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# Lower bounds on bearing capacity of shells and plates loaded at the edges by distributed moments

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PRESENTED here is the theory which enables finding the lower bounds on the bearing capacity of thin-walled shells and plates loaded by continuously distributed bending and twisting moments. By way of illustration, three particular problems are solved.

Podano teorię dla wyznaczania dolnej oceny nośności granicznej cienkościennych powłok i płyt obciążonych na krawędzi rozłożonymi w sposób ciągły momentami zginającymi i skręcającymi. Jako ilustrację przedstawiono kilka rozwiązań szczegółowych.

Дается теория для определения нижней оценки предельной нагрузки тонкостенных оболочек и плит, нагруженных по краям распределенными, непрерывным образом, изгибающими и скручивающими моментами. Для иллюстрации представлено несколько подробных решений.

## 1. Introduction

THE COMPLETE solution to the problem of bearing capacity of shells and plates should satisfy all static and kinematic conditions. Such a complete solution consists in determining the statically admissible plastic stress field and the kinematically admissible collapse mechanism corresponding to this field. Since the system of equations of the theory of bearing capacity of shells and plates is complex, construction of complete solutions is in most practical cases very difficult. Therefore, only solutions to the relatively simple cases are available. Thus the main effort has been devoted to incomplete solutions, providing lower and upper estimates of the unknown true value of the bearing capacity. The practical importance of these incomplete solutions results from the two basic limit design theorems of plasticity [1, 2]. In more general cases the method of kinematically admissible collapse mechanisms providing upper bounds on the limit load has been proved effective. The upper estimate of the bearing capacity is, however, on the unsafe side and, therefore, construction of the statically admissible stress systems providing lower bounds is always expedient, since it makes it possible to obtain the estimate of the possible excess of the two bounds from the unknown exact value of the limit load.

In the previous work [3] the concept of discontinuous piecewise homogeneous statically admissible plastic fields of moments was proposed for determining the lower bounds on limit load for plates of complex configuration subject to pure bending. In the present work, the concept of nonhomogeneous statically admissible continuous plastic fields of moments is introduced. Using such fields, we can find estimates of lower bounds on the bearing capacity for thin-walled shells of arbitrary double curvature loaded at the edges by distributed moments. As the particular case, the fields of moments for plates are obtained from the general theory.

## 2. Plastic fields of moments for shells

Consider a thin-walled shell of arbitrary double curvature. The shape of the shell is described in the Cartesian coordinate system by the parametric equations:

$$x = x(\alpha, \beta), \quad y = y(\alpha, \beta), \quad z = z(\alpha, \beta).$$

The two parameters $\alpha$ and $\beta$ can be considered as the curvilinear system of coordinates on the shell surface. In our considerations, this system will coincide with the lines of principal curvatures of the shell.

Assume that the external loading of the shell is limited to the continuously distributed moments acting at the edges. We may assume, therefore, the internal forces to be reduced

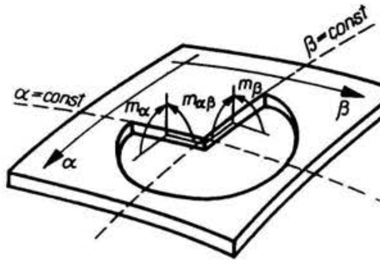


FIG. 1.

to the bending moments $m_\alpha$, $m_\beta$ and the twisting moment $m_{\alpha\beta}$ acting in any cross-section along an $\alpha$ = const or a $\beta$ = const line, respectively (Fig. 1).

The conditions of internal equilibrium are expressed by the equations

$$(2.1)\quad \begin{aligned} &(m_\beta - m_\alpha)\frac{\partial A}{\partial \beta} + A\frac{\partial m_\beta}{\partial \beta} + 2m_{\alpha\beta}\frac{\partial B}{\partial \alpha} + B\frac{\partial m_{\alpha\beta}}{\partial \alpha} = 0, \\ &(m_\alpha - m_\beta)\frac{\partial B}{\partial \alpha} + B\frac{\partial m_\alpha}{\partial \alpha} + 2m_{\alpha\beta}\frac{\partial A}{\partial \beta} + A\frac{\partial m_{\alpha\beta}}{\partial \beta} = 0, \end{aligned}$$

where $A$ and $B$ are the coefficients of the first quadratic form. This means that the length $ds$ of an infinitisemal element on the shell surface is determined by the equation:

$$ds^2 = A^2 d\alpha^2 + B^2 d\beta^2.$$

We discuss below separately the theory of statically admissible plastic fields of moments for the Huber-Mises and for the Tresca yield criteria.

### 2.1. Fields of moments for the Huber-Mises yield criterion

According to the Huber-Mises yield criterion, the plastic state of a cross-section of the shell is reached if the following equation is satisfied:

$$(2.2)\quad m_\alpha^2 - m_\alpha m_\beta + m_\beta^2 + 3m_{\alpha\beta}^2 = m_0^2,$$

where $m_0$ denotes the limit moment for pure bending. For shell of thickness $2h$, the limit moment is equal to $m_0 = \sigma_{pl} h^2$, where $\sigma_{pl}$ is the yield locus of the material. If a sandwich plate is considered, and if the distance between the two layers each of the thickness $\delta$ is equal to $2h$, the limit moment has the value $m_0 = 2\sigma_{pl}\delta h$.

The two equations of equilibrium (2.1) and the yield criterion (2.2) constitute a system of three equations with three unknowns functions $m_\alpha$, $m_\beta$ and $m_{\alpha\beta}$. Let us introduce an auxiliary function $\omega(\alpha, \beta)$, defined by the following relations:

$$m_1+m_2 = 2m_0\cos\omega, \quad m_1-m_2 = \frac{2}{\sqrt{3}} m_0\sin\omega; \tag{2.3}$$

$m_1$ and $m_2$ are the principal bending moments. The subscript notations have been so chosen that always $m_1 \geqslant m_2$.

If the angle which makes the normal to the cross-section where the greater principal moment $m_1$ is acting with the local direction of the $\alpha = \text{const}$ line is denoted by $\varphi$ (Fig. 2),

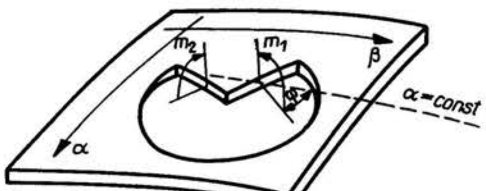


FIG. 2.

the moments $m_\alpha$, $m_\beta$, $m_{\alpha\beta}$ may be expressed in terms of the two auxiliary functions $\omega(\alpha, \beta)$ and $\varphi(\alpha, \beta)$ by the relations([1])

$$\begin{aligned} m_\alpha &= \frac{m_0}{\sqrt{3}}\left(\sqrt{3}\cos\omega - \sin\omega\cos 2\varphi\right), \\ m_\beta &= \frac{m_0}{\sqrt{3}}\left(\sqrt{3}\cos\omega + \sin\omega\cos 2\varphi\right), \\ m_{\alpha\beta} &= \frac{m_0}{\sqrt{3}}\sin\omega\sin 2\varphi. \end{aligned} \tag{2.4}$$

Introduction of the relations (2.4) into the equations of equilibrium (2.1) leads to a system of two quasi-linear partial differential equations with sought for functions $\omega$ and $\varphi$, and two independent variables $\alpha$ and $\beta$. This system is hyperbolic if $\pi/6 < \omega < 5\pi/6$, or $7\pi/6 < \omega < 11\pi/6$. Differential equations of the characteristics of that system are

$$\frac{d\alpha}{d\beta} = \frac{B}{A}\tan(\varphi+\psi), \quad d\varphi + d\chi = \frac{1}{A}\frac{\partial B}{\partial\alpha}d\beta - \frac{1}{B}\frac{\partial A}{\partial\beta}d\alpha, \tag{2.5a}$$

for the lines of the first family, and

$$\frac{d\alpha}{d\beta} = \frac{B}{A}\tan(\varphi-\psi), \quad d\varphi - d\chi = \frac{1}{A}\frac{\partial B}{\partial\alpha}d\beta - \frac{1}{B}\frac{\partial A}{\partial\beta}d\alpha, \tag{2.5b}$$

([1]) These relations are similar to those used in the plane stress analysis of the theory of plasticity [4].

for the lines of the second family (2). For the sake of brevity two auxiliary notations

$$2\psi = \pi - \arccos\frac{\cot\omega}{\sqrt{3}},$$

$$\chi = -\frac{1}{2}\int\limits_{\pi/6}^{\omega}\frac{\sqrt{3-4\cos^2\omega}}{\sin\omega}\,d\omega,$$

used by V. V. SOKOLOVSKII [4] in the plastic plane stress theory, have been introduced. Equations (2.5) are identical with the equations of characteristics obtained in the author's previous work [5] for the quite different problem of the plastic flow of a thin layer resting initially on the surface of a rigid block of arbitrary double curvature.

For particular cases, statically admissible plastic fields of moments can be obtained by solving appropriate boundary-value problems for the equations of characteristics (2.5). In general, the well known numerical Massau procedure has to be used.

### 2.2. Fields of moments for the Tresca yield criterion

If the Tresca yield criterion is used, two different cases have to be distinguished. Consider first the case where the principal bending moments have opposite signs. The plastic state of the cross-section is reached if the following equation is satisfied

$$(m_\beta - m_\alpha)^2 + 4m_{\alpha\beta}^2 = m^2{}_0, \tag{2.6}$$

where $m_0$ represents the limit bending moment defined identically as in the previous Section.

The difference and the sum of principal bending moments can be expressed by means of the new auxiliary function $\chi$ and the limit moment $m_0$:

$$m_1 - m_2 = m_0, \qquad m_1 + m_2 = 2m_0\chi + m^*. \tag{2.7}$$

$m^*$ is a constant to be chosen arbitrarily. In most practical calculations we assume $m^* = 0$.

For the moments $m_\alpha$, $m_\beta$, $m_{\alpha\beta}$, the following expressions can be written:

$$\begin{aligned} m_\alpha &= m^* + m_0\left(\chi - \frac{1}{2}\cos 2\varphi\right),\\ m_\beta &= m^* + m_0\left(\chi + \frac{1}{2}\cos 2\varphi\right),\\ m_{\alpha\beta} &= \frac{1}{2}m_0 \sin 2\varphi. \end{aligned} \tag{2.8}$$

They satisfy identically the yield criterion (2.6). If the expressions (2.8) are introduced into the equations of equilibrium (2.1), a quasi-linear system of partial differential equations of the hyperbolic type is obtained. The equations of characteristics of this system have the form:

$$\frac{d\alpha}{d\beta} = \frac{B}{A}\tan\left(\varphi + \frac{\pi}{4}\right), \quad d\varphi + d\chi = \frac{1}{A}\frac{\partial B}{\partial\alpha}d\beta - \frac{1}{B}\frac{\partial A}{\partial\beta}d\alpha, \tag{2.9a}$$

(2) The equations of characteristics can easily be generalized for shells of non-uniform thickness. In such a general case the limit moment $m_0$ is not constant over the shell surface, but is a given function $m_0(\alpha, \beta)$ of the coordinates. Thus in equations resulting from introducing the relations (2.4) into (2.1), the derivatives $\partial m_0/\partial\alpha$ and $\partial m_0/\partial\beta$ will appear, and equations of characteristics will take a more complex form.

for the characteristics of the first family, and

$$\frac{d\alpha}{d\beta} = \frac{B}{A}\tan\left(\varphi - \frac{\pi}{4}\right), \qquad d\varphi - d\chi = \frac{1}{A}\frac{\partial B}{\partial\alpha}d\beta - \frac{1}{B}\frac{\partial A}{\partial\beta}d\alpha, \tag{2.9b}$$

for the characteristics of the second family.

The characteristics form an orthohonal mesh on the surface of the shell. The mesh of characteristics and the field of moments can be found by solving respective boundary-value problems for the Eqs. (2.9).

If the principal bending moments have the same signs, the plastic state of the cross-section of the shell is reached provided that the following equation is satisfied:

$$(m_\beta - m_\alpha)^2 + 4m_{\alpha\beta}^2 = [2m_0 - |m_\beta + m_\alpha|]^2. \tag{2.10}$$

Let us introduce the new variable $\lambda$, determined by the relations(³):

$$m_1 - m_2 = 2m_0, \qquad |m_1 + m_2| = 2m_0(1-\lambda). \tag{2.11}$$

Now for the moments $m_\alpha$, $m_\beta$, $m_{\alpha\beta}$ we can write

$$\begin{aligned} m_\alpha &= m_0[\varkappa(1-\lambda) - \lambda\cos 2\varphi],\\ m_\beta &= m_0[\varkappa(1-\lambda) + \lambda\cos 2\varphi],\\ m_{\alpha\beta} &= m_0\lambda\sin 2\varphi, \end{aligned} \tag{2.12}$$

where $\varkappa = \operatorname{sign} m_\alpha = \operatorname{sign} m_\beta$.

Substituting (2.12) into the equations of equilibrium (2.1), we obtain the system of equations:

$$\begin{aligned} &A\sin 2\varphi\frac{\partial\varphi}{\partial\beta} - B(\varkappa + \cos 2\varphi)\frac{\partial\varphi}{\partial\alpha} = \sin 2\varphi\frac{\partial B}{\partial\alpha} + (\varkappa + \cos 2\varphi)\frac{\partial A}{\partial\beta},\\ &-A(\varkappa - \cos 2\varphi)\frac{\partial\ln\lambda}{\partial\beta} + B\sin 2\varphi\frac{\partial\ln\lambda}{\partial\alpha} - (\varkappa - \cos 2\varphi)\frac{\partial\varphi}{\partial\alpha}\\ &\qquad = -\sin 2\varphi\frac{\partial B}{\partial\alpha} + (\varkappa - \cos 2\varphi)\frac{\partial A}{\partial\beta}. \end{aligned} \tag{2.13}$$

This system is of the parabolic type and has, therefore, one family of characteristics determined by the equations

$$\frac{d\beta}{A\sin 2\varphi} = \frac{d\alpha}{-B(\varkappa + \cos 2\varphi)} = \frac{d\varphi}{\sin 2\varphi\frac{\partial B}{\partial\alpha} + (\varkappa + \cos 2\varphi)\frac{\partial A}{\partial\beta}}. \tag{2.14}$$

### 3. Working examples for shells

Consider a thin-walled shell of the shape described by the parametric equations

$$\begin{aligned} x &= \frac{a^2\cos^2\beta}{\sqrt{a^2\cos^2\beta + b^2\sin^2\beta}} - r\cos\alpha\cos\beta,\\ y &= \frac{b^2\sin^2\beta}{\sqrt{a^2\cos^2\beta + b^2\sin^2\beta}} - r\cos\alpha\sin\beta,\\ z &= r\sin\alpha. \end{aligned} \tag{3.1}$$

(³) The procedure used here is similar to that used in the plane stress theory of plastic bodies [4].

The orthogonal system of coordinates $\alpha$, $\beta$ coincides with the lines of principal curvatures of the shell. The coeffcients of the first quadratic form are

$$A = r, \quad B = \frac{a^2b^2}{\sqrt{(a^2\cos^2\beta+b^2\sin^2\beta)^3}} - r\cos\alpha.$$

Let the shell be clamped at the edge $\alpha = \pi/2$, and loaded by the uniformly distributed bending moments $m_0 = \sigma_{pl}h^2$ at the edge $\alpha = 0$ (Fig. 3).

For the Huber-Mises yield criterion (2.2), along the edge $\alpha = 0$ we have $m_1 = m_0$ and $m_2 = 0$, if the plastic statically admissible field of moments is to be found. Thus,

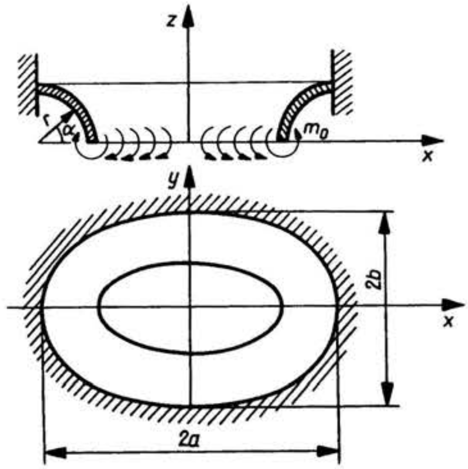


FIG. 3.

according to (2.3) and the definition of the angle $\varphi$, introduced in Sec. 2.1, the two auxiliary functions $\omega$ and $\varphi$ have the following constant values

$$\omega = \frac{\pi}{3}, \quad \varphi = \frac{\pi}{2} \tag{3.2}$$

at any point of the loaded edge $\alpha = 0$. The Cauchy boundary value problem for the equations of characteristics (2.5) is, therefore, uniquely defined, and the plastic field of moments can be found for the entire shell. Figure 4 shows the mesh of characteristics for a quadrant of the shell calculated for the particular case, where $a = 1.0$, $b = 0.6$ and $r = 0.2$. Thus the statically admissible field of moments for the entire shell has been constructed. The shell is able to carry the loading by uniformly distributed bending moments equal at least to $m_0$.

As the next example, the hyperboloidal shell described by the parametric equations

$$x = a\frac{\cos\beta}{\cos\alpha}, \quad y = b\frac{\sin\beta}{\cos\alpha}, \quad z = c\tan\alpha, \tag{3.3}$$

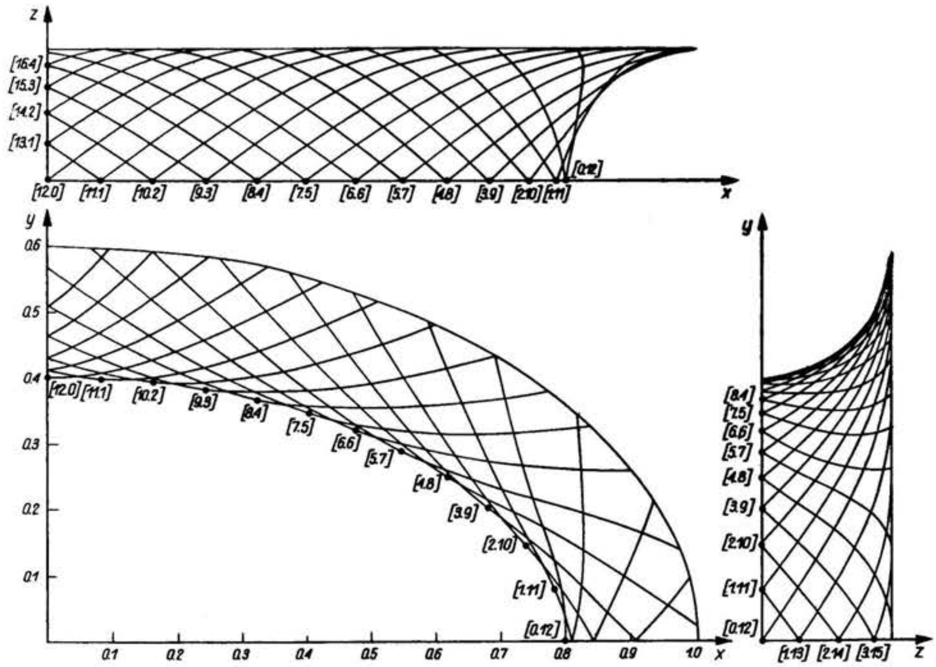


FIG. 4.

is considered. The coefficients of the first quadratic form are:

$$A = \frac{1}{\cos^2\alpha}\sqrt{\sin^2\alpha(a^2\cos^2\beta+b^2\sin^2\beta)+c^2},$$

$$B = \frac{1}{\cos\alpha}\sqrt{a^2\sin^2\beta+b^2\cos^2\beta}.$$

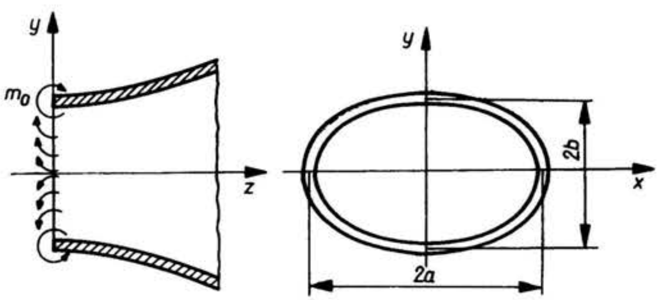


FIG. 5.

Assume that the shell is loaded at the edge $\alpha = 0$, as in the previous case, by the uniformly distributed bending moments $m_0 = \sigma_{pl}h^2$ (Fig. 5). Now, however, the problem will be solved by assuming the Tresca yield criterion.

If the plastic field of moments is to be found, along the edge $\alpha = 0$ the principal bending moments must have the values $m_1 = m_0$, $m_2 = 0$. Thus from (2.8) we obtain that the auxiliary function $\chi$ and the angle $\varphi$ have the constant values

$$\varphi = \frac{\pi}{2}, \quad \chi = \frac{1}{2}, \tag{3.4}$$

along the edge $\alpha = 0$. The constant $m^*$ appearing in (2.8) is assumed to be equal to zero. The field of moments for the entire shell can now be found by solving the Cauchy boundary-value problem for the equations of characteristics (2.9). The mesh of characteristics for

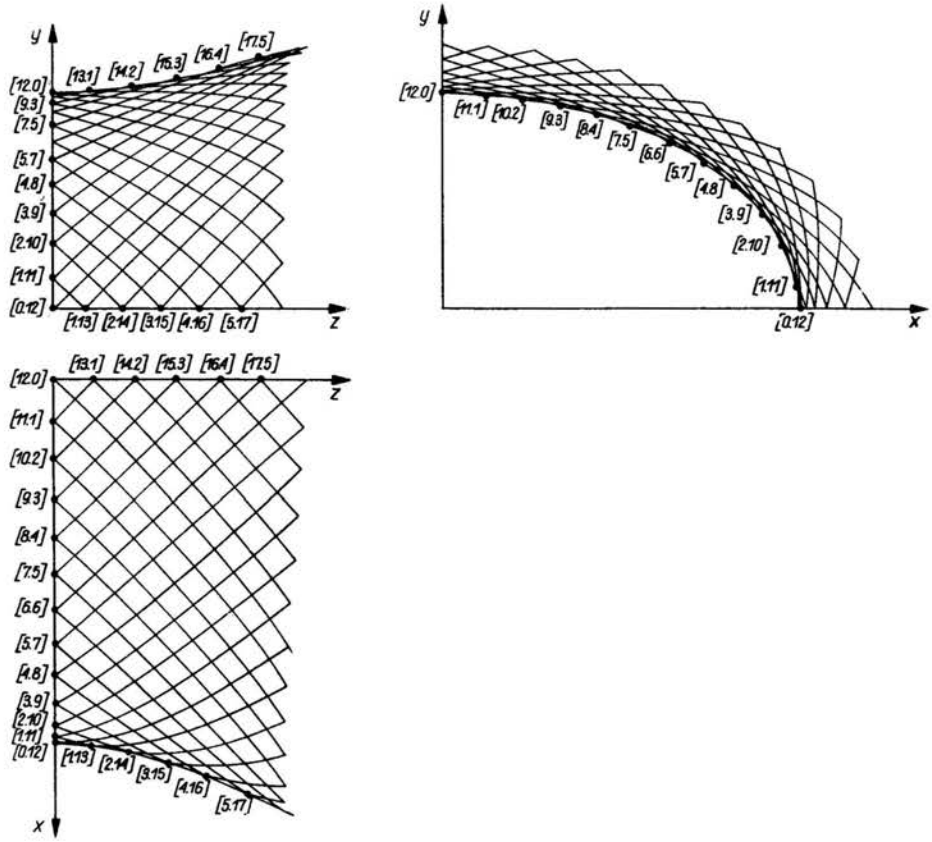

FIG. 6.

a quadrant of the shell calculated for the particular case, where $a = 1.0$, $b = 0.6$ and $c = 1.0$, is shown in Fig. 6. Our solution indicates, therefore, that the shell is able to carry at least the loading shown in Fig. 5.

## 4. Plastic fields of moments for plates

The equations of plastic fields of moments for plates can be obtained from the corresponding equations for shells by assuming that $A = B = 1$, and replacing the coordinates $\beta$ and $\alpha$ by $x$ and $y$ respectively.

For the Huber-Mises yield criterion, the equations of characteristics result directly from (2.5). Thus we have

$$\frac{dy}{dx} = \tan(\varphi+\psi), \quad \varphi+\chi = \text{const}, \tag{4.1a}$$

for the characteristics of the first family, and

$$\frac{dy}{dx} = \tan(\varphi-\psi), \quad \varphi-\chi = \text{const}, \tag{4.1b}$$

for the characteristics of the second family.

These equations are identical with those of the plane stress theory derived by V. V. SOKOLOVSKII [4]([4]). In the Eqs. (4.1) $\varphi$ denotes the angle which makes the normal to the cross-section where the greater principal bending moment is acting with the $x$-axis (Fig. 7).

As a typical example of application consider a rectangular plate clamped at the edges (Fig. 8). The plate of the thickness $2h$ has a central elliptical hole, whose edge is loaded

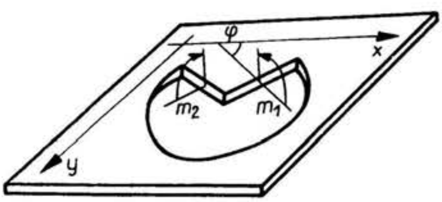


FIG. 7.

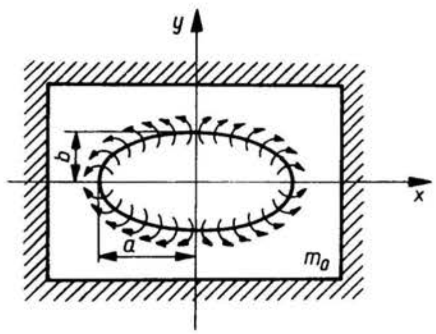


FIG. 8.

by the uniformly distributed bending moments $m_0 = \sigma_{\text{pl}} h^2$. If the equations of the ellipse are written in the parametric form

$$\begin{aligned} x &= \frac{a^2 \sin\gamma}{\sqrt{a^2\sin^2\gamma+b^2\cos^2\gamma}}, \\ y &= -\frac{b^2\cos^2\gamma}{\sqrt{a^2\sin^2\gamma+b^2\cos^2\gamma}}, \end{aligned} \tag{4.2}$$

the boundary conditions for the functions $\varphi$ and $\omega$ along the edge of the hole are

$$\varphi = \frac{\pi}{2}+\gamma, \quad \omega = \frac{\pi}{3}. \tag{4.3}$$

Thus the Cauchy boundary-value problem for the equations of characteristics (4.1) is defined, and the statically admissible field of moments can be numerically calculated for the entire plate. The mesh of characteristics is identical with that given in Sokolovskii's

([4]) Equations (4.1) can be generalized for plates with non-uniform thickness in the same manner as in the case of equations for shells.

book [4] and has not been shown here. The present solution shows that the bearing capacity of the clamped plate is not smaller than that assumed above.

In the same manner, numerous similar problems for plates can be solved providing lower bounds on their bearing capacity under loading by distributed moments at the edges.

## 5. Concluding remarks

Solutions presented in this work can provide complementary data if more practical problems including simultaneous loading of shells and plates by distributed moments and membrane forces acting at the edges have to be considered. Assume for example that the edge of the shell shown in Fig. 3 is loaded by uniformly distributed membrane forces $p$ and uniformly distributed bending moments $m$. The limit states corresponding to various combinations of the two loading parameters $p$ and $m$ can be represented in the $m$-$p$ plane as a limit curve. The procedure presented in the previous sections can be useful in estimation of this curve in the vicinity of the $m$-axis where $p = 0$.

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