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Large macro-homogeneous strain in a random micro-nonhomogeneous elastic space

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ADDITIONAL displacements produced by the micro-nonhomogeneity are assumed to be small. The metric tensor matrices and Christoffel symbols of the convective coordinate system are obtained. Higher order terms are disregarded due to the assumption that Grad u is an extremely small magnitude. The problem considered is stochastically non-linear and may be solved by means of the perturbation method. The solution is expressed in terms of the Green tensor of linear elasticity. The correlation tensor for additional displacement is obtained as also the expression for the expected value of stress.

W pracy przyjęto, że dodatkowe przemieszczenia wywołane mikroniejednorodnością są małe. Otrzymano macierze tensorów metrycznych i symbole Christoffela konwekcyjnego układu współrzędnych. Przyjmując, że Grad u jest wielkością bardzo małą, pominięto człony wyższego rzędu. Omawiane zagadnienie jest stochastycznie nieliniowe i rozwiązano je metodą małego parametru. Rozwiązanie może być przedstawione za pomocą tensora Greena liniowej teorii sprężystości. Otrzymano tensor korelacji dla dodatkowego przemieszczenia oraz wyrażenie na wartość oczekiwaną naprężenia.

В работе принято, что дополнительные перемещения, вызванные микронеоднородностью, малы. Получены матрицы метрических тензоров и символы Кристоффеля сопутствующей системы координат. Принимая, что Grad u является величиной очень малой, отброшены члены высшего порядка. Обсуждаемая проблема стохастически нелинейна и решена методом малого параметра. Решение может быть представлено при помощи тензора Грина линейной теории упругости. Получены тензор корреляций для дополнительного перемещения и выражение для ожидаемого значения напряжения.

1. Geometrical preliminaries

THE PROBLEM of stresses in micro-nonhomogeneous elastic bodies within the linear theory of elasticity was considered in the papers [1, 2]. In this paper is considered the problem arising when the macro-strain is large but the additional displacements due to the random nonhomogeneity are infinitesimal.

In the undeformed state, the positions of the points are described by the Cartesian coordinate system x. In the deformed state, an auxillary Cartesian coordinate system is taken related to the Cartesian coordinate system in the unstrained state by the following equation:

$$Y^i = a^i_{.j} x^j, \quad x^n = b^n_{.m} Y^m, \tag{1.1}$$

where the matrix of transformation **A** with the components $a^i_{.j}$ is a nonsingular set of numbers and $b^n_{.m}$ are components of its inverse. Let us assume that for a homogeneous body the Eqs. (1.1) describe the deformation of the body.

The Cartesian coordinate system in the strained micro-nonhomogeneous body is related to the auxiliary coordinate system Y_i by the equations:

$$Z^i = Y^i + \varepsilon u^i(Y^1, Y^2, Y^3), \tag{1.2}$$

where $u^i(Y^1, Y^2, Y^3)$ are components of random additional displacements due to the nonhomogeneity and ε is a nonrandom parameter. This description does not so far introduce any approximations, but it is suitable only in the case in which the additional displacements are small, and the nonhomogeneities are small and are at least almost statistically homogeneous in space.

The position vector in the deformed state is

$$\mathbf{R} = Z^i\mathbf{E}_i, \tag{1.3}$$

where $\mathbf{E}^i$ are Cartesian base vectors. The Cartesian components are related by the Eqs. (1.2) and (1.1) to the Cartesian components x^i in the undeformed state. Let us consider a convected coordinate system θ^i which in the undeformed state corresponds to the Cartesian coordinate system x^i. The base vectors of the convected coordinate system are

$$\mathbf{G}_k = \frac{\partial R}{\partial \theta^K} = \left[\delta^i_j + \varepsilon\frac{\partial u_i}{\partial Y^j}\right] a^j_{.k} E_i. \tag{1.4}$$

The covariant metric tensor of the convected coordinate system is

$$G_{mn} = a^l_{.m}\left[\delta_{lp} + \varepsilon\left(\frac{\partial u_l}{\partial Y^p} + \frac{\partial u_p}{\partial Y^l}\right) + \varepsilon^2\frac{\partial u^i}{\partial Y^l}\frac{\partial u^j}{\partial Y^p}\delta_{ij}\right] a^p_{.n}. \tag{1.5}$$

Let us introduce the notation

$$\operatorname{Grad}\mathbf{u} = \frac{\partial u_l}{\partial Y^p}\ulcorner E^l E^p \urcorner. \tag{1.6}$$

In this notation and denoting by $\mathbf{A}$ the matrix with coefficients $a^i_{.j}$, the expression for the matrix tensor (1.5) assumes the following matrix form:

$$\mathbf{G} = \mathbf{A}^T[\mathbf{I} + 2\varepsilon(\operatorname{Grad}\mathbf{u})^S + \varepsilon^2(\operatorname{Grad}\mathbf{u})^T(\operatorname{Grad}\mathbf{u})]\mathbf{A}. \tag{1.7}$$

Now, let us assume the gradient of the additional displacement field to be so small that its squares may be disregarded. Within this assumption the expression for the metric tensor of the convected coordinate system reduces to the following linear form in the displacement gradient:

$$\mathbf{G} = \mathbf{A}^T[\mathbf{I} + 2\varepsilon(\operatorname{Grad}\mathbf{u})^S]\mathbf{A}. \tag{1.8}$$

The matrix of the contravariant components of the metric tensor of the convected system is the inverse matrix. Thus, considering linear terms in ε only:

$$\mathbf{G}^{-1} = \mathbf{A}^{-1}[\mathbf{I} - 2\varepsilon(\operatorname{Grad}\mathbf{u})]^S(\mathbf{A}^{-1})^T. \tag{1.9}$$

The cobase vectors of the convected coordinate system, taking into account the linear terms in ε, are

$$\mathbf{G}^m = \left[\delta^l_i - \varepsilon\frac{\partial u_l}{\partial Y^i}\right] b^m_{.l}\mathbf{E}^i. \tag{1.10}$$

The Christoffel symbols of the convected coordinate system are:

$$\Gamma^r_{ij} = \mathbf{G}^r \cdot \mathbf{G}_{i,j} = \varepsilon b^{rp}\frac{\partial^2 u_p}{\partial Y^s \partial Y^t} a^t_{.j} a^s_{.i}. \tag{1.11}$$

2. The state of stress

The invariants of strain are:

$$
\begin{aligned}
I_{\mathrm{I}} &= g^{rs}G_{rs} = \operatorname{tr}\{\mathbf{A}^T[\mathbf{I}+2\varepsilon(\operatorname{Grad}\mathbf{u})^S]\mathbf{A}\},\\
I_{\mathrm{III}} &= G/g = \det\{\mathbf{A}^T[\mathbf{I}+2\varepsilon(\operatorname{Grad}\mathbf{u})^S]\mathbf{A}\},\\
I_{\mathrm{II}} &= g_{rs}G^{rs}I_s = I_{\mathrm{III}}\operatorname{tr}\{\mathbf{A}^{-1}[\mathbf{I}-2\varepsilon(\operatorname{Grad}\mathbf{u})^S](\mathbf{A}^{-1})^T\}.
\end{aligned}
\tag{2.1}
$$

Following the methods used in the study of superposition of infinitesimal strains on finite deformation [3], the invariants may be expanded in series of powers in ε. Retaining linear terms only, it follows

$$
\begin{aligned}
I_{\mathrm{I}} &= \operatorname{tr}\mathbf{A}^T\mathbf{A}+\varepsilon I'_{\mathrm{I}},\\
I_{\mathrm{II}} &= (\det\mathbf{A})^2\operatorname{tr}[(\mathbf{A}^{-1})(\mathbf{A}^{-1})^T]+\varepsilon I'_{\mathrm{II}},\\
I_{\mathrm{III}} &= (\det\mathbf{A})^2+\varepsilon I'_{\mathrm{III}},
\end{aligned}
\tag{2.2}
$$

where

$$
\begin{aligned}
I'_{\mathrm{I}} &= 2\operatorname{tr}[\mathbf{A}^T(\operatorname{Grad}\mathbf{u})^S\mathbf{A}],\\
I'_{\mathrm{II}} &= 2(\det\mathbf{A})^2\{\operatorname{tr}[(\mathbf{A}^{-1})(\mathbf{A}^{-1})^T]\operatorname{tr}(\operatorname{Grad}\mathbf{u})^S+\operatorname{tr}[\mathbf{A}^{-1}(\operatorname{Grad}\mathbf{u})^S(\mathbf{A}^{-1})^T]\},\\
I'_{\mathrm{III}} &= 2(\det\mathbf{A})^2\operatorname{tr}(\operatorname{Grad}\mathbf{u})^S.
\end{aligned}
\tag{2.3}
$$

The increments of the strain invariants I'_K are linear functions in the symmetrical parts of the additional displacement gradients.

For a homogeneous isotropic elastic body in finite deformation, the stresses in the convected coordinate system are:

$$
\tau^{ij} = \Phi g^{ij}+\Psi B^{ij}+pG^{ij},
\tag{2.4}
$$

where

$$
\Phi = \frac{2}{\sqrt{I_{\mathrm{III}}}}\frac{\partial W}{\partial I_{\mathrm{I}}},\qquad \Psi = \frac{2}{\sqrt{I_{\mathrm{III}}}}\frac{\partial W}{\partial I_{\mathrm{II}}},\qquad p = 2\sqrt{I_{\mathrm{III}}}\frac{\partial W}{\partial I_{\mathrm{III}}},
$$

$$
B^{ij} = I_{\mathrm{I}}g^{ij}-g^{ir}g^{js}G_{rs}
$$

and

$$
W = W(I_{\mathrm{I}}, I_{\mathrm{II}}, I_{\mathrm{III}}).
$$

For homogeneous strain, the functions Φ, Ψ and p are constant. When there are additional displacements, the function W is a function of the invariants I_K and their increments I'_K. Thus it follows that up to linear terms in ε:

$$
\Phi = \Phi(I_{\mathrm{I}}, I_{\mathrm{II}}, I_{\mathrm{III}})+\varepsilon\frac{\partial\Phi}{\partial I_K}(I_{\mathrm{I}}, I_{\mathrm{II}}, I_{\mathrm{III}})I'_K.
\tag{2.5}
$$

The same is true for Ψ and p.

Let us assume that the material functions are random fields in space. The functions may be represented by the following equations:

$$
\Phi = \langle\Phi\rangle+\Delta\Phi,\qquad \Psi = \langle\Psi\rangle+\Delta\Psi,\qquad p = \langle p\rangle+\Delta p,
\tag{2.6}
$$

where $\langle\Phi\rangle, \langle\Psi\rangle, \langle p\rangle$ are the expected values, and $\Delta\Phi$, $\Delta\Psi$, Δp are the fluctuations. The expected values of the fluctuations are zero. In an analogous way, the derivatives with respect to the strain invariants may be represented as sums of the expected values and fluctuations. For example,

$$\Phi_{,K} = \langle \Phi_{,K} \rangle + \Delta\Phi_{,K} \tag{2.7}$$

where $\Phi_{,K} = \dfrac{\partial \Phi}{\partial I_K}$.

The tensors B^{ij} and G^{ij} may be expanded as functions of ε. It follows in matrix notation, using previous results, that

$$\mathbf{B} = \underset{0}{\mathbf{B}} + \varepsilon \mathbf{B}', \quad \mathbf{G}^{-1} = \underset{0}{\mathbf{G}}^{-1} + \varepsilon (\mathbf{G}')^{-1}, \tag{2.8}$$

where

$$\underset{0}{\mathbf{B}} = I_{\rm I}\mathbf{I} - \mathbf{A}^T\mathbf{A}, \quad \mathbf{B}' = I'_{\rm I}\mathbf{I} - 2\mathbf{A}^T(\mathrm{Grad}\,\mathbf{u})^S\mathbf{A},$$

$$\underset{0}{\mathbf{G}}^{-1} = \mathbf{A}^{-1}(\mathbf{A}^{-1})^T, \quad (\mathbf{G}')^{-1} = -2\mathbf{A}^{-1}(\mathrm{Grad}\,\mathbf{u})^S(\mathbf{A}^{-1})^T.$$

When all the notations are used, the stress tensor in the convected coordinate system when only linear terms in ε are considered, is given by the following relation:

$$\begin{aligned} \boldsymbol{\tau} = \underset{0}{\boldsymbol{\tau}} + \Delta\Phi\mathbf{I} + \Delta\Psi\underset{0}{\mathbf{B}} + \Delta p\,\underset{0}{\mathbf{G}}^{-1} + \varepsilon[\langle\psi\rangle\mathbf{B}' + \langle p\rangle(\mathbf{G}')^{-1} + \Delta\Psi\mathbf{B}' + \Delta p(\mathbf{G}')^{-1}] \\ + \varepsilon[\langle\Phi_{,K}\rangle I'_K\mathbf{I} + \langle\Psi_{,K}\rangle\mathbf{I}'_K\underset{0}{\mathbf{B}} + \langle p_{,K}\rangle I'_K\underset{0}{\mathbf{G}}^{-1}] + \varepsilon[\Delta\Phi_{,K}I'_K\mathbf{I} + \Delta\Psi_{K,}I'_K\underset{0}{\mathbf{B}} + \Delta p_{,K}I'_K\underset{0}{\mathbf{G}}^{-1}]. \end{aligned} \tag{2.9}$$

The tensor components are given in the convected coordinate system. Thus, in absolute notation the stress tensor is defined by

$$\boldsymbol{\tau} = \tau^{ij}\mathbf{G}_i \oplus \mathbf{G}_j. \tag{2.10}$$

For physical discussion, the stress components in the convected coordinate system have no direct meaning. It is therefore useful to substitute the base vectors expressed in terms of Cartesian base vectors $\mathbf{E}_i$ (1.4) in the Eq. (2.10). Finally, the matrix of the physical components is expressed by the following equation:

$$\underset{\sim}{\mathcal{T}} = \mathbf{A}\boldsymbol{\tau}\mathbf{A}^T + 2\varepsilon\mathbf{A}\boldsymbol{\tau}\mathbf{A}^T(\mathrm{Grad}\,\mathbf{u})^S, \tag{2.11}$$

where $\underset{\sim}{\mathcal{T}}$ is the matrix of the physical components.

When the Eqs. (2.3) are substituted into (2.9) and then into (2.11), the final result is

$$\mathcal{T}^{ij} = \underset{0}{\mathcal{T}}^{ij} + \Delta\underset{0}{\mathcal{T}}^{ij} + \varepsilon\langle c^{ijkl}\rangle u_{k,l} + \varepsilon\Delta c^{ijkl}u_{l,k}, \tag{2.12}$$

where $\underset{0}{\mathcal{T}}^{ij}$ are the physical components in large deformation of a homogeneous body, $\Delta\mathcal{T}^{ij}$ depends on the fluctuation and is a random tensor field, $\langle c^{ijkl}\rangle$ are the expected values of material tensors, Δc^{ijkl} is a random tensor field depending upon the fluctuations.

For a neo-Hookean body, the function W is expressed by

$$W = c(I_{\rm I} - 3), \tag{2.13}$$

and the body is incompressible which means that

$$\det\mathbf{A} = 1, \quad \mathrm{tr}(\mathrm{Grad}\,\mathbf{u})^S = 0. \tag{2.14}$$

In this case, the expressions in the Eq. (2.12) assume the following simple form:

$$\underset{\tilde{0}}{\mathscr{T}} = 2\langle c\rangle \mathbf{A}\mathbf{A}^T + p_0\mathbf{I},$$

$$\Delta\underset{\tilde{0}}{\mathscr{T}} = \frac{\Delta c}{\langle c\rangle}(\underset{0}{\mathscr{T}} - p_0\mathbf{I}) + \Delta p\mathbf{I},$$

$$\langle c^{ijkl}\rangle = \frac{1}{2}[(\underset{0}{\mathscr{T}}^{ik} - p_0\delta^{ik})\delta^{lj} + (\underset{0}{\mathscr{T}}^{il} - p_0\delta^{il})\delta^{kj} + (\underset{0}{\mathscr{T}}^{jk} - p_0\delta^{jk})\delta^{il} + (\underset{0}{\mathscr{T}}^{lj} - p_0\delta_{lj})\delta^{ik}],$$

$$\Delta c^{ijkl} = \frac{\Delta c}{\langle c\rangle}\langle c^{ijkl}\rangle. \tag{2.15}$$

3. Differential equations of the problem

The equilibrium equations in the convected coordinate system θ^i are

$$\tau^{ij}|_j = 0. \tag{3.1}$$

When the expressions for Christoffel symbols (1.11) are used, it follows that

$$\frac{\partial\tau^{ij}}{\partial Y^k}a^K_{.j} + \varepsilon b^{ip}\frac{\partial^2 u_p}{\partial Y^s\partial Y^t}a^t_{.m}a^s_{.j}\tau^{mj} + \varepsilon b^{jp}\frac{\partial^2 u_p}{\partial Y^s\partial Y^t}a^t_{.m}a^s_{.j}\tau^{im} = 0. \tag{3.2}$$

Let us multiply both sides of the Eqs. (3.2) by $a^l_{.i}$. Simple manipulations yield

$$\frac{\partial}{\partial Y^k}(a^l_{.j}\tau^{ij}a^k_{.j}) + \varepsilon\frac{\partial^2 u_l}{\partial Y^s\partial Y^t}a^t_{.m}\tau^{mj}a^s_{.j} + \varepsilon\frac{\partial^2 u_s}{\partial Y^s\partial Y^t}a^l_{.i}\tau^{im}a^t_{.m} = 0. \tag{3.3}$$

From the Eq. (2.11) when only linear terms in ε are considered, it follows that

$$\mathbf{A}\boldsymbol{\tau}\mathbf{A} = \underset{\sim}{\mathscr{T}}[\mathbf{I} - 2\varepsilon(\mathrm{Grad}\,\mathbf{u})^s]. \tag{3.4}$$

Thus the equations of equilibrium, when only linear terms in ε are taken, in terms of the physical components, $\mathscr{T}^{ij}$ become

$$[\mathscr{T}^{lk} - \varepsilon\mathscr{T}^{ls}(u_{s,k} + u_{k,s})]_{,k} + \varepsilon b^{lpsk}u_{p,sk} = 0, \tag{3.5}$$

where

$$b^{lpsk} = \left[\mathscr{T}^{sk}\delta^{lp} + \frac{1}{2}\mathscr{T}^{ls}\delta^{kp} + \frac{1}{2}\mathscr{T}^{lk}\delta^{sp}\right].$$

Expressing the stress components by the displacements by means of the Eq. (2.12) yields the differential displacement equations. The problem is especially simple for the neo-Hookean solid. In this case, the expressions in the square brackets in the Eq. (3.5) in matrix notation assume the following form:

$$\underset{\sim}{\mathscr{T}} - 2\varepsilon\underset{\sim}{\mathscr{T}}(\mathrm{Grad}\,\mathbf{u})^S = \underset{\tilde{0}}{\mathscr{T}} + \frac{\Delta c}{\langle c\rangle}(\underset{\tilde{0}}{\mathscr{T}} - p_0\mathbf{I}) + \Delta p\mathbf{I} - 2p_0(\mathrm{Grad}\,\mathbf{u})^S. \tag{3.6}$$

Thus, the differential equations of the problem are

$$\varepsilon\left[1+\frac{\Delta c}{\langle c\rangle}\right]a^{lpsk}u_{p,sk}+\Delta p_{,l}+\frac{\Delta c_{,k}}{\langle c\rangle}(\underset{0}{\mathcal{T}}^{lk}-p_0\delta^{lk})=0, \tag{3.7}$$

where

$$a^{lpsk}=(\underset{0}{\mathcal{T}}^{sk}-p_0\delta^{sk})\delta^{lp}+\frac{1}{2}(\underset{0}{\mathcal{T}}^{ls}-p_0\delta^{ls})\delta^{pk}+\frac{1}{2}(\underset{0}{\mathcal{T}}^{lk}-p_0\delta^{lk})\delta^{ps}.$$

This equation can be expressed in terms of the large deformations, when it is considered from the first of the Eqs. (2.15) that

$$\underset{0}{\mathcal{T}}-p_0\mathbf{I}=2\langle c\rangle\mathbf{A}\mathbf{A}^T. \tag{3.8}$$

In the Eqs. (3.7), there are three unknown components of the additional displacement vector $\mathbf{u}$ and the unknown scalar function Δp. These equations must be supplemented by the condition of incompressibility (2.14), which yields the fourth equation

$$u_{s,s}=0. \tag{3.9}$$

In view of the incompressibility condition (3.9), the Eq. (3.7) simplifies to the following form:

$$\varepsilon\left[1+\frac{\Delta c}{\langle c\rangle}\right](\underset{0}{\mathcal{T}}^{sk}-p_0\delta^{sk})u_{l,sk}+\Delta p_{,l}=-\frac{\Delta c_{,k}}{\langle c\rangle}(\underset{0}{\mathcal{T}}^{lk}-p_0\delta^{lk}). \tag{3.10}$$

By means of the condition (3.9), the additional displacements can be eliminated from the set of equations (3.10) to obtain a single equation for the unknown function Δp. After differentiating and adding, the author obtained

$$\Delta p_{,ll}=-(\underset{0}{\mathcal{T}}^{lk}-p_0\delta^{lk})\frac{\Delta c_{,kl}}{\langle c\rangle}. \tag{3.11}$$

Thus the unknown function Δp can be found from the Poisson Eq. (3.11), and when the solution is substituted into the Eq. (3.10), we obtain a set of three differential equations for the unknown additional displacements.

For further discussion, it will be useful to substitute

$$\Delta p=q+p_0\frac{\Delta c}{\langle c\rangle}. \tag{3.12}$$

When this notation is used, the Eqs. (3.10) and (3.11) become

$$\begin{gathered}\varepsilon\left(1+\frac{\Delta c}{\langle c\rangle}\right)(\underset{0}{\mathcal{T}}^{sk}-p_0\delta^{sk})u_{l,sk}+q_{,l}=-\frac{\Delta c_{,k}}{\langle c\rangle}\underset{0}{\mathcal{T}}^{lk},\\ q_{,ll}=-\underset{0}{\mathcal{T}}^{lk}\frac{\Delta c_{,kl}}{\langle c\rangle}.\end{gathered} \tag{3.13}$$

The equations are linear in the random additional displacements and random function q.

4. Solution of the problem

Particularly simple is the solution in the case in which we consider uniform pressure in the body. In this case, the stresses in the homogeneous body are

$$\underset{0}{\mathscr{T}}^{ij} = [2\langle c\rangle + p_0]\,\delta^{ij} \tag{4.1}$$

and the Eqs. (3.10), (3.9) are satisfied if we take $u_l = 0$ and

$$\Delta p = -\frac{\Delta c}{\langle c\rangle}\,[2\langle c\rangle]\,\delta^{ij}. \tag{4.2}$$

When we substitute into the Eqs. (2.15) and (2.12), it emerges that in this case the inhomogeneities have no influence upon the stresses.

So far it has been assumed that the additional displacements are small, but nothing has been said about the fluctuations of the material constants. Let us assume that the increments in material functions are of the order ε and solve the problem by the perturbation method. A direct solution is difficult because, although the equations are geometrically linear, they are statistically non-linear, since they include a product of the random field Δc and the random displacement field. Let us expand the unknown displacements and the function q in terms of the small parameter ε;

$$\mathbf{u} = \sum_{n=0} \varepsilon^n \underset{n}{u}, \qquad q = \sum_{n=0} \varepsilon^n q_n. \tag{4.3}$$

After substitution into the Eqs. (3.10) and (3.9), the author obtained for the first approximation

$$\begin{gathered}(\underset{0}{\mathscr{T}}^{sk} - p_0\,\delta^{sk})\underset{0}{u}_{l,sk} + \underset{0}{q}_{,l} = -\frac{\Delta c_{,k}}{\langle c\rangle}\underset{0}{\mathscr{T}}^{lk},\\ \underset{0}{u}_{s,s} = 0,\end{gathered} \tag{4.4}$$

and for the subsequent steps

$$\begin{gathered}(\underset{0}{\mathscr{T}}^{sk} - p_0\,\delta^{sk})\underset{n}{u}_{l,sk} + \underset{n}{q}_{,l} = -\frac{\Delta c}{\langle c\rangle}(\underset{0}{\mathscr{T}}^{sk} - p_0\,\delta^{sk})\underset{n-1}{u}_{l,sk},\\ \underset{0}{u}_{s,s} = 0.\end{gathered} \tag{4.5}$$

According to the Eqs. (4.4) and (4.5), the problem reduces to the solution of successive equations for an anisotropic elastic incompressible body within the linear theory.

For an infinite body, the equations may easily be solved by means of the Green's tensors for the anisotropic body. It follows for the first approximation that

$$\begin{gathered}\underset{0}{u}_p(\underset{1}{Y}^k) = -\int_V K_{pl}(\underset{2}{Y}^j - \underset{1}{Y}^j)\frac{\Delta C_{,k}}{\langle c\rangle}(\underset{2}{Y}^j)\underset{0}{\mathscr{T}}^{lk}d\underset{2}{V},\\ \underset{0}{q}(\underset{1}{Y}^k) = -\int_V K_l(\underset{2}{Y}^j - \underset{1}{Y}^j)\frac{\Delta c_{,k}}{\langle c\rangle}(\underset{2}{Y}^j)\underset{0}{\mathscr{T}}^{lk}d\underset{2}{V}.\end{gathered} \tag{4.6}$$

When we apply a Green identity, the Eqs. (4.6) can be transformed to

$$(4.7)\qquad \begin{aligned} \underset{0}{u_p}(\underset{1}{Y^k}) &= \int\limits_V \frac{\Delta c}{\langle c\rangle}(\underset{2}{Y^j})\, h_p(\underset{2}{Y^j}-\underset{1}{Y^j})\, d\underset{2}{V}, \\ \underset{0}{q}(\underset{1}{Y^k}) &= \int\limits_V \frac{\Delta c}{\langle c\rangle}(\underset{2}{Y^j})\, h(\underset{2}{Y^j}-\underset{1}{Y^j})\, d\underset{2}{V}, \end{aligned}$$

where

$$h_p(\underset{2}{Y^j}-\underset{1}{Y^j}) = \underset{0}{\mathcal{T}^{lk}} \frac{\partial}{\partial \underset{2}{Y^k}} [K_{pl}(\underset{2}{Y^j}-\underset{2}{Y^j})],$$

$$h(\underset{2}{Y^j}-\underset{1}{Y^j}) = \underset{0}{\mathcal{T}^{lk}} \frac{\partial}{\partial \underset{2}{Y^k}} [K_l(\underset{2}{Y^j}-\underset{1}{Y^j})].$$

When we restrict our discussion to the first approximation, it follows that the expected value of the additional displacements and the function q are zero because we have assumed that the expected value of Δc is zero.

To find the correlation tensors between the additional displacement vector and the function describing the inhomogeneities, let us multiply the first of the expressions (4.7) by $\Delta c(\underset{3}{Y^j})$ on both sides and take the expected values:

$$(4.8)\qquad \underset{u,\Delta c}{R_p} = \langle \underset{0}{u_p}(\underset{1}{Y^k})\Delta c(\underset{3}{Y^k})\rangle = \langle c\rangle^{-1} \int\limits_V \underset{\Delta c,\Delta c}{R}(\underset{3}{Y^j}, \underset{2}{Y^j})\, h_p(\underset{2}{Y^j}-\underset{1}{Y^j})\, d\underset{2}{V},$$

where $\underset{\Delta c,\Delta c}{R}$ is the correlation function for the function Δc, and $d\underset{2}{V} = d\underset{2}{Y^1}d\underset{2}{Y^2}d\underset{2}{Y^3}$. When the material functions are statistically homogeneous, their correlation function depends on the difference of coordinates of points:

$$(4.9)\qquad \underset{\Delta c,\Delta c}{R} = \underset{\Delta c,\Delta c}{R}(\underset{3}{Y^j}-\underset{2}{Y^j}).$$

In this case, the Eq. (4.8) reduces to the following form:

$$(4.10)\qquad \underset{u,\Delta c}{R_p}(\underset{3}{Y^j}-\underset{1}{Y^j}) = \langle c\rangle^{-1} \int\limits_V \underset{\Delta c,\Delta c}{R}(\underset{3}{Y^j}-\underset{1}{Y^j}-y^j)\, h_p(y^j)\, dV,$$

where

$$y^j = \underset{2}{Y^j}-\underset{1}{Y^j}, \qquad dV = dy'dy^2dy^3.$$

Thus, the correlation tensor function depends upon the difference of coordinates only.

The correlation tensors between the displacement vectors at point 1 and 3 by definition are

$$(4.11)\qquad \underset{u,u}{R_{pr}}(\underset{3}{Y^j}, \underset{1}{Y^j}) = \langle \underset{1}{u_p}\underset{3}{u_r}\rangle .$$

When we substitute displacements expressed by the solutions (4.7), it follows that:

$$R_{\underset{u,u}{pr}}(\underset{3}{Y^j}, \underset{1}{Y^j}) = \langle c\rangle^{-2} \int_V \int_V \underset{\Delta c,\Delta c}{R} (\underset{4}{Y^j}, \underset{2}{Y^j}) h_p(\underset{2}{Y^j} - \underset{1}{Y^j}) h_r(\underset{4}{Y^j} - \underset{3}{Y^j}) d\underset{2}{V} d\underset{4}{V}, \tag{4.12}$$

where

$$d\underset{2}{V} = d\underset{2}{Y^1} d\underset{2}{Y^2} d\underset{2}{Y^3}, \quad d\underset{4}{V} = d\underset{4}{Y^1} d\underset{4}{Y^2} d\underset{4}{Y^3}.$$

When the field Δc is statistically homogeneous:

$$R_{\underset{u,u}{pr}}(\underset{3}{Y^j} - \underset{1}{Y^j}) = \langle c\rangle^{-2} \int_V \int_V \underset{\Delta c,\Delta c}{R} (\underset{3}{Y^j} - \underset{1}{Y^j} + \underset{4}{y^j} - \underset{2}{y^j}) h_p(\underset{2}{y^j}) h_r(\underset{4}{y^j}) d\underset{2}{V} d\underset{4}{V}, \tag{4.13}$$

where

$$\underset{2}{y^j} = \underset{2}{Y^j} - \underset{1}{Y^j}, \quad \underset{4}{y^j} = \underset{4}{Y^j} - \underset{3}{Y^j},$$

and

$$d\underset{2}{V} = d\underset{2}{y^1} d\underset{2}{y^2} d\underset{2}{y^3}, \quad d\underset{4}{V} = d\underset{4}{y^1} d\underset{4}{y^2} d\underset{4}{y^3}.$$

Let us calculate the stresses due to the first approximation. According to the Eq. (2.12) it follows

$$\mathscr{T}^{ij} = \underset{0}{\mathscr{T}^{ij}} + \frac{\Delta c}{\langle c\rangle} \underset{0}{\mathscr{T}^{ij}} + \underset{0}{q} \delta^{ij} + \langle c^{ijkl}\rangle \int_V \left[1 + \frac{\Delta c}{\langle c\rangle}(\underset{1}{Y^j})\right] \frac{\Delta c}{\langle c\rangle}(\underset{2}{Y^j}) \frac{\partial}{\partial \underset{1}{Y^l}} h_k(\underset{2}{Y^j} - \underset{1}{Y^j}) d\underset{2}{V}. \tag{4.14}$$

For practical calculations, the expected value of the stress is of great importance. When the expected value is taken in the Eq. (4.14), it follows that

$$\langle \mathscr{T}^{ij}\rangle = \underset{0}{\mathscr{T}^{ij}} + \langle c^{ijkl}\rangle \int_V \langle c\rangle^{-2} \underset{\Delta c,\Delta c}{K} (\underset{2}{Y^p}, \underset{1}{Y^p}) \frac{\partial}{\partial \underset{2}{Y^l}} h_k(\underset{2}{Y^p} - \underset{3}{Y^p}) d\underset{2}{V}. \tag{4.15}$$

Thus, even in the first approximation, the expected value of the stress depends upon the distribution of nonhomogeneities in space, and therefore it is not described by the field quantities at the point. If the random field of the material properties is statistically homogeneous, then the Eq. (4.15) reduces to

$$\langle \mathscr{T}^{ij}\rangle = \underset{0}{\mathscr{T}^{ij}} + \langle c^{ijkl}\rangle \int_V \langle c\rangle^{-2} \underset{\Delta c,\Delta c}{K}(y^p) h_{k,l}(y^p) dV. \tag{4.16}$$

In this case, the expected stress field is constant.

It must be noted that in the integrals the Green's functions decrease rapidly when we move from the point at which we calculate the quantities, thus only the contributions of the neighbourhoods of the points are of importance. In the calculations, the next steps can be taken. Then, in general, the expected values of the additional displacements will be different from zero.

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