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On the accuracy of approximations of the Huber yield condition

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It is well known that the differences between the Huber-Mises-Hencky and the Tresca-Guest yield conditions do not exceed 13.4% when referred to the HMH condition. However, if we consider the processes as a whole, the final differences depend on the flow rule assumed and may be much larger. The approximation of the HMH by the TG yield condition is here considered in the general case in two variants: combined with the Prandtl-Reuss rule of similarity of deviators, and with the associated flow rule. It is found that the latter case gives larger errors and in practice the associated flow rule should be considered as a next step in approximation.

Powszechnie wiadomo, że różnice pomiędzy warunkami plastyczności Hubera-Misesa-Hencky'ego i Treski-Guesta nie przekraczają 13,4%, gdy odnosimy je do warunku HMH. Jednakże przy rozpatrywaniu procesów jako całości, końcowe różnice zależą od przyjętego prawa płynięcia plastycznego i mogą być znacznie większe. W pracy rozpatrywano ogólny przypadek aproksymacji warunku HMH warunkiem TG w dwóch wariantach: przy połączeniu go z prawem Prandtla-Reussa podobieństwa dewiatorów oraz przy zastosowaniu stowarzyszonego prawa płynięcia. W ostatnim przypadku występują z reguły większe błędy i stosowanie stowarzyszonego prawa płynięcia należy tu traktować jako dokonanie następnego kroku przybliżenia.

Известно, что разницы между условиями пластичности Губера-Мизеса-Генки и Трески-Геста не превышают 13,4%, когда отнесем их к условию ГМГ. Однако при рассмотрении процессов как целого конечные разницы зависят от принятого закона пластического течения и могут быть значительно большими. В работе рассмотрен общий случай аппроксимации условия ГМГ условием ТГ в двух вариантах: при соединении его с законом Прандтля-Рейсса подобия девиаторов и при применении ассоциированного закона течения. В последнем случае выступают, как правило, больше ошибки и применение ассоциированного закона течения следует здесь трактовать как следующий шаг приближения.

1. Statement of the problem

MANY reasons are known for combining the yield condition with the associated flow rule, expressing the normality of the plastic strain rate vector to the yield surface in the stress space. They have been discussed by R. MISES [9], D. C. DRUCKER [2], D. R. BLAND [1] and others. Several experiments confirm such a conception although certain deviations for various materials and various processes have also been observed.

An entirely different situation occurs if an approximate yield condition is used instead of the exact one. Then the choice of a suitable flow rule should be considered from the point of view of the approximation errors. Such a situation appears, for example, if we replace for a perfectly plastic body–the non-linear Huber-Mises-Hencky (HMH) yield condition, experimentally better confirmed — by a linearized one; the Tresca-Guest (TG) yield condition is here the most typical, but certain other proposals are offered, too (maximal deviatoric stress condition, R. SCHMIDT [13], D. D. IVLEV [6], R. M. HAYTHORNTHWAITE [3]).

J. A. KÖNIG [7] considered the problem of approximation of the HMH yield condition by the TG yield condition combined with the classical Prandtl-Reuss equations and with the associated flow rule. He analyzed the deflections of plastic plates and shells and found smaller errors in the first variant of approximation. The present paper will discuss the problem of such an approximation in a much more general way — namely whether the process is arbitrarily prescribed in principal strains, or whether it is prescribed in a mixed manner, some strains and some stresses (the latter case is typical for plates and shells, where the distributions of ε_1, ε_2, and $\sigma_3 = 0$ are usually considered as given). The errors for various directions of the strain rate vector $\dot{\varepsilon}$ will be compared and mean square errors will be evaluated. Some examples will also be given.

2. The HMH yield condition and the Prandtl-Reuss (associated) flow rule as the exact description of the process

In order to obtain simple results using the TG yield condition in the following sections, we confine ourselves in the paper as a whole to processes with fixed and known principal directions. Assume the process to be prescribed in the strain space — i.e., the functions $\varepsilon_1 = \varepsilon_1(t)$, $\varepsilon_2 = \varepsilon_2(t)$ and $\varepsilon_3 = \varepsilon_3(t)$ are given; then the stresses may be evaluated. The problem will here be analyzed locally: for the given strain rate vector, the stress rates will be calculated and compared.

The functions ε_1, ε_2, and ε_3 are not independent in the case of an incompressible body; thus we discuss (in Sec. 2 through 5) elastically compressible bodies, but the errors of approximation will be independent of the Poisson's ratio ν and the incompressibility may be analyzed by the limiting process.

In the equations of the similarity of the stress deviator s_{ij} and the plastic strain rate deviator $\dot{e}^p_{ij}$ (the Prandtl-Reuss equations),

$$\frac{\dot{e}^p_1}{s_1} = \frac{\dot{e}^p_2}{s_2} = \frac{\dot{e}^p_3}{s_3}, \tag{2.1}$$

together with the differentiated HMH yield condition

$$s_1\dot{s}_1 + s_2\dot{s}_2 + s_3\dot{s}_3 = 0, \tag{2.2}$$

we regard the total strains, their rates and the stresses as known, and seek the stress rates $\dot{\sigma}_{ij}$. Substitute

$$\dot{e}^p_1 = \dot{\varepsilon}^p_1 = \dot{\varepsilon}_1 - \frac{1}{E}[\dot{\sigma}_1 - \nu(\dot{\sigma}_2 + \dot{\sigma}_3)], \qquad s_1 = \frac{1}{3}(2\sigma_1 - \sigma_2 - \sigma_3), \tag{2.3}$$

and similar relations for $\dot{e}^p_2$, $\dot{e}^p_3$, s_2, and s_3, and solve the system of three equations obtained, linear with respect to $\dot{\sigma}_1$, $\dot{\sigma}_2$, and $\dot{\sigma}_3$. After some simple but lengthy transformations (and making use of the original HMH yield condition), we obtain:

$$\frac{\dot{\sigma}_1}{E} = \frac{2\dot{\varepsilon}_1 - \dot{\varepsilon}_2 - \dot{\varepsilon}_3 + 2\nu(\dot{\varepsilon}_1 + 4\dot{\varepsilon}_2 + 4\dot{\varepsilon}_3)}{6(1+\nu)(1-2\nu)} + \frac{1}{2(1+\nu)\sigma_0^2}[(\sigma_2 - \sigma_3)^2\dot{\varepsilon}_1 + (\sigma_1 - \sigma_2)^2\dot{\varepsilon}_2 + (\sigma_1 - \sigma_3)^2\dot{\varepsilon}_3], \tag{2.4}$$

where σ_0 denotes the yield point in simple tension. The formulas for $\dot{\sigma}_2$ and $\dot{\sigma}_3$ may be here obtained by cyclic interchange of the suffixes.

The formulas derived express stress rates in terms of three strain rates and of three stresses. To reduce this large number of independent variables, we confine ourselves to $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$ and introduce at first the Lode parameter for stresses:

$$\mu_\sigma = \frac{2\sigma_2-\sigma_1-\sigma_3}{\sigma_1-\sigma_3}, \quad -1 \leqslant \mu_\sigma \leqslant 1. \tag{2.5}$$

Thus we may write:

$$\sigma_1-\sigma_2 = \frac{1-\mu_\sigma}{2}(\sigma_1-\sigma_3), \quad \sigma_2-\sigma_3 = \frac{1+\mu_\sigma}{2}(\sigma_1-\sigma_3); \tag{2.6}$$

the HMH yield condition takes the form:

$$\sigma_1-\sigma_3 = \frac{2}{\sqrt{3+\mu_\sigma^2}}\sigma_0, \tag{2.7}$$

and instead of (2.4) we obtain:

$$\begin{aligned}
\frac{\dot\sigma_1}{G} &= \frac{1}{1-2\nu}(\dot\varepsilon_1+\dot\varepsilon_2+\dot\varepsilon_3)-\dot\varepsilon_2+\frac{2\mu}{3+\mu_\sigma^2}\left[\dot\varepsilon_1-\dot\varepsilon_2+\frac{\mu_\sigma}{3}(\dot\varepsilon_1+\dot\varepsilon_2-2\dot\varepsilon_3)\right],\\
\frac{\dot\sigma_2}{G} &= \frac{2\nu}{1-2\nu}(\dot\varepsilon_1+\dot\varepsilon_2+\dot\varepsilon_3)+2\dot\varepsilon_2+\frac{2\mu_\sigma}{3+\mu_\sigma^2}\left[-\dot\varepsilon_1+\dot\varepsilon_3+\frac{\mu_\sigma}{3}(\dot\varepsilon_1-2\dot\varepsilon_2+\dot\varepsilon_3)\right],\\
\frac{\dot\sigma_3}{G} &= \frac{1}{1-2\nu}(\dot\varepsilon_1+\dot\varepsilon_2+\dot\varepsilon_3)-\dot\varepsilon_2+\frac{2\mu_\sigma}{3+\mu_\sigma^2}\left[\dot\varepsilon_2-\dot\varepsilon_3+\frac{\mu_\sigma}{3}(-2\dot\varepsilon_1+\dot\varepsilon_2+\dot\varepsilon_3)\right],
\end{aligned} \tag{2.8}$$

G being the Kirchhoff modulus. All the three formulas are given here, since μ_σ is unsymmetrical with respect to the suffixes and the symmetry of notation is lost.

Further reduction of the number of variables may be obtained by the introduction of spherical coordinates in the space of strain rates. We introduce three parameters $\dot\varepsilon$, φ, and ϑ, such that

$$\dot\varepsilon_1 = \dot\varepsilon\sin\vartheta\sin\varphi, \quad \dot\varepsilon_2 = \dot\varepsilon\cos\vartheta, \quad \dot\varepsilon_3 = \dot\varepsilon\sin\vartheta\cos\varphi. \tag{2.9}$$

These parameters are introduced in such a way as to distinguish ε_2, corresponding to the intermediate stress σ_2. Finally, (2.8) will be rewritten in the form:

$$\begin{aligned}
\frac{\dot\sigma_1}{G\dot\varepsilon} &= \frac{1}{1-2\nu}(\sin\vartheta\sin\varphi+\sin\vartheta\cos\varphi+\cos\vartheta)-\cos\vartheta\\
&\quad+\frac{2\mu}{3+\mu_\sigma^2}\left[\sin\vartheta\sin\varphi-\cos\vartheta+\frac{\mu}{3}(\sin\vartheta\sin\varphi+\cos\vartheta-2\sin\vartheta\cos\varphi)\right],\\
\frac{\dot\sigma_2}{G\dot\varepsilon} &= \frac{2\nu}{1-2\nu}(\sin\vartheta\sin\varphi+\sin\vartheta\cos\varphi+\cos\vartheta)+2\cos\vartheta\\
&\quad+\frac{2\mu}{3+\mu_\sigma^2}\left[-\sin\vartheta\sin\varphi+\sin\vartheta\cos\varphi+\frac{\mu}{3}(\sin\vartheta\sin\varphi-2\cos\vartheta+\sin\vartheta\cos\varphi)\right],\\
\frac{\dot\sigma_3}{G\dot\varepsilon} &= \frac{1}{1-2\nu}(\sin\vartheta\sin\varphi+\sin\vartheta\cos\varphi+\cos\vartheta)-\cos\vartheta\\
&\quad+\frac{2\mu_\sigma}{3+\mu_\sigma^2}\left[\cos\vartheta-\sin\vartheta\cos\varphi+\frac{\mu_\sigma}{3}(-2\sin\vartheta\sin\varphi+\cos\vartheta+\sin\vartheta\cos\varphi)\right].
\end{aligned} \tag{2.10}$$

The formulas (2.10) will be treated as exact, and the approximation errors will be evaluated with respect to them.

The range of validity of the formulas derived is limited by the equations of the neutral process. This process may be determined by the differentiated yield condition (2.2) in which $\dot{s}_1$, $\dot{s}_2$, and $\dot{s}_3$ are calculated from Hooke's law:

$$s_1(\dot{\varepsilon}_1-\dot{\varepsilon}_m)+s_2(\dot{\varepsilon}_2-\dot{\varepsilon}_m)+s_3(\dot{\varepsilon}_3-\dot{\varepsilon}_m) = 0. \tag{2.11}$$

The terms with the mean strain rate $\dot{\varepsilon}_m$ vanish. Introducing the Lode parameter μ_σ and substituting

$$s_1 = \frac{3-\mu_\sigma}{6}(\sigma_1-\sigma_3), \quad s_2 = \frac{\mu_\sigma}{3}(\sigma_1-\sigma_3), \quad s_3 = -\frac{3+\mu}{6}(\sigma_1-\sigma_3), \tag{2.12}$$

we obtain:

$$(3-\mu\ )\dot{\varepsilon}_1+2\mu_{,}\dot{\varepsilon}_2-(3+\mu\ )\dot{\varepsilon}_3 = 0, \tag{2.13}$$

or, in spherical coordinates (2.9):

$$\operatorname{tg}\vartheta = \frac{2\mu_\sigma}{(3+\mu_\sigma)\cos\varphi-(3-\mu_\sigma)\sin\varphi}. \tag{2.14}$$

The regions bounded by (2.14) will be shown later, together with the bounds following from the TG yield condition.

3. The first variant of approximation: the TG yield condition with the similarity of deviators as the flow rule

Consider now the combination of the Tresca-Guest yield condition and the Prandtl-Reuss equations. Under the assumption of $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$, we write the yield condition in the form:

$$\sigma_1-\sigma_3 = \sigma_0, \tag{3.1}$$

and combine it with (2.1). Substitution of (2.3) and differentiation of (3.1) with respect to the time leads to the system of three equations linear in $\dot{\sigma}_1$, $\dot{\sigma}_2$, and $\dot{\sigma}_3$. Its solution may be written as follows:

$$\begin{aligned}\frac{\dot{\sigma}_1}{G} = \frac{\dot{\sigma}_3}{G} &= \frac{1}{1-2\nu}(\dot{\varepsilon}_1+\dot{\varepsilon}_2+\dot{\varepsilon}_3)-\dot{\varepsilon}_2+\frac{1}{3}(\dot{\varepsilon}_1-\dot{\varepsilon}_3)\left[1-\frac{2(\sigma_1-\sigma_2)}{\sigma_0}\right],\\ \frac{\dot{\sigma}_2}{G} &= \frac{2\nu}{1-2\nu}(\dot{\varepsilon}_1+\dot{\varepsilon}_2+\dot{\varepsilon}_3)+2\dot{\varepsilon}_2-\frac{2}{3}(\dot{\varepsilon}_1-\dot{\varepsilon}_3)\left[1-\frac{2(\sigma_1-\sigma_2)}{\sigma_0}\right].\end{aligned} \tag{3.2}$$

Introducing the Lode parameter μ_σ, (2.5), and the spherical coordinates in the strain rate space, (2.9), we rewrite (3.2) in the form:

$$\begin{aligned}\frac{\dot{\sigma}_1}{G\dot{\varepsilon}} = \frac{\dot{\sigma}_3}{G\dot{\varepsilon}} &= \frac{1}{1-2\nu}(\sin\vartheta\sin\varphi-\sin\vartheta\cos\varphi+\cos\vartheta)-\cos\vartheta+\frac{\mu_\sigma}{3}\sin\vartheta(\sin\varphi-\cos\varphi),\\ \frac{\dot{\sigma}_2}{G\dot{\varepsilon}} &= \frac{2\nu}{1-2\nu}(\sin\vartheta\sin\varphi+\sin\vartheta\cos\varphi+\cos\vartheta)+2\cos\vartheta-\frac{2\mu_\tau}{3}\sin\vartheta(\sin\varphi-\cos\varphi).\end{aligned} \tag{3.3}$$

The range of validity of these formulas is limited by the equations of the neutral process, which here take the form:

$$\dot{\varepsilon}_1-\dot{\varepsilon}_3 = 0. \tag{3.4}$$

or

$$(3.5) \qquad \sin\varphi - \cos\varphi = 0, \quad \varphi = \frac{\pi}{4} \quad \text{and} \quad \varphi = \frac{5}{4}\pi.$$

For active processes (3.3) (loading), we have $\dot{\varepsilon}_1 - \dot{\varepsilon}_3 > 0$, hence $\frac{\pi}{4} < \varphi < \frac{5}{4}\pi$. This region is independent of ϑ and of μ . We regard it as an approximation of the region bounded by (2.14); they coincide for $\mu_\sigma = 0$, but for other values of μ_σ the differences may be significant, Fig. 1.

Figure 1 contains also the lines describing simple processes: simple loading (proportional increase of the components of the strain deviator) and simple unloading. For

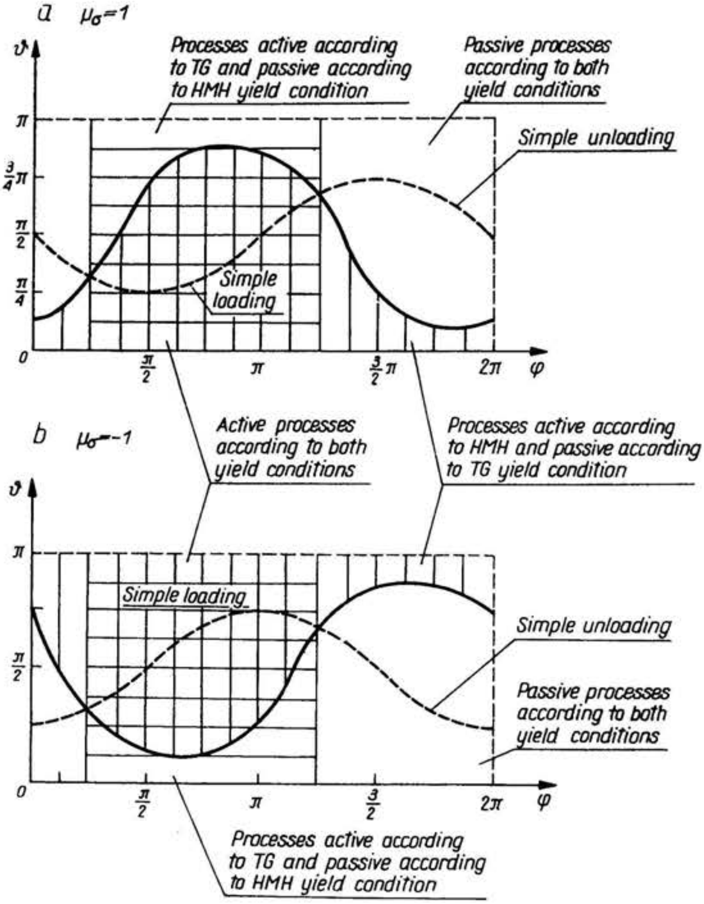

FIG. 1.

simple processes, the Lode parameters are equal (at least according to the Prandtl-Reuss equations):

$$(3.6) \qquad \mu_\sigma = \mu_\varepsilon = \mu_{\dot{\varepsilon}} = \frac{2\dot{\varepsilon}_2 - \dot{\varepsilon}_1 - \dot{\varepsilon}_3}{\dot{\varepsilon}_1 - \dot{\varepsilon}_3};$$

hence, introducing the spherical coordinates (2.9),

$$\operatorname{tg}\vartheta = \frac{2}{(1+\mu_\sigma)\sin\varphi+(1-\mu\ )\cos\varphi}. \tag{3.7}$$

It is seen from Fig. 1 that the simple processes are active (loading) or passive (unloading) in the same regions according to the two yield conditions used.

4. The second variant of approximation; the TG yield condition with the associated flow rule

Combine now the approximate yield condition (3.1) with the associated flow rule. The normality of the vector of plastic flow gives:

$$\dot{\varepsilon}_1^p = -\dot{\varepsilon}_3^p, \quad \dot{\varepsilon}_2^p = 0. \tag{4.1}$$

These equations with substituted (2.3) and together with differentiated (3.1) determine the stress rates:

$$\begin{aligned} \frac{\dot{\sigma}_1}{G} &= \frac{\dot{\sigma}_3}{G} = \frac{1}{1-2\nu}(\dot{\varepsilon}_1+\dot{\varepsilon}_2+\dot{\varepsilon}_3)-\dot{\varepsilon}_2, \\ \frac{\dot{\sigma}_2}{G} &= \frac{2\nu}{1-2\nu}(\dot{\varepsilon}_1+\dot{\varepsilon}_2+\dot{\varepsilon}_3)+2\dot{\varepsilon}_2. \end{aligned} \tag{4.2}$$

These equations are simpler than (3.2), but the errors with respect to (2.8) will as a rule be larger. Introducing (2.5) and (2.9), we rewrite (4.2) in the form:

$$\begin{aligned} \frac{\dot{\sigma}_1}{G\dot{\varepsilon}} &= \frac{\dot{\sigma}_3}{G\dot{\varepsilon}} = \frac{1}{1-2\nu}(\sin\vartheta\sin\varphi+\sin\vartheta\cos\varphi+\cos\vartheta)-\cos\vartheta, \\ \frac{\dot{\sigma}_2}{G\dot{\varepsilon}} &= \frac{2\nu}{1-2\nu}(\sin\vartheta\sin\varphi+\sin\vartheta\cos\varphi+\cos\vartheta)+2\cos\vartheta. \end{aligned} \tag{4.3}$$

The range of validity of (4.3) is $\frac{\pi}{4} < \varphi < \frac{5}{4}\pi$ as derived in Sec. 3.

5. Analysis of the approximation errors

Denote the differences between (3.3) and the exact formulas (2.10) by Δ_{js}, and between (4.3) and (2.10) by Δ_{ja}, $j = 1, 2, 3$. Several terms are equal and vanish in the differences; we obtain for the similarity of deviators:

$$\begin{aligned} \Delta_{1s} &= \Big\{-\frac{2}{3+\mu_\sigma^2}\Big[\sin\vartheta\sin\varphi-\cos\vartheta+\frac{\mu_\sigma}{3}(\sin\vartheta\sin\varphi+\cos\vartheta-2\sin\vartheta\cos\varphi)\Big] \\ &\qquad +\frac{1}{3}\sin\vartheta(\sin\varphi-\cos\varphi)\Big\}\mu_\sigma, \\ \Delta_{2s} &= \Big\{-\frac{2}{3+\mu_\sigma^2}\Big[-\sin\vartheta\sin\varphi+\sin\vartheta\cos\varphi+\frac{\mu_\tau}{3}(\sin\vartheta\sin\varphi-2\cos\vartheta+\sin\vartheta\cos\varphi)\Big] \\ &\qquad -\frac{2}{3}\sin\vartheta(\sin\varphi-\cos\varphi)\Big\}\mu_\sigma, \end{aligned} \tag{5.1}$$

$$\Delta_{3s} = \left\{-\frac{2}{3+\mu_\sigma^2}\left[\cos\vartheta - \sin\vartheta\cos\varphi + \frac{\mu}{3}(-2\sin\vartheta\sin\varphi + \cos\vartheta + \sin\vartheta\cos\varphi)\right] + \frac{1}{3}\sin\vartheta(\sin\varphi - \cos\varphi)\right\}\mu_\sigma,$$

and for the associated flow rule:

$$\Delta_{1a} = -\frac{2}{3+\mu_\sigma^2}\left[\sin\vartheta\sin\varphi - \cos\vartheta + \frac{\mu_\sigma}{3}(\sin\vartheta\sin\varphi + \cos\vartheta - 2\sin\vartheta\cos\varphi)\right]\mu_\sigma,$$

$$\text{(5.2)}\quad \Delta_{2a} = -\frac{2}{3+\mu_\sigma^2}\left[-\sin\vartheta\sin\varphi + \sin\vartheta\cos\varphi + \frac{\mu_\sigma}{3}(\sin\vartheta\sin\varphi - 2\cos\vartheta + \sin\vartheta\cos\varphi)\right]\mu_\sigma,$$

$$\Delta_{3a} = -\frac{2}{3+\mu_\sigma^2}\left[\cos\vartheta - \sin\vartheta\cos\varphi + \frac{\mu_\sigma}{3}(-2\sin\vartheta\sin\varphi + \cos\vartheta + \sin\vartheta\cos\varphi)\right]\mu_\sigma.$$

The formulas for Δ_{3s} and Δ_{3a} may be obtained from the formulas for Δ_{1s} and Δ_{1a} by the interchange of $\sin\varphi$ and $\cos\varphi$ with simultaneous change of the sign of μ_σ; thus in what follows we analyze Δ_1 and Δ_2 only.

The analysis will be carried out in the regions in which the processes are active according to the two yield conditions used, Fig. 1. Within the regions estimated as active by one and as passive by the other yield condition, the whole approximation is doubtful in both variants.

In order to determine the regions of larger $|\Delta_{ja}|$ and of larger $|\Delta_{js}|$, we equate the absolute values of these errors, separately for $j = 1$ and for $j = 2$. The equations $\Delta_{js} = \Delta_{ja}$ give $\varphi = \frac{\pi}{4}$ and $\varphi = \frac{5}{4}\pi$, i.e., the lines coinciding with the boundary of the region considered. Thus the separation lines sought for will be obtained from the equations $\Delta_{js} = -\Delta_{ja}$. For Δ_1 we have:

$$\text{(5.3)}\qquad \operatorname{tg}\vartheta_1 = \frac{12-4\mu_\sigma}{(9+4\mu_\sigma-\mu_\sigma^2)\sin\varphi + (3-8\mu_\sigma+\mu_\sigma^2)\cos\varphi},$$

and for Δ_2

$$\text{(5.4)}\qquad \operatorname{tg}\vartheta_2 = \frac{-4\mu_\sigma}{(3-2\mu_\sigma-\mu_\sigma^2)\sin\varphi - (3+2\mu_\sigma-\mu_\sigma^2)\cos\varphi}.$$

The separation lines are presented in Fig. 2 for Δ_1 ($\mu_\sigma = 1$ and $\mu_\sigma = -1$), and in Fig. 3 for Δ_2. It is seen that the law of similarity of deviators is slightly better if Δ_1 is taken into account, and is much better if we discuss Δ_2. In Fig. 4, the relation (5.4) is presented as a function joining three variables: φ, ϑ and μ .

To obtain more synthetic results, we calculate and compare the mean square deviations for (5.1) and (5.2). These deviations $\bar{\Delta}_j$ will be defined as follows:

$$\text{(5.5)}\qquad \bar{\Delta}_j = \sqrt{\frac{1}{2\pi}\int_0^\pi \sin\vartheta\, d\vartheta \int_{\pi/4}^{5\pi/4} \Delta_j^2(\varphi,\vartheta)\, d\varphi}\ ;$$

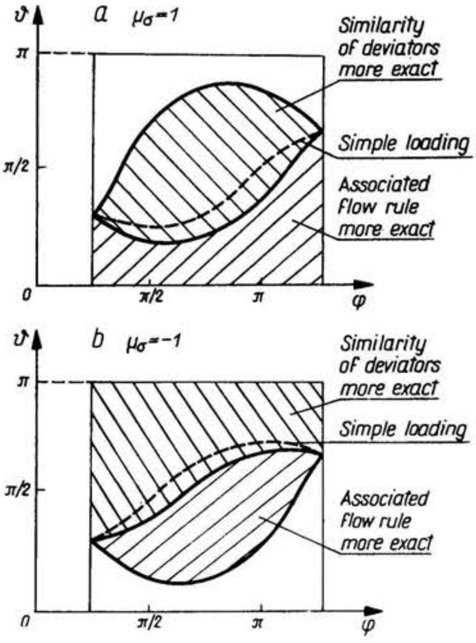

FIG. 2.

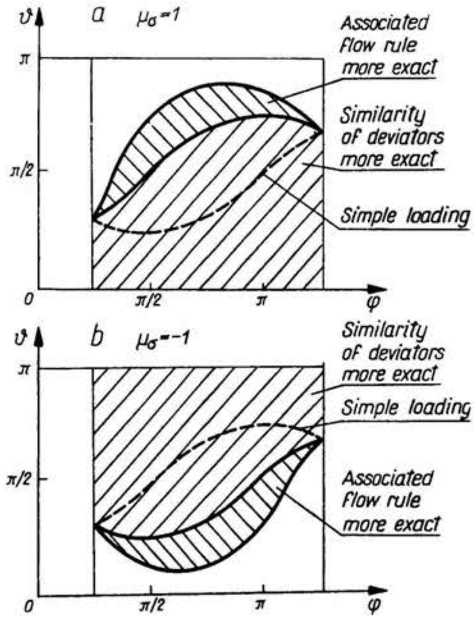

FIG. 3.

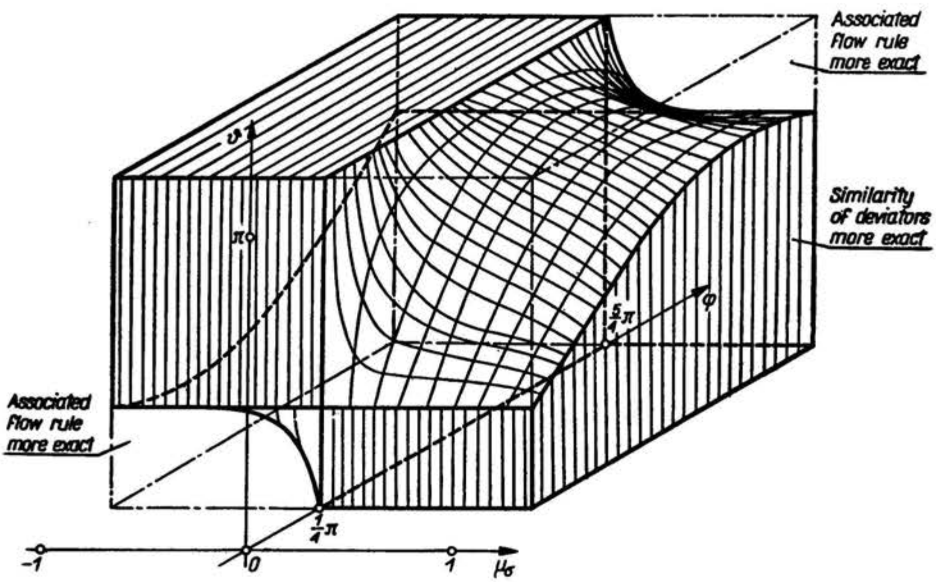

FIG. 4.

in the interest of simplicity, the integration is here taken over the surface of the half-sphere $0 \leqslant \vartheta \leqslant \pi$, $\frac{\pi}{4} \leqslant \varphi \leqslant \frac{5}{4}\pi$, corresponding to the regions of active processes as described by the TG yield condition. The following general formula for the definite integral appearing in (5.5) may be derived:

$$\frac{1}{2\pi}\int\limits_0^{\pi} \sin\vartheta d\vartheta \int\limits_{\pi/4}^{5\pi/4} (A_1 \sin\vartheta \sin\varphi + A_2 \cos\vartheta + A_3 \sin\vartheta \cos\varphi)^2 d\varphi = \frac{1}{3}(A_1^2 + A_2^2 + A_3^2). \tag{5.6}$$

As a matter of fact, the integrals appearing in (5.5) are of the type (5.6): the constants A_i may be found comparing (5.5) with (5.1) and (5.2). Hence for the subsequent mean square errors we obtain:

$$\bar{\Delta}_{1s} = \frac{(3-\mu_\sigma)\,|\mu_\sigma|\sqrt{2}}{3\sqrt{3}\,\sqrt{3+\mu_\sigma^2}}, \quad \bar{\Delta}_{2s} = \frac{2\sqrt{2}\,\mu_\sigma^2}{3\sqrt{3}\,\sqrt{3+\mu_\sigma^2}}, \tag{5.7}$$

$$\bar{\Delta}_{1a} = \bar{\Delta}_{2a} = \frac{2\sqrt{2}\,|\mu\,|}{3\sqrt{3+\mu^2}}. \tag{5.8}$$

Figure 5 presents the ratios $\bar{\Delta}_{js}/\bar{\Delta}_{ja}$ as the functions of μ . These ratios are, as a rule, much less than unity, showing clearly that the flow rule expressing the similarity of deviators

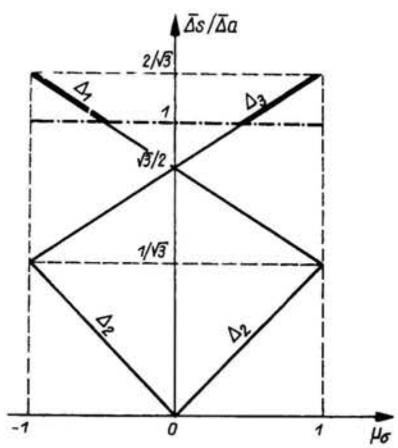

FIG. 5.

is, on the average, more exact. Even within the small intervals, where the calculated ratios $\bar{\Delta}_{js}/\bar{\Delta}_{ja}$ are larger than unity (shown by the thick lines in Fig. 5), the associated flow rule can not be regarded as more exact: for example, this occurs for Δ_1 in the vicinity of $\mu_\sigma = -1$, but Fig. 2b shows that the area of integration in which Δ_{1s} is larger than Δ_{1a} should be much smaller than the approximately assumed half-sphere (in the lowest part of Fig. 2b the processes are passive according to the HMH yield condition and the approximation is practically inadmissible).

6. Differential equations for the Lode parameter μ_σ

Another synthetic comparison of the accuracy of the approximations discussed may be achieved by means of the derivation of the differential equations for μ. The formulas obtained for $\dot{\sigma}_j$ make it possible to determine the rate of μ. The differentiation of (2.5) gives:

$$\dot{\mu}_\sigma = \frac{2\dot{\sigma}_2-\dot{\sigma}_1-\dot{\sigma}_3}{\sigma_1-\sigma_3} - \frac{(2\sigma_2-\sigma_1-\sigma_3)(\dot{\sigma}_1-\dot{\sigma}_3)}{(\sigma_1-\sigma_3)^2}. \tag{6.1}$$

Substituting here (2.8), we obtain for the HMH yield condition:

$$\frac{\dot{\mu}_\sigma \sigma_0}{G} = \sqrt{3+\mu_\sigma^2}\,(\mu_{\dot{\varepsilon}}-\mu_\sigma)(\dot{\varepsilon}_1-\dot{\varepsilon}_3), \tag{6.2}$$

where $\mu_{\dot{\varepsilon}}$ denotes the Lode parameter for the strain rates. Similar differential equations based on the TG yield condition are:

With the Prandtl-Reuss flow rule, (3.2) substituted:

$$\frac{\dot{\mu}\,\sigma_0}{G} = 2(\mu_{\dot{\varepsilon}}-\mu)(\dot{\varepsilon}_1-\dot{\varepsilon}_3), \tag{6.3}$$

and with the associated flow rule, (4.2) substituted;

$$\frac{\dot{\mu}_\sigma\sigma_0}{G} = 2\mu_{\dot{\varepsilon}}(\dot{\varepsilon}_1-\dot{\varepsilon}_3). \tag{6.4}$$

The similarity of (6.2) and (6.3) is evident. For $\mu_{\dot{\varepsilon}} = \text{const}$, the solutions in both cases are $\mu_\sigma = \mu_{\dot{\varepsilon}} = \text{const}$ (simple loading). On the other hand this solution does not in general satisfy (6.4) — the differences with respect to (6.2) are here essential.

7. Example of a process prescribed in a mixed manner

An interesting comparison and discussion will be connected with a plate subject to prescribed deformations $\varepsilon_1 = \varepsilon_1(t)$ and $\varepsilon_2 = \varepsilon_2(t)$ in its plane. Since in such a thin plate $\sigma_3 = \sigma_3(t) = 0$, the whole process may be regarded as prescribed in a mixed manner. Assume the incompressibility of material and proportional increase of the strains:

$$\varepsilon_1 = \alpha(t)\frac{\sigma_0}{E}, \quad \varepsilon_2 = m\alpha(t)\frac{\sigma_0}{E}, \quad \varepsilon_3 = -(1+m)\alpha(t)\frac{\sigma_0}{E}, \tag{7.1}$$

where $\alpha(t)$ stands for a certain monotonically increasing function of time, and m taken from the interval $-\frac{1}{2} < m < 1$ denotes a constant. This interval of m corresponds to $\sigma_1 > \sigma_2 > \sigma_3$.

In the elastic range, the stresses are equal to

$$\begin{aligned}\sigma_1 &= \frac{4}{3}E(\varepsilon_1+\frac{1}{2}\varepsilon_2) = \frac{4}{3}\alpha\sigma_0(1+\frac{m}{2}),\\ \sigma_2 &= \frac{4}{3}E(\varepsilon_2+\frac{1}{2}\varepsilon_1) = \frac{4}{3}\alpha\sigma_0(m+\frac{1}{2}),\end{aligned} \tag{7.2}$$

and it may be seen that $\sigma_1 > \sigma_2 > \sigma_3 = 0$.

Assume at first the HMH yield condition treated here as exact. Substituting (7.2) into this condition, we determine first the boundary value of α, separating the elastic and the plastic range:

(7.3)
$$\alpha = \bar{\alpha} = \frac{\sqrt{3}}{2(1+m+m^2)},$$

and the corresponding stresses are equal to

(7.4)
$$\sigma_1 = \frac{2+m}{\sqrt{3(1+m+m^2)}}\sigma_0, \quad \sigma_2 = \frac{2m+1}{\sqrt{3(1+m+m^2)}}\sigma_0.$$

The Prandtl-Reuss equations take the form:

(7.5)
$$\begin{aligned} \dot{\varepsilon}_1 - \dot{\varepsilon}_3 &= \lambda\sigma_1 + \frac{3}{2E}\dot{\sigma}_1, \\ \dot{\varepsilon}_2 - \dot{\varepsilon}_3 &= \lambda\sigma_2 + \frac{3}{2E}\dot{\sigma}_2, \end{aligned}$$

and are satisfied by constant values of the stresses (7.4). This stabilization of stresses will be treated as the exact solution.

Consider now the first variant of the Tresca-Guest approximation — combined with the flow rule (7.5). The yield condition gives here simply $\sigma_1 = \sigma_0$, and the boundary value of α, separating the elastic and the plastic range, equals:

(7.6)
$$\alpha = \bar{\alpha} = \frac{3}{2(2+m)}.$$

The corresponding stresses are:

(7.7)
$$\sigma_1 = \sigma_0, \quad \sigma_2 = \frac{2m+1}{m+2}\sigma_0.$$

The Eqs. (7.5) are satisfied by constant values of the stresses (7.7); thus during the whole process the numerical differences with respect to (7.4) do not exceed 13.4% (the largest error corresponds to $m = 0$).

The second variant of approximation combines TG yield condition with the associated flow rule, which gives here:

(7.8)
$$\dot{\varepsilon}_2^p = 0.$$

Hence the change of ε_2 is purely elastic:

(7.9)
$$\dot{\varepsilon}_2 = m\dot{\alpha}\frac{\sigma_0}{E} = \frac{\dot{\sigma}_2}{E} - \frac{1}{2}\frac{\dot{\sigma}_1}{E} = \frac{\dot{\sigma}_2}{E},$$

since $\dot{\sigma}_1 = 0$; this equation determines σ_2 — namely, after integration:

(7.10)
$$\sigma_2 = \frac{2m+1}{m+2}\sigma_0 + m(\alpha - \bar{\alpha})\sigma_0 = \left(\frac{1}{2} + m\alpha\right)\sigma_0.$$

This formula is valid in a certain interval of time only, as long as $0 < \sigma_2 < \sigma_0$. The further part of the process depends on the value of the constant m. If $-1/2 < m < 0$, then σ_2 decreases to zero and reaches this value for $\alpha = -1/(2m)$. This point corresponds to the corner of the TG yield hexagon and the flow rule (7.8) is at this point no longer valid: the stresses simply remain constant, $\sigma_1 = \sigma_0$, $\sigma_2 = \sigma_3 = 0$. If $0 < m < 1$, then σ_2 increases to σ_0, reaches this value at $\alpha = 1/(2m)$, and then remains constant. Finally, if $m = 0$, then $\sigma_2 = \sigma_0/2 = \text{const}$.

The processes discussed are illustrated in Fig. 6, showing the motion in the plane $\varepsilon_1 \varepsilon_2$ of the limit curves separating the regions of active and passive processes. In the first two cases, the curves move in the direction of the radius determining the loading path in this plane. In the third case — TG yield condition with the associated flow rule — the curve moves initially in the direction of ε_1 and then in the second period in the direc-

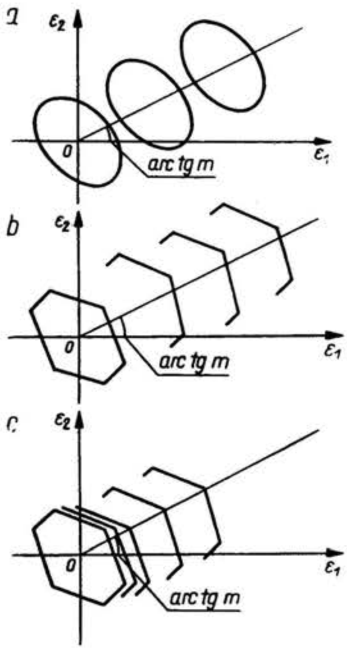

FIG. 6.

tion of the radius. A certain analogy may here be observed with the concepts of kinematical strain-hardening: the cases (a) and (b) (similarity of deviators) correspond to the ZIEGLER hardening rule [14], whereas the case (c) corresponds to the Melan-Ishlinsky-Prager hardening rule, [8, 5, 12] (similar motion is observed there in the stress space, and in our case the strain space is considered).

Other graphical interpretations of the case (c) (TG yield condition with the associated flow rule) are shown in Figs. 7 and 8. Figure 7 divides the strain plane $\varepsilon_1 \varepsilon_2$ into the regions in which the stresses are constant (for simple loading processes), and the regions of varying stresses (the vectors show then the direction of $d\sigma$). Figure 8 shows the trajectories of motion (in the stress plane) of the points representing the processes which are simple in the strain plane. These trajectories are broken lines, running to the corners. A certain instability in the Lapunov sense may here be observed: small change of the initial condition (of the value m, for example in the vicinity of $m = 0$) may result in a very large change of the trajectory. The lines, corresponding to instability, are marked by dashed lines. So the associated flow rule eliminates the instability in Drucker's sense, but may cause instability of the trajectory in Lapunov's sense.

Compare now the numerical results of (7.10) with the result (7.4), treated as exact. The largest error occurs for $m \approx 0$, positive or negative. For very small positive m and for sufficiently large α we obtain $\sigma_2 = \sigma_0$ instead of $\sigma_3 = \sigma_0/\sqrt{3}$; thus the error is 73% from above. For very small negative m and for sufficiently large α we obtain $\sigma_2 = 0$

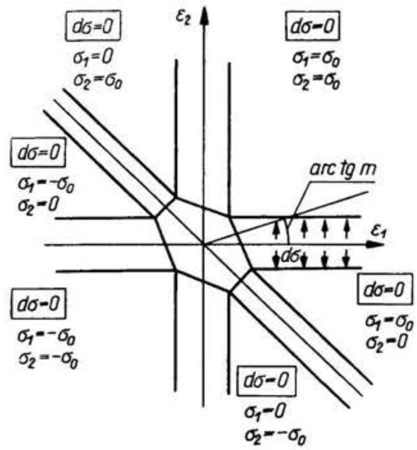

FIG. 7.

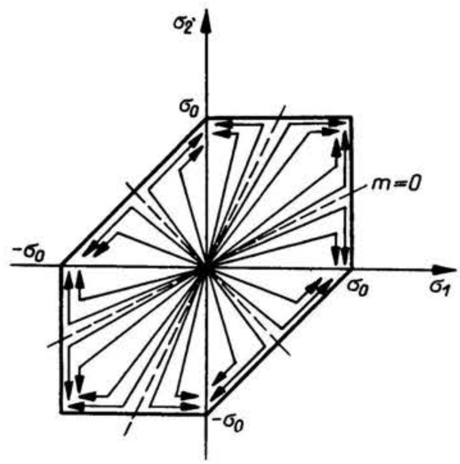

FIG. 8.

instead of $\sigma_2 = \sigma_0/\sqrt{3}$ — thus the error may be called 100% from below. Hence the errors are here much larger then in the first variant of approximation (TG yield condition with the similarity of deviators).

8. Example of the stress distribution in the cross-section of a surface structure

Consider now the purely plastic state in the cross-section of an incompressible plate or shell. Assuming the Love-Kirchhoff hypothesis of straight normals we may write, in principal directions:

$$\varepsilon_1 = \varkappa_1 z + \lambda_1, \quad \varepsilon_2 = \varkappa_2 z + \lambda_2, \tag{8.1}$$

where $\varkappa_j$ denote the increments of the curvatures (or the rates of the curvatures according to the theory of plastic flow), λ_j — the elongations or the rates of elongations of the middle surface. The law of similarity of deviators and the assumption $\sigma_3 = 0$ make it possible to determine the stress distribution — namely,

$$\begin{aligned} \sigma_1 &= \frac{1}{\varphi}[(2\varkappa_1 + \varkappa_2)z + (2\lambda_1 + \lambda_2)], \\ \sigma_2 &= \frac{1}{\varphi}[(2\varkappa_2 + \varkappa_1)z + (2\lambda_2 + \lambda_1)]. \end{aligned} \tag{8.2}$$

Substitution of (8.2) into the HMH yield condition (A. A. ILYUSHIN [4]) determines finally the unknown function φ:

$$\varphi = \frac{\sqrt{3}}{\sigma_0}[(\varkappa_1^2 + \varkappa_1\varkappa_2 + \varkappa_2^2)z^2 + (2\varkappa_1\lambda_1 + 2\varkappa_2\lambda_2 + \varkappa_1\lambda_2 + \varkappa_2\lambda_1)z + (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)]^{1/2}. \tag{8.3}$$

The formulas (8.2) with (8.3) substituted will be considered as exact.

The first variant of approximation by the TG yield condition combines it with (8.2). For example, in the case $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 = 0$, we have

$$\sigma_1 = \sigma_0, \quad \sigma_2 = \frac{(2\varkappa_2+\varkappa_1)z+(2\lambda_2+\lambda_1)}{(2\varkappa_1+\varkappa_2)z+(2\lambda_1+\lambda_2)}\sigma_0. \tag{8.4}$$

In the neighbouring case $\sigma_2 \geqslant \sigma_1 \geqslant \sigma_3 = 0$, the stresses amount to:

$$\sigma_1 = \frac{(2\varkappa_1+\varkappa_2)z+(2\lambda_1+\lambda_2)}{(2\varkappa_2+\varkappa_1)z+(2\lambda_2+\lambda_1)}\sigma_0, \quad \sigma_2 = \sigma_0. \tag{8.5}$$

The boundary value of the variable z, separating the regions of validity of (8.4) and (8.5), may be evaluated by equating σ_1 and σ_2:

$$z = z_b = -\frac{\lambda_1-\lambda_2}{\varkappa_1-\varkappa_2}. \tag{8.6}$$

Similarly, we may determine the stress distributions corresponding to other sides of the TG yield hexagon. They are either linear or hyperbolic. Further details are given in [15].

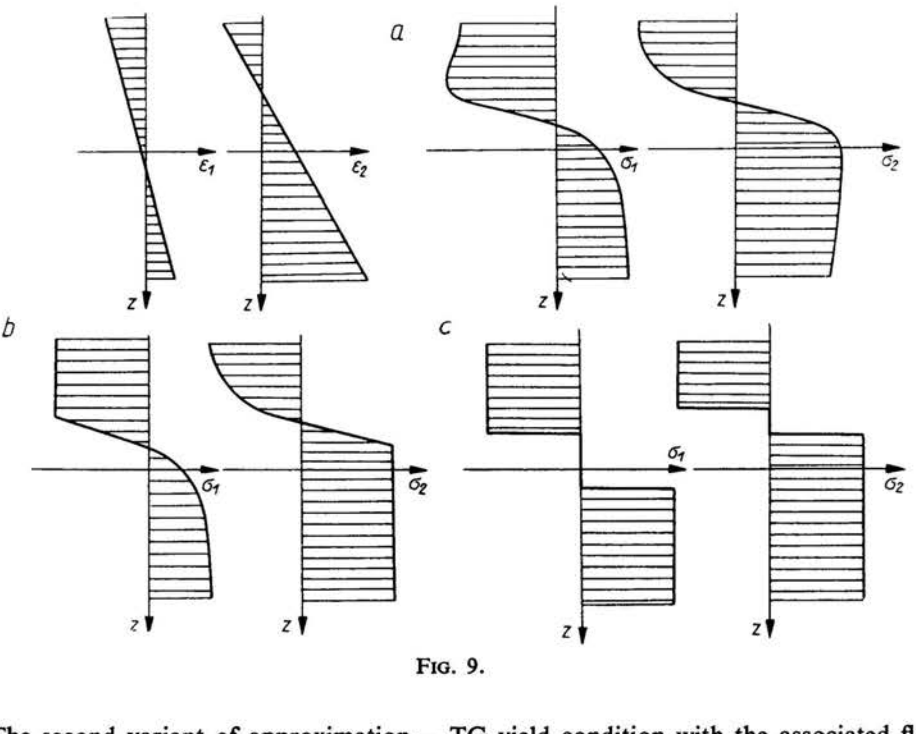

FIG. 9.

The second variant of approximation — TG yield condition with the associated flow rule — suggested by S. M. FEYNBERG and D. C. DRUCKER, was developed by E. T. ONAT and W. PRAGER [10, 11]. It is much simpler, since all the stress distributions are linear, piecewise constant. In fact, the requirement of orthogonality of the strains or strain rates to the subsequent sides of the yield hexagon is — as a rule — in contradiction with (8.1), thus the states of stresses are represented by the corners of the hexagon only.

Figure 9 shows an example of the strain (or strain rate) distribution and of the corresponding stress distributions: (a) according to HMH, (b) TG with the Prandtl-Reuss rule of similarity of deviators, (c) TG with the associated flow rule. The last diagrams are much simpler, but also less exact.

9. Conclusions

The approximation of the HMH yield condition with the (associated) Prandtl-Reuss flow rule by the TG yield condition is, in general, much more exact if we retain the classical Prandtl-Reuss equations than if we make use of the flow rule associated with the TG yield condition. The latter procedure, which is usually simpler, should be considered as a further, less exact step of approximation.

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