbf{r}$ is related to the deformation and to the Lagrange's multipliers λ^i, $i = 1, \ldots, p$, by means of the Eq. (4.8). This is a system of $3+p$ equations for $3+p$ unknown functions $\chi^m(\mathbf{X}, t)$, $m = 1, 2, 3$, $\lambda^i(\mathbf{X}, t)$, $i = 1, 2, \ldots, p$, and can be determined only if $p \leqslant 2$; in the case $p > 3$ the Eq. (4.1) are over-determined with respect to the unknowns χ^1, χ^2, χ^3, and the eouations (4.9) (after taking into consideration (1.4), (4.8)) are under-determined with respect to the unknowns $\lambda^1, \ldots, \lambda^p$.

Let us investigate now the boundary conditions for the system of field equations (4.1), (4.9), (1.4), (4.8). Let on the portion $\tilde{\partial}\varkappa(\mathscr{B})$ of the boundary $\partial\varkappa/(\mathscr{B})$ the values of functions $\chi^m{}_{,\beta^L}$, $L = 0, 1, \ldots, k-1$, be known (these values cannot be prescribed independently of each other). It follows that $\delta\chi^m{}_{,\beta^L} = 0, L = 0, 1, \ldots, k-1$, on $\tilde{\partial}\varkappa(\mathscr{B})$. Thus the relation (4.5), after taking into account (4.6), reduces to the form:

$$\int_{\mathring{\partial}\varkappa(\mathscr{B})} s_m \delta\chi^m ds_R + \int_{\mathring{\partial}\varkappa(\mathscr{B})} \sum_{L=0}^{k-1} N_m{}^{\alpha\beta^L} \delta\chi^m{}_{,\beta^L} n_\alpha ds_R = 0, \tag{4.10}$$

where on the portion $\mathring{\partial}\varkappa(\mathscr{B})$ of the boundary $\partial\varkappa(\mathscr{B})$ the external surface tractions $\mathbf{p}(\mathbf{X}, t)$, $\mathbf{X} \in \mathring{\partial}\varkappa(\mathscr{B})$ are prescribed. To simplify the calculations, let us confine ourselves to the cases $k = 2, k = 1$.

4.1. The case $\mathbf{k = 2}$.

Let us denote by $\mathbf{n}_R = (n_\alpha)$ the unit normal to $\partial\varkappa(\mathscr{B})$ and let $\partial\chi \equiv \nabla\chi\mathbf{n}_R$ be a normal derivative on the boundary. Morever, let $\xi: \mathring{\partial}\varkappa(\mathscr{B}) \to R^2$ be a local coordinate system on $\partial\varkappa(\mathscr{B})$, where $\xi = (\xi^A)$ and the index "A" runs over the sequence $1, 2$. We shall also denote by $\mathbf{a}^A = (a^A_\alpha)$ the contravariant base vectors on $\mathring{\partial}\varkappa(\mathscr{B})$. Using these denotations we are able, after some calculations, to transform (4.10) to the form

$$\int\limits_{\mathring{\partial}\varkappa(\mathscr{B})} \{[s_m + N_m{}^\alpha n_\alpha - (N_m{}^{\alpha\beta} a^A_\beta n_\alpha)|_A]\,\delta\chi^m + N_m{}^{\alpha\beta} n_\alpha n_\beta \partial(\delta\chi^m)\}\, ds_R + \oint\limits_{\partial[\mathring{\partial}\varkappa(\mathscr{B})]} N_m{}^{\alpha\beta} n_\alpha u_\beta \delta\chi^m d\lambda_R = 0, \tag{4.11}$$

where $\mathring{\partial}[\partial\varkappa(\mathscr{B})]$ is a boundary of $\mathring{\partial}\varkappa(\mathscr{B})$ with the unit normal $\mathbf{u} = (u_\alpha)$ (vector $\mathbf{u}$ being tangent to $\mathring{\partial}\varkappa(\mathscr{B})$), and where the vertical line denotes the covariant differentiaton on $\mathring{\partial}\varkappa(\mathscr{B})$ with respect to the surface coordinates ξ^A and in the metric $\mathbf{a}_A \cdot \mathbf{a}_B$, $\mathbf{a}_A$ being covariant base vectors on $\mathring{\partial}\varkappa(\mathscr{B})$. Since on the portion $\tilde{\partial}\varkappa(\mathscr{B})$ of the boundary $\partial\varkappa(\mathscr{B})$ the values of functions $\boldsymbol{\chi}$ and $\partial\boldsymbol{\chi}$ are known, then $\delta\boldsymbol{\chi} = \mathbf{0}$ on $\tilde{\partial}\varkappa(\mathscr{B})$ and the last term in (4.11) must be equal to zero, provided that $\delta\boldsymbol{\chi}$ are continuous. Moreover, the functions $\delta\boldsymbol{\chi}$ and $\partial(\delta\boldsymbol{\chi})$ on $\tilde{\partial}\varkappa(\mathscr{B})$ may be treated as independent of each other and arbitrary when the Lagrange multipliers approach is applied. It follows that on $\mathring{\partial}\varkappa(\mathscr{B})$ the following conditions hold:

$$s_m = -N_m{}^\alpha n_\alpha + (N_m{}^{\alpha\beta} a^A_\beta n_\alpha)|_A, \quad N_m{}^{\alpha\beta} n_\alpha n_\beta = 0. \tag{4.12}$$

Using $(2.2)_2$ we arrive finally at the following boundary conditions:

$$\begin{aligned} &(T_m{}^\alpha + N_m{}^\alpha) n_\alpha - (N_m{}^{\alpha\beta} a^A_\beta n_\alpha)|_A = p^R_m, \quad N_m{}^{\alpha\beta} n_\alpha n_\beta = 0 \quad \text{on} \quad \mathring{\partial}\varkappa(\mathscr{B}) \times R, \\ &\boldsymbol{\chi} = \tilde{\boldsymbol{\chi}}, \quad \partial\boldsymbol{\chi} = \tilde{\boldsymbol{\psi}} \quad \text{on} \quad \tilde{\partial}\varkappa(\mathscr{B}) \times R, \end{aligned} \tag{4.13}$$

where $\tilde{\boldsymbol{\chi}}$, $\tilde{\boldsymbol{\psi}}$ are known. All quantities in $(4.13)_1$ are related to the reference configuration. Since the equation of motion (4.9) can also be written in the form

$$(T_m{}^\alpha + N_m{}^\alpha)_{,\alpha} + \varrho_R b_m + a_m = \varrho_R \ddot{\chi}_m, \tag{4.14}$$

then the sums $T_m{}^\alpha + N_m{}^\alpha$ can be interpreted, from the formal point of view, as "total" stress components if the conditions $(N_m{}^{\alpha\beta} a^A_\beta n_\alpha)|_A = 0$ hold. The quantities in the Eqs. (4.11)–(4.14) are related to the reference configuration.

4.2. The case $\mathbf{k} = 1$.

This is a case of simple constraints $\gamma_i(\mathbf{X}, t, \boldsymbol{\chi}, \nabla\boldsymbol{\chi}) = 0$. The boundary conditions (4.13) reduce to the form:

$$\begin{aligned} (T_m{}^\alpha + N_m{}^\alpha) n_\alpha &= p^R_m \quad \text{on} \quad \mathring{\partial}\varkappa(\mathscr{B}) \times R, \\ \boldsymbol{\chi} &= \tilde{\boldsymbol{\chi}} \quad \text{on} \quad \tilde{\partial}\varkappa(\mathscr{B}) \times R, \end{aligned} \tag{4.15}$$

and the sums $T_m{}^\alpha + N_m{}^\alpha$ can always be interpreted, from the formal point of view, as components of a certain "total" stress tensor. Such an approach is commonly used if the simple constraints are taken into account; the stress tensor $\mathbf{T}$ determined by the history of the deformation gradient, in accordance with (1.4), was called in [1] "the extra stress" and the tensor $\mathbf{N} \equiv (JN_m{}^\alpha \chi_{n,\alpha})$ was interpreted as a part of stress that does no work in any motion satisfying the constraints. The latter condition holds only if the functions γ_i do not depend explicitly on t and $\boldsymbol{\chi}$; it follows that $a_m = 0$ in the equations of motion

(4.14). Moreover, if the functions γ_i are also invariant under arbitrary rotation of the physical space, then the tensor $\mathbf{N}$ in the case of simple constraints is symmetric. This special case of simple constraints, in which they are assumed to be frame indifferent, has been investigated in several papers (cf. [1, 2]).

The Eqs. (4.14), which are valid for an arbitrary k, where N_m^{α}, a_m are given by (4.4), will be called Lagrange equations of the first kind for the constrained continuum. The alternative form of these equations, which is related not to the reference but to the actual configuration, will be obtained after substituting the right-hand side of (4.8) into (4.9).

5. Examples of constrained continua

The well known examples of constrained continua are those of incompressible bodies (given by $\det \nabla\boldsymbol{\chi} - 1 = 0$), bodies which are inextensible in the direction of $\mathbf{e}$ in $\boldsymbol{\varkappa}$ (equations of constraints have the form $(\nabla\boldsymbol{\chi}^T\nabla\boldsymbol{\chi}\mathbf{e})\cdot\mathbf{e} - 1 = 0$) and rigid bodies, where $\nabla\boldsymbol{\chi}^T\nabla\boldsymbol{\chi} - \mathbf{1} = \mathbf{0}$, [1]. Those constraints are simple and real. An example of simple imaginary constraints can be given by a rod-like body with cross-sections which are inextensible; denoting by $\mathbf{e}$ a vector normal to the cross-section and putting $\mathbf{e}_A\cdot\mathbf{e} = 0$, $(\nabla\mathbf{e})\mathbf{e}_A = \mathbf{0}$, we shall obtain the equations of constraints $(\nabla\boldsymbol{\chi}^T\nabla\boldsymbol{\chi}\mathbf{e}_A - \mathbf{e}_A)\cdot\mathbf{e}_B = 0$; $A, B \in \{1, 2\}$. Examples of non-simple constraints given by $\nabla(\nabla\boldsymbol{\chi}\mathbf{e})\mathbf{e} = 0$, where $(\nabla\mathbf{e})\mathbf{e} = 0$, are used in the theory of shells, $\mathbf{e}$ being the direction of the material fibre normal to the midsurface of the shell in the reference configuration. In the separate parts (which are called the finite elements) of discretized bodies, we introduce constraints of the form $\nabla(\nabla\boldsymbol{\chi}^T\nabla\boldsymbol{\chi}) = \mathbf{0}$. In rod-like bodies which are assumed to preserve plane cross-sections, normal to the vector $\mathbf{e}$ in the reference configuration $\boldsymbol{\varkappa}$, we have to introduce the constraints $\nabla(\nabla\boldsymbol{\chi}\mathbf{e})\mathbf{e}_A = 0$, $(\nabla\boldsymbol{\chi}\mathbf{e})\cdot\mathbf{e}_A = 0$. In shell-like bodies preserving straight-line material fibres in the direction given by a vector $\mathbf{e}$ in $\boldsymbol{\varkappa}$, the constraints have the form $\nabla(\nabla\boldsymbol{\chi}\mathbf{e})\mathbf{e} = \mathbf{0}$, where $(\nabla\mathbf{e})\mathbf{e} = \mathbf{0}$. The special cases of constraints given above are of great importance mainly in engineering mechanics and will be studied in a separate communication. Other kinds of constraints, including integrable constraints, we shall investigate in a further part of this paper.

Appendix

Alternative form of basic axioms for the constrained continuum mechanics

The formulation of the mechanics of a constrained continuum is given in Sec. 1 in terms of primitive concepts $\mathscr{B}, \boldsymbol{\chi}, \varrho, \mathbf{b}, \mathbf{p}, \mathbf{T}$ and the axioms given by the equation of constraints (1.1), the dynamical principle (1.3) and the stress relation (1.1). Now, we shall give an alternative formulation of constrained continuum mechanics, by introducing other primitive concepts and basic axioms; both approaches, however, will be equivalent.

As the primitive concepts we shall take $\mathscr{B}, \boldsymbol{\chi}, \varrho$ and the system of forces which will be characterized by the conditions (i) and (ii) of Sec. 1 and by the following two conditions: (iii). There exist in $\chi_t(\mathscr{B}_a)$, $a = 0, 1, \ldots, r$ the vector valued functions $\mathbf{t}_{(n)}(\mathbf{X}, t)$, defined for each unit vector $\mathbf{n}$, which are called stress vectors. The stress vector is interpreted as acting across the oriented surface element with normal $\mathbf{n}$, which is situated at the place $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ in the physical space.

(iv) There exist fields $\mathbf{r} = \mathbf{r}(\mathbf{X}, t)$; $\mathbf{X} \in \varkappa(\mathscr{B}_a)$; $a = 0, 1, \ldots, r$; $t \in R$, of the density per unit mass of the reaction body force.

As the kinematic axiom we shall take the existence of manifold $\mathscr{A}$ of admissible motions, $\boldsymbol{\chi} \in \mathscr{A}$, which is given a priori by the equations of constraints (1.1). More general manifolds may also be taken into account.

As the dynamical axioms, we shall take the principle of momentum (3.1) and that of moment of momentum (3.3).

We shall also assume that the stress relation holds in the form (1.4); to this end we have to prove, by means of the two dynamical axioms, the existence of the symmetric Cauchy stress tensor $\mathbf{T}$ such that $\mathbf{t}_{(\mathbf{n})} = \mathbf{T}\mathbf{n}$.

From the principle of momentum the equations of motion follow in the form $(2.2)_1$. We shall also define the fields of the surface reaction forces by means of $(2.2)_{2,3}$.

The meaning of the system of reaction forces $\{\mathbf{r}, \mathbf{s}\}$ is obvious; it has to maintain the motion $\boldsymbol{\chi}$ of the body in the manifold $\mathscr{A}$ of admissible motions([4]). If the manifold $\mathscr{A}$ constitutes an uncountable set of admissible motions, then there is an infinite number of systems of reaction forces which may satisfy this condition. Thus, following the known approach of analytical mechanics, we assume that the constraints are ideal — i.e., that (2.1) holds for every $\delta\boldsymbol{\chi}$; this is final axiom of the formulation of the constrained continuum mechanics given in this Appendix.

References

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Received June 29, 1973.

([4]) Following [1] we denote motion $\mathscr{B} \times R \to E^3$ and the deformation function $\varkappa(\mathscr{B}) \times R \to E^3$ by the same symbol $\boldsymbol{\chi}$.