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Concentric supersonic thermal sources in a perfect gas

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THE PROBLEM of a small spherical body of perfect gas subject to the action of a concentric supersonic thermal wave is solved. The solution obtained is valid until the thermal wave has reached the neighbourhood of the centre of the gas ball, in which the thermal wave is caught up by the shock wave. The results of the present paper, together with the solution of the problem of a subsonic thermal wave now in preparation, furnish, in addition to direct technical application, a point of departure for the obtainment of averaged equations of laser heating and compression of plasma, taking into consideration thermal and shock wave fronts and the recovery of the fusion energy.

W pracy podano rozwiązanie dynamiki kulki gazu doskonałego, poddanego działaniu koncentrycznej naduderzeniowej fali termicznej. Rozwiązanie skonstruowano do momentu osiągnięcia przez falę termiczną otoczenia centrum kulki, w którym następuje wyprzedzenie fali termicznej przez uderzeniową. Wyniki pracy niniejszej, łącznie z przygotowywanym rozwiązaniem dla poduderzeniowej fali termicznej, poza ich bezpośrednią aplikacją techniczną stanowić będą punkt wyjścia dla konstrukcji uśrednionych równań laserowego nagrzewania i kompresji plazmy przy uwzględnieniu frontów fal termicznych i uderzeniowych oraz odzysku energii syntezy.

В работе дается решение динамики шарика идеального газа подвергнутого действию концентрической, сверхударной тепловой волны. Решение построено для момента достижения тепловой волной окрестности центра шарика, в которой наступает выход тепловой волны перед ударную волну. Результаты настоящей работы, совместно с приготовливаемым решением для доударной тепловой волны, кроме их непосредственного технического применения будут составлять исходную точку для построения усредненных уравнений лазерного нагрева и сжатия плазмы при учете фронтов тепловых и ударных волн, а также выхода энергии синтеза.

1. Introduction

REFERENCE [1, 2, 3, 4] were devoted to the problem of plane super sub and transonic waves produced in a perfect gas, by a thermal wave moving at constant speed. In addition to the possibility of direct application, these works furnished a simplified model constituting a point of departure for the establishment of averaged equations of laser heating of plasma taking into account the infleunce of thermal and shock wave fronts and the recovered fusion energy [5, 6].

The analogous problem for concentric spherical waves is still more important for practice. For elastic waves, a number of solutions were obtained in [7, 8].

Some averaged descriptions for problems of laser heating and concentric compression of plasma were studied in Refs. [9 10, 11, 12]. In these averaged descriptions electron-type heat conduction [13] waves and hydrodynamic expansion waves [14, 15, 16] are considered separately. A joint analysis of plane thermal and shock waves was performed on the grounds of the averaged description in Refs. [5, 6].

To realize an analogous averaged description for a concentric spherical wave we must first construct, similarly to Refs. [2, 3, 4], solutions for a concentric spherical wave in perfect gas — that is, a concentric super and subsonic thermal wave.

This problem will be dealt with in the present and the subsequent paper. The present paper will be devoted to the problem of concentric supersonic spherical thermal wave, for which we shall determine the pressure, velocity and density in function of r, t and the point at which the thermal wave is caught up by the shock wave (in the neighbourhood of the centre of the gas ball).

In the subsequent paper, an analogous problem will be considered for a subsonic thermal wave. Similarly to Refs. [2, 3, 4], the two works, in addition to having direct applications, provide a point of departure for the establishment of a theory of averaged laser compression of plasma taking into consideration the influence of thermal and shock wave fronts.

In Sec. 2, we shall formulate the fundamental equations of the problem. In Sec. 3, we shall discuss a procedure for numerical solution of the equations, and Sec. 4 will be devoted to the numerical analysis of the solution.

In Sec. 5, a particular closed-form solution on the line $t = r_0/c$ will be obtained as a means of control of the numerical solution and a theoretical contribution to the methods for obtaining such solutions. In the concluding remarks we summarize the results.

2. The fundamental equations

Let us analyse the concentric motion of a perfect gas contained within a sphere the initial radius of which is r_0. In this body of gas a thermal wave front moves concentrically at

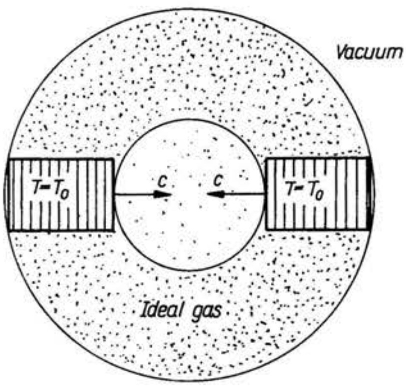

FIG. 1.

a constant supersonic speed c, the intensity T_0 being also constant (Fig. 1). This gas ball is located in vacuum.

We shall confine ourselves to the analysis of the parameters of state of the gas connected with the incident thermal wave and the action of the free surface of the gas. Problems of reflection of wave fronts from the centre will not be dealt with, since they require consideration of the problem of catching up of the thermal wave by the shock wave which takes place close to the centre.

This problem will be dealt with in a separate paper devoted to the problem of subsonic velocity of a thermal wave. Our analysis will be made in Lagrangian coordinates r, t.

The problem referred to is formulated in an unambiguous manner by the equations

$$\varrho_0 v_{,t} = -\left(\frac{r+u}{r}\right)^2 p_{,r}, \tag{2.1}$$

$$1+u_{,r} = \left(\frac{r}{r+u}\right)^2 \frac{\varrho_0}{\varrho}, \tag{2.2}$$

$$p = RT_0\varrho = \varrho a^{*2}, \quad a^{*2} = RT_0, \tag{2.3}$$

from which we find, after some manipulations, the following quasi-linear equation

$$u_{,tt} - \frac{a^{*2}}{(1+u_{,r})^2} u_{,rr} + 2a^{*2} \frac{u-ru_{,r}}{r(r+u)(1+u_{,r})} = 0. \tag{2.4}$$

In the set of coordinates assumed, the local velocity of perturbation is

$$a(u_{,r}) = \frac{a^*}{1+u_{,r}} = \left(\frac{r+u}{r}\right)^2 \frac{p}{\varrho_0 a^*}. \tag{2.5}$$

The Eq. (2.4) can be replaced by the equivalent set of equations along the characteristics

$$dv = -a^* \frac{dp}{p} - 2a^* \frac{v}{r+u} dt \tag{2.6}$$

for $dr = a(u_{,r})\,dt$ and

$$dv = a^* \frac{dp}{p} + 2a^* \frac{v}{r+u} dt \tag{2.7}$$

for $dr = -a(u_{,r})dt$.

The relations (2.6) and (2.7) take, on integrating, the finite form

$$\begin{aligned} v &= -a^* \ln p - 2a^* \int \frac{v}{r+u} dt \quad \text{for} \quad dr = adt, \\ v &= a^* \ln p + 2a^* \int \frac{v}{r+u} dt \quad \text{for} \quad dr = -adt. \end{aligned} \tag{2.8}$$

To complete the set of Eqs. (2.8) use must be made of the relations of kinematic and dynamic continuity at the thermal wave front

$$\varrho_0 c = \varrho(c-v), \quad \varrho_0 c v_1 = p_1 - p_0, \tag{2.9}$$

where c is the velocity of the thermal wave front. This completes the formulation of the problem.

3. Numerical solution of the equations

The following dimensionless quantities will be introduced in the interests of simplicity:

$$R = \frac{r}{r_0}, \quad T = \frac{a^* t}{r_0}, \quad U = \frac{u}{r_0}, \quad V = \frac{v}{a^*},$$
$$P = \frac{p}{\varrho_0 a^{*2}}, \quad A = \frac{a}{a^*}, \quad C = \frac{c}{a^*} = n. \tag{3.1}$$

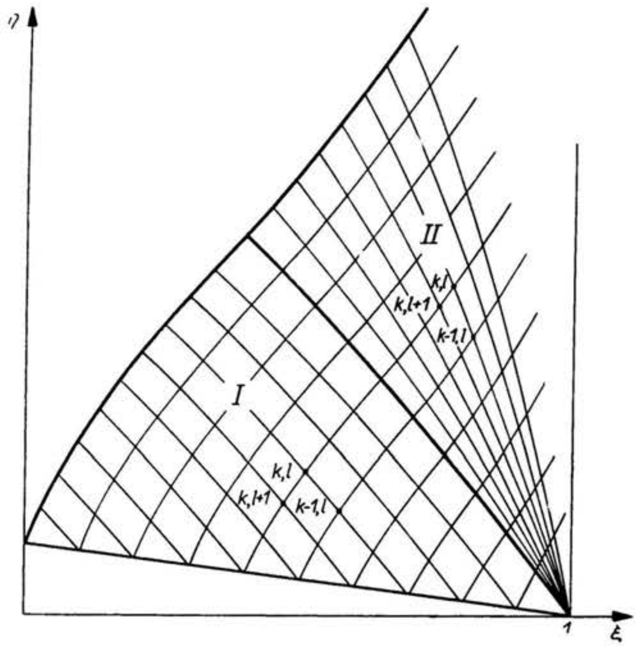

FIG. 2.

The solution will be found by means of the method of characteristics [17]. The characteristic pattern is as shown in Fig. 2. The recurrence equations for particular values of the parameters of the problem will, in particular regions, take the form

Region I

$$T_{k,l} = \frac{1}{A_{k,l+1} + A_{k-1,l}} [R_{k-1,l} - R_{k,l+1} + A_{k,l+1} T_{k,l+1} + A_{k-1,l} T_{k-1,l}],$$

$$R_{k,l} = \frac{1}{2} [R_{k,l+1} + R_{\varkappa-1,l} + A_{k,l+1}(T_{k,l} - T_{k,l+1}) - A_{k-1,l}(T_{k,l} - T_{k-1,l})], \tag{3.2}$$

$$V_{k,l} = \frac{1}{2} \Bigg\{ V_{k,l+1} + V_{k-1,l} + \ln P_{k,l+1} - \ln P_{k-1,l} + 2 \Bigg[\left(\frac{V}{R+U} \right)_{k-1,l} (T_{k,l} - T_{k-1,l})$$
$$- \left(\frac{V}{R+U} \right)_{k,l+1} (T_{k,l} - T_{k,l+1}) \Bigg] \Bigg\},$$

[3.2] *cont.*

$$P_{kl} = \exp\left\{\frac{1}{2}[V_{k,l+1} - V_{k-1,l} - \ln P_{k,l-1} + \ln P_{k-1,l}] - \left[\left(\frac{V}{R+U}\right)_{k-1,l}(T_{k,l} - T_{k-1,l}) + \left(\frac{V}{R+U}\right)_{k,l+1}(T_{k,l} - T_{k,l+1})\right]\right\},$$

$$U_{k,l} = U_{k,l+1} + \frac{V_{k,l} + V_{k,l+1}}{2}(T_{k,l} - T_{k,l+1}),$$

$$A_{k,l} = \left(1 + \frac{U}{R}\right)^2_{k,l} P_{k,l}.$$

The initial values of the parameters will be found from the thermal wave front. In agreement with [2] we have, bearing in mind (3.1):

$$P_C = \frac{C^2}{2}\left(1 - \sqrt{1 - \frac{4}{C^2}}\right) = \frac{\varrho_C}{\varrho_0},$$
(3.3)
$$V_C = -\frac{P_C}{C} = C\left(\frac{1}{P_C} - 1\right), \quad U_C = 0.$$

By applying the indicial notation in agreement with the scheme of Fig. 2, we have

$$P_{k,k} = \frac{C^2}{2}\left(1 - \sqrt{1 - \frac{4}{C^2}}\right),$$
(3.4)
$$V_{k,k} = -\frac{P_{k,k}}{C}, \quad U_{k,k} = 0, \quad A_{k,k} = P_{k,k}$$
$$T_{k,k} = \frac{1 - R_{k,k}}{C}, \quad R_{k,k} = 1 - (k-1)h,$$

where h is the computation step.

Region II

In this region, the gas is expanded. A set of characteristic lines pass through the point 1.1 (which means that the solution is not unique). In order to make the problem unique, we assume initial values of the parameters $A_{1,l} = P_{1,l}$, where the index l denotes successive natural numbers.

Within Region II the computation is carried out according to the scheme (3.2) except for the points 2,1, for which we have

$$T_{2,l} = \frac{1}{A_{2,l+1} + A_{1,l}}(R_{1,l} - R_{2,l+1} + A_{2,l+1}T_{2,l+1} + A_{1,l}T_{1,l}),$$

$$R_{2,l} = \frac{1}{2}[R_{2,l+1} + R_{1,l} + A_{2,l+1}(T_{2,l} - T_{2,l+1}) - A_{1,l}(T_{2,l} - T_{1,l})],$$
(3.5)
$$V_{2,l} = V_{2,l+1} - 2\left(\frac{V}{R+U}\right)_{2,l+1}(T_{2,l} - T_{2,l+1}), \quad P_{2,l} = P_{1,l},$$

$$U_{2,l} = U_{2,l+1}\frac{V_{2,l} + V_{2,l+1}}{2}(T_{2,l} - T_{2,l+1}),$$

A routine was prepared for the above equations for an EMC and the computation was carried out. In agreement with Ref. [17] the second approximation was used, giving accuracy of order $O(h^3)$. The results are discussed in the next section.

4. Computation results and analysis

As indicated in the foregoing section, numerical computation was carried out by means of an EMC. The results and the data are represented in Figs. 3, 4, 5 and 6.

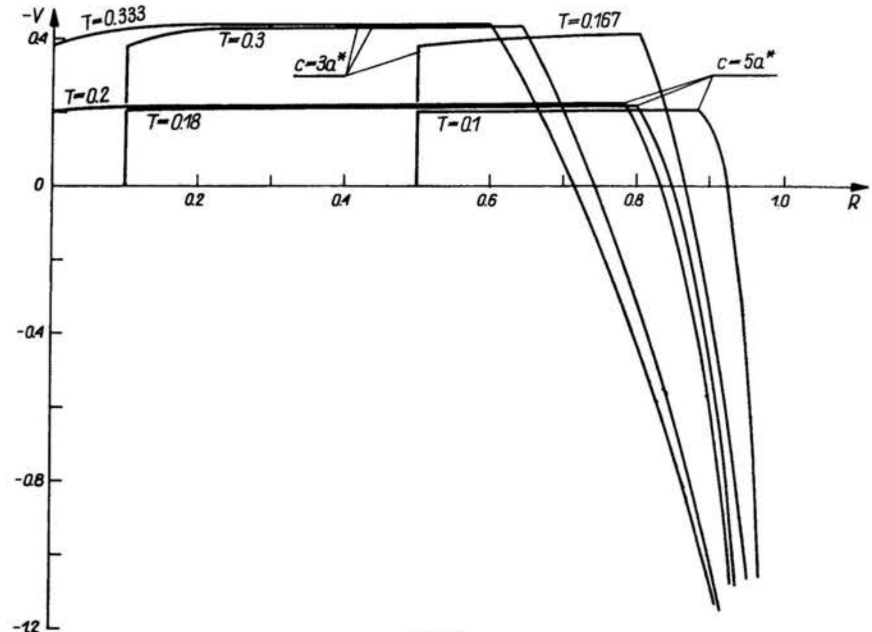

FIG. 3.

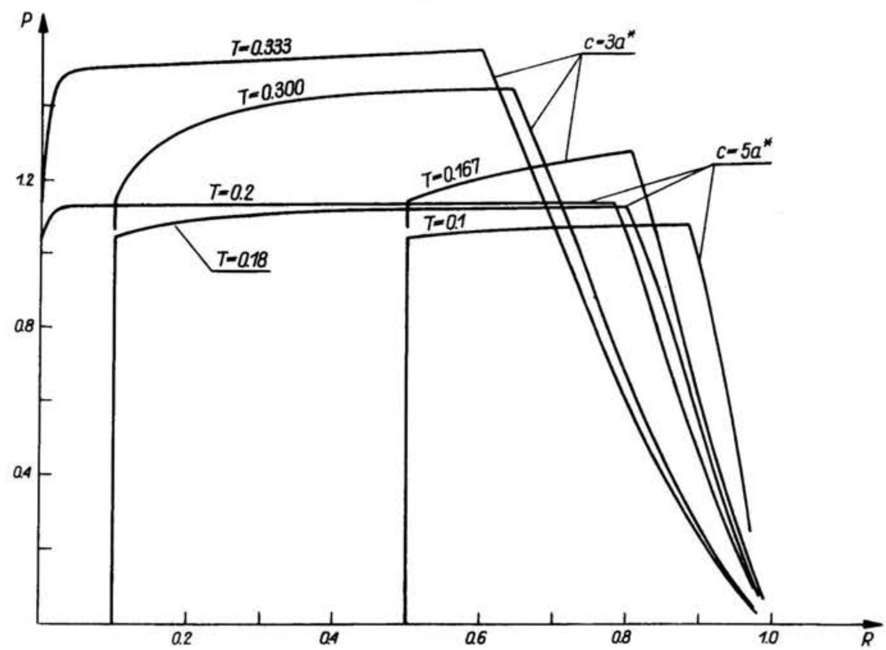

FIG. 4.

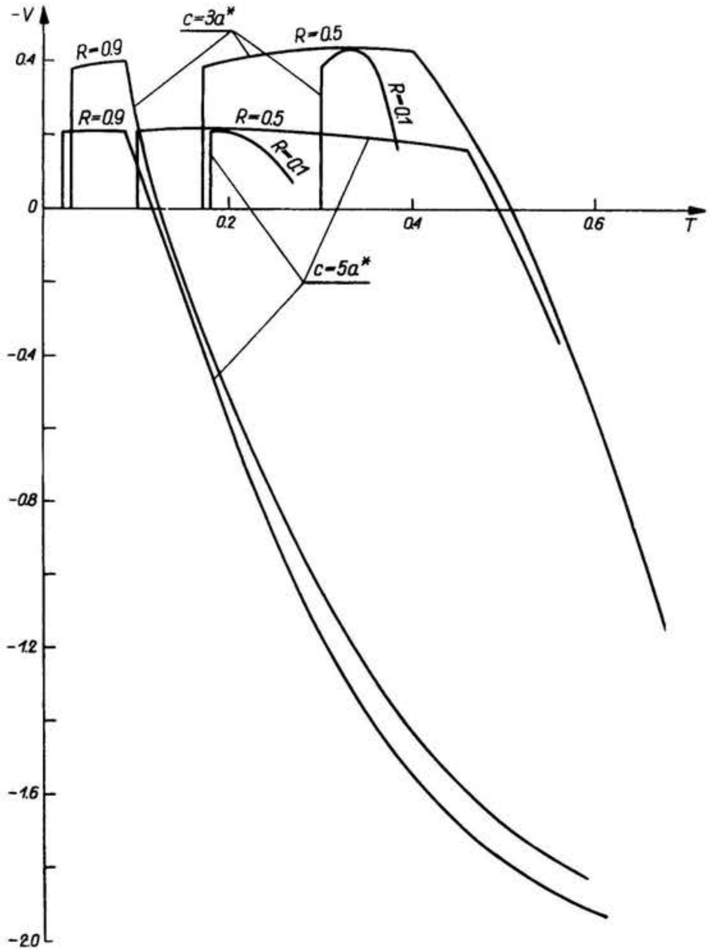
-V
0.4
0
-0.4
-0.8
-1.2
-1.6
-2.0
0.2
0.4
0.6
T
R=0.9
R=0.9
c=3a*
R=0.5
R=0.5
R=0.1
R=0.1
c=5a*

FIG. 5.

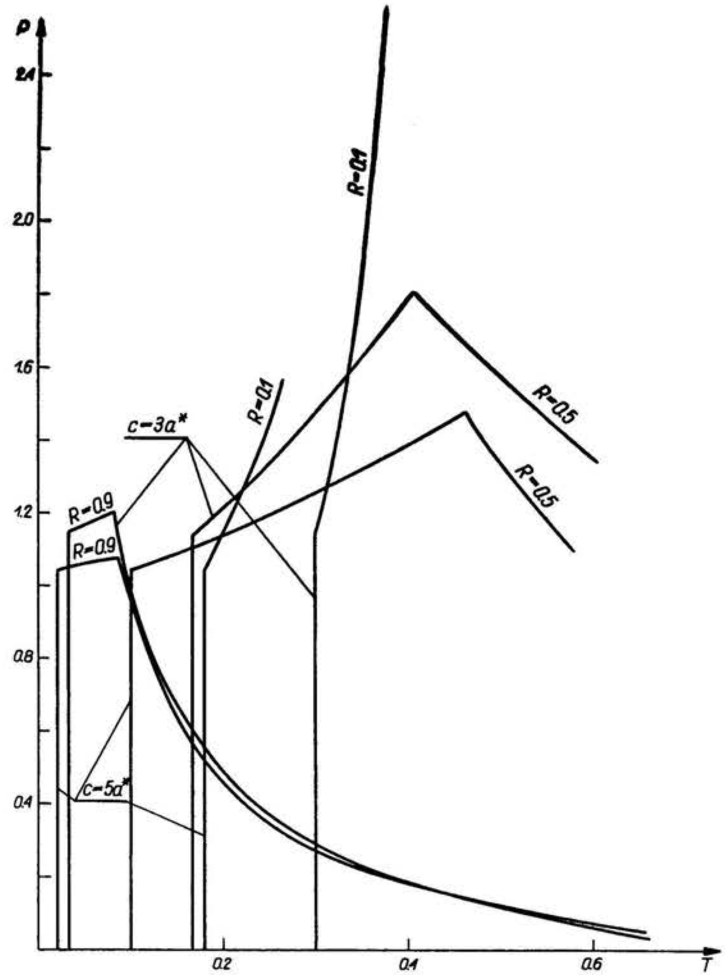
p
2.4
2.0
1.6
1.2
0.8
0.4
0.2
0.4
0.6
T
R=0.1
R=0.1
R=0.5
R=0.5
c=3a*
R=0.9
R=0.9
c=5a*

FIG. 6.

Figures 3 and 4 show the variation of the dimensionless quantities v and $\varrho(p \sim \varrho)$ in given sections t = const of the phase plane. Figures 5 and 6 show analogous diagrams for r = const and various n.

In addition to the numerical results (which are verified additionally in Sec. 5) we can draw from these figures the following conclusions.

1. The values of ϱ and v at the thermal wave front remain constant similarly to the case of the plane wave [2].

2. Directly behind the thermal wave front we have isothermal compression of the gas the rate of which increases as the wave front approaches the centre.

3. When the thermal wave front approaches the centre, the density distribution approaches a constant distribution along r except the zone of rarefaction which is separated by an unloading wave. The action of the free surface is in our case (in which vacuum is assumed outside the gas ball) very rapid.

4. The velocity distribution shows similar features.

5. When the centre is approached, the increase in density and velocity behind the thermal front is considerable (about 100%).

6. If the shock wave front approaches and exceeds the speed of the thermal wave front (in numerical computations it is manifested by intersection between the characteristics and the thermal wave front), the variability of v and ϱ becomes of a different type, singularities occurring in a sufficiently small neighbourhood of the centre.

7. Zones of durable constant parameters occur in the neighbourhood of $r_0/2$.

8. The above results lead to very promising conclusions concerning the averaging method in connection with the remark made in p. 3 and 7.

5. A particular closed-form solution for $t = r_0/c$

It will be shown that an (analytic) closed-form solution can be found for the line $t = r_0/c$. This will enable verification of the numerical analysis and, in addition, a method will be obtained for constructing such solutions.

We shall start out from the equations of the problem expressed in Eulerian coordinates

$$(5.1)\qquad \begin{aligned} &\dot{\varrho}+\varrho' v+\varrho v'+\frac{2}{r}\varrho v = 0,\\ &\varrho(\dot{v}+vv')+A\varrho' = 0, \end{aligned}$$

where $p = A\varrho$, $A = RT_0 = a^{*2}$, the dot and the prime denoting differentiation with respect to time and r, respectively.

In the neighbourhood of the point $ct \approx r_0$, the coefficient $2/r$ in the Eqs. (5.1) can be replaced by

$$(5.2)\qquad \frac{2}{r} \approx \frac{2}{r+ct-r_0}.$$

Then, a solution of the set of equations (5.1) may be sought in the form

$$(5.3)\qquad \varrho = \varrho(ct+r-r_0) = \varrho(z), \quad v = v(ct+r-r_0) = v(z).$$

Our equations become

(5.4)
$$(c+v)(\dot{\ln\varrho})+\dot{v}+\frac{2}{z}v = 0,$$
$$(c+v)\dot{v}+A(\dot{\ln\varrho}) = 0,$$

the dot denoting now differentiation with respect to the argument. On eliminating $\dot{\ln\varrho}$ from (5.4), we find:

(5.5)
$$\frac{dv}{v}[(c+v)^2-A] = 2A\frac{dz}{z}$$

or, on integrating and determining the integration constant from the condition

(5.6)
$$v = v_0 \quad \text{for} \quad z \to \varepsilon,$$

the equation

(5.7)
$$\frac{a^{*2}}{v_0^2}\left(\frac{c^2}{a^{*2}}-1\right)\ln u+\frac{2c}{v_0}(u-1)+\frac{u^2-1}{2}-2\frac{a^{*2}}{v_0^2}\ln\frac{z}{\varepsilon} = 0,$$

where

(5.8)
$$u = \frac{v}{v_0}.$$

From (5.4) and the condition of $\varrho = \varrho_f$ for $u = 1$, we find ϱ:

(5.9)
$$\varrho = \varrho_f e^{\frac{v_0^2}{a^{*2}}\left[\frac{c}{v_0}(1-u)+\frac{1}{2}(1-u^2)\right]}.$$

The condition (5.6) expresses the fact that the shock wave catches up with the thermal wave before reaching the centre; therefore it has been written for the close neighbourhood of the centre. In this sense, the relation (5.2) is approximate. Thus, by analogy to the similarity theory [15] our solution preserves its sense for r or order ε, that is, for

(5.10)
$$r \leqslant 2\varepsilon.$$

Let us denote

(5.11)
$$c = na^*.$$

Then, on the basis of the analogous solution for the thermal wave front (Ref. [2]), we have

(5.12)
$$\frac{v_0}{a^*} = \frac{n}{2}\left[1-\sqrt{1-\frac{4}{n^2}}\right].$$

In this connection, the expression for ϱ takes the form

(5.13)
$$\varrho = \varrho_f e^{\frac{n^2}{4}\left(1-\sqrt{1-\frac{4}{n^2}}\right)^2\left[\frac{2-u^2}{2}+\frac{2(u-1)}{1-\sqrt{1-\frac{4}{n^2}}}\right]},$$

and we obtain the following equation for u:

$$\ln u+\frac{n^2}{n^2-1}\left[1-\sqrt{1-\frac{4}{n^2}}\right](u-1)+\frac{u^2-1}{8}\frac{n^2}{n^2-1}\left(1-\sqrt{1-\frac{4}{n^2}}\right)^2 - \frac{2}{n^2-1}\ln\frac{z}{\varepsilon}=0. \tag{5.14}$$

By finding u from (5.14), we obtain ϱ from (5.13) and hence p. This solution enables verification of the numerical solution, and indicates a method which is of use in problems of spherical and cylindrical waves.

Let us verify the solutions. We assume

$$n = 3. \tag{5.15}$$

Then we can find for $\varepsilon = r_0/10$ a solution according to (5.10) for $x = r/r_0 \leqslant 0.2$.

The Eq. (5.14) now takes the form

$$\ln u+0.287(u-1)+\frac{u^2-1}{109}=\frac{1}{4}\ln 2.0. \tag{5.16}$$

Hence $u = 1.14$, which accurately coincides (after changing the variable u into V) with the result expressed in Fig. 3. For $x = 0.1$ we have $u = 1$, in agreement with the initial condition (5.6). For $\varepsilon = r_0/5$ (which means a rough approximation, ε being considerable) and $x = 0.5$ (more strictly the boundary lies at 0.4). We find $u = 1.18$ instead of the value of 1.16 obtained previously by the numerical solution (conversely $\varepsilon = r_0/4$; $x = 0.5$ or $\varepsilon = r_0/5$; $x = 0.4$. $u = 1.145$). It is seen that the difference between the present solution and the numerical solution above is insignificant.

If we plot v and ϱ in function of x, they will be of the same type as the diagrams in Figs. 3 to 6.

6. Final remarks

Summing up the results of the present paper, the problem may be considered to have been solved numerically. A fragment of it has been solved in an analytic manner, which enables certain conclusions of a qualitative nature to be drawn on the type of the distribution of above all, ϱ and v, and also the expansion of gas away from the surface and the point at which the thermal wave caught up by the shock wave (close to the centre). It can also be easily shown that in limiting cases (omitted in the present paper) our solutions become the corresponding solutions for the plane wave [2]. They have definite properties, the same as the solution for the linear elastic problem.

In addition to the direct technical application, the solutions obtained will (similarly to the results of Refs. [2, 3, 4] for the plane wave) constitute a point of departure for the construction of averaged equations of concentric laser heating and compression of plasma, taking into consideration the influence of thermal and shock wave fronts (Refs. [5, 6] for the plane problem).

To this end, it is necessary to obtain analogous solutions for subsonic concentric thermal waves which, by the different method of solution will be dealt with in a separate publication in view.

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