Archives of Mechanics • Archiwum Mechaniki Stosowanej • 26, 1, pp. 13–24, Warszawa 1974

Energy method of analysis of dynamic stability of a cylindrical shell subjected to torsion

J. LEYKO and S. SPRYSZYŃSKI (ŁÓDŹ)

IN THIS paper is presented an approximate energy method of analysis in a non-linear approach as regards dynamic stability of a thin elastic cylindrical shell subjected to torsion. The particular case in which the torque applied increases proportionally to the time is considered in detail. Also given are the results of numerical calculations.

W pracy przedstawiono przybliżoną metodę energetyczną analizy stateczności dynamicznej — w ujęciu nieliniowym — cienkościennej sprężystej powłoki walcowej, poddanej skręcaniu. Rozwiązano szczegółowo przypadek, gdy moment skręcający powłokę wzrasta proporcjonalnie do czasu. Przedstawiono wyniki przykładów liczbowych, dotyczących tego ostatniego przypadku.

В работе представлен приближенный, энергетический метод анализа динамической устойчивости — в нелинейной трактовке — тонкостенной упругой цилиндрической оболочки подвергнутой скручиванию. Решен частный случай, когда момент скручивающий оболочку возрастает пропорционально времени. Представлены результаты числовых примеров касающихся этого последнего случая.

1. Introduction

WE SHALL consider the dynamic stability of a thin elastic isotropic shell subjected to the action of rapidly growing torque applied at the ends of the shell (Fig. 1). The analysis

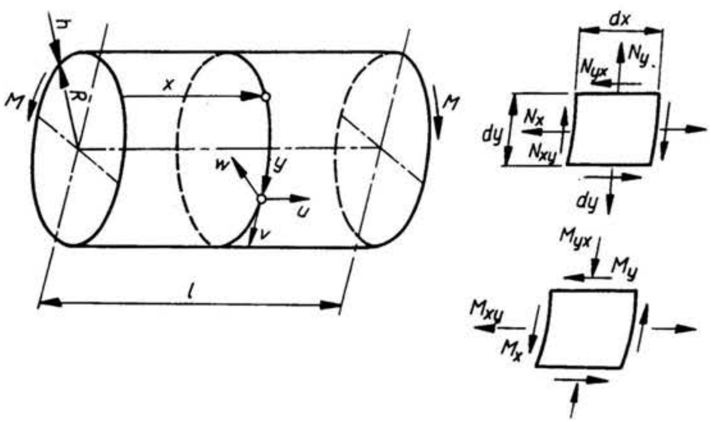

FIG. 1.

of the stability will be carried out on the ground of non-linear shallow shell theory, and small initial displacements of the middle surface of the shell from the ideal cylindrical surface will be taken into account. In such a case, when the longitudinal and the tangential components of inertial forces are disregarded, we obtain two non-linear partial differential equations for the deflection of the middle surface and sectional forces function

$$D\nabla^2\nabla^2(w-w_0) = L(w,\Phi)+\frac{1}{R}\frac{\partial^2\Phi}{\partial x^2}-\varrho h\frac{\partial^2 w}{\partial t^2}, \tag{1.1}$$

$$\frac{1}{Eh}\nabla^2\nabla^2\Phi = -\frac{1}{2}[L(w,w)-L(w_0,w_0)]-\frac{1}{R}\frac{\partial^2(w-w_0)}{\partial x^2}. \tag{1.2}$$

In these equations the following notations are used:

$w(x, y, t)$ total normal deflection measured from ideal cylindrical surface,
$w_0(x, y)$ initial normal deflection,
x, y coordinates defining the position of the point in the middle surface, as shown in Fig. 1,
t time,
$\Phi(x, y, t)$ sectional forces function,
R radius of ideal cylindrical surface,
h thickness of the shell,
ϱ density,
$D = \dfrac{Eh^3}{12(1-\nu^2)}$ — flexural rigidity of the shell,
E Young modulus,
ν Poisson's ratio,
$\nabla^2 = \dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$ — Laplace operator.

The symbol $L(,)$ means a non-linear operator defined as follows:

$$L(w,\Phi) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2\Phi}{\partial x^2}-2\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2\Phi}{\partial x\,\partial y}. \tag{1.3}$$

The sectional forces in the middle surface of the shell and the sectional moments are expressed by the following formulae:

$$N_x = \sigma_x h = \frac{\partial^2\phi}{\partial y^2},\quad N_y = \sigma_y h = \frac{\partial^2\phi}{\partial x^2},\quad N_{xy} = \tau_{xy}h = -\frac{\partial^2\Phi}{\partial x\,\partial y}, \tag{1.4}$$

$$M_x = -D\left[\frac{\partial^2(w-w_0)}{\partial x^2}+\nu\frac{\partial^2(w-w_0)}{\partial y^2}\right],\quad M_y = -D\left[\frac{\partial^2(w-w_0)}{\partial y^2}+\nu\frac{\partial^2(w-w_0)}{\partial x^2}\right], \tag{1.5}$$
$$M_{xy} = D(1-\nu)\frac{\partial^2(w-w_0)}{\partial x\,\partial y}.$$

The differential Eq. (1.1) was obtained from the conditions of dynamic equilibrium of an element cut from the shell, and the Eq. (1.2) from the condition of compatibility for the components of strain of the middle surface

$$\varepsilon_x = \frac{\partial u}{\partial x}+\frac{1}{2}\left[\left(\frac{\partial w}{\partial x}\right)^2-\left(\frac{\partial w_0}{\partial x}\right)^2\right],\quad \varepsilon_y = \frac{\partial v}{\partial y}-\frac{w-w_0}{R}+\frac{1}{2}\left[\left(\frac{\partial w}{\partial y}\right)^2-\left(\frac{\partial w_0}{\partial y}\right)^2\right], \tag{1.6}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}.$$

In these formulae, u is the longitudinal and v the circumferential component of displacement of the middle surface.

2. The method of solution

In order to obtain an approximate solution of the title problem, we take for normal deflection w of the shell, a function fulfilling the kinematic boundary conditions, and having the form of the series

$$w(x, y, t) = f_1(t)\, W_1(x, y) + f_2(t)\, W_2(x, y) + \ldots \tag{2.1}$$

in which $W_1(x, y)$, $W_2(x, y)$, ... are functions fulfilling the condition

$$W_i(x, y) = W_i(x, y+2\pi R), \quad i = 1, 2, \ldots$$

and $f_1(t), f_2(t)$, ... are unknown functions of the time t. These functions will be taken as generalized coordinates for which Lagrangean equations of motion will be established.

To obtain these equations, we must express the elastic energy of the shell in terms of generalized coordinates, the kinetic energy in terms of generalized velocities, and we must find the generalized forces corresponding to the generalized coordinates.

Introducing the expression (2.1) for $w(x, y, t)$ into the right side of the Eq. (1.2) and treating the function $w_0(x, y)$ as given, we obtain the linear differential equation for the sectional forces function Φ. The solution of this equation must fulfil static boundary conditions, and can be represented as follows:

$$\Phi = \tilde{\Phi} - s_{xy} xy, \tag{2.2}$$

where s_{xy} is the mean value of tangential sectional force, and

$$\tilde{\Phi} = \tilde{\Phi}_0 + f_1 \tilde{\Phi}_1 + f_2 \tilde{\Phi}_2 + \ldots + f_1^2 \tilde{\Phi}_{11} + f_2^2 \tilde{\Phi}_{22} + \ldots + f_1 f_2 \tilde{\Phi}_{12} + \ldots \tag{2.3}$$

In this last equation $\tilde{\Phi}_0, \tilde{\Phi}_1$, ... are functions of x and y.

According to (1.4) we have now

$$N_x = \frac{\partial^2 \tilde{\Phi}}{\partial y^2}, \quad N_y = \frac{\partial^2 \tilde{\Phi}}{\partial x^2}, \quad N_{xy} = s_{xy} - \frac{\partial^2 \tilde{\Phi}}{\partial x \partial y}. \tag{2.4}$$

The mean tangential sectional force $s_{x,y}$ is determined from the formula

$$s_{xy} = \frac{M}{2\pi R^2} \tag{2.5}$$

in which M is the torque applied.

It is easy to note that, in this case, for an arbitrary x the following conditions must be fulfilled:

$$\int_0^{2\pi R} \frac{\partial^2 \tilde{\Phi}}{\partial x \partial y} dy = 0, \quad \int_0^{2\pi R} \frac{\partial^2 \tilde{\Phi}}{\partial y^2} dy = 0. \tag{2.6}$$

The elastic energy of the cylindrical shell of length l and radius R can be represented in following form [1]:

$$V = \frac{1}{2Eh} \int_0^l \int_0^{2\pi R} [(\nabla^2 \Phi)^2 - (1+\nu) L(\Phi, \Phi)]\, dx\, dy + \frac{D}{2} \int_0^l \int_0^{2\pi R} \{[\nabla^2 (w-w_0)]^2 - (1-\nu) L(w-w_0, w-w_0)\}\, dx\, dy.$$

The first integral on the right side represents the elastic energy due to stretching of the middle surface, and the second — the energy of the bending. Substituting for Φ the expression (2.2) and taking into account the condition (2.6), we obtain

$$(2.7)\qquad V = \frac{1}{2Eh}\int\limits_0^l\int\limits_0^{2\pi R}[(\nabla^2\tilde{\Phi})^2-(1+\nu)L(\tilde{\Phi},\tilde{\Phi})]\,dx\,dy+\frac{D}{2}\int\limits_0^l\int\limits_0^{2\pi R}\{[\nabla^2(w-w_0)]^2 - (1-\nu)L(w-w_0,w-w_0)\}\,dx\,dy+2\frac{1+\nu}{Eh}s_{xy}^2\pi RL.$$

In order to find the generalized force Q_i corresponding to the generalized coordinate f_i we give to this coordinate a virtual increment δf_i and we find the work done by the torque. Denoting by $\delta\theta$ the corresponding virtual angle of twist of the shell, we have

$$(2.8)\qquad Q_i\,\delta f_i = M\delta\theta.$$

The angle of twist of the shell is given by the formula [1],

$$\theta = -\frac{1}{2\pi R^2}\left[\int\limits_0^l\int\limits_0^{2\pi R}\left(\frac{1}{Gh}\frac{\partial^2\Phi}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right)dx\,dy\right].$$

Taking into account (1.8) and (1,12). we obtain:

$$\theta = -\frac{1}{2\pi R^2}\left[\int\limits_0^l\int\limits_0^{2\pi R}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right)dx\,dy-\frac{2\pi RL}{Gh}s_{xy}\right].$$

Hence

$$\delta\theta = -\frac{1}{2\pi R^2}\frac{\partial}{\partial f_i}\left[\int\limits_0^l\int\limits_0^{2\pi R}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}-\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right)dx\,dy-\frac{2\pi RL}{Gh}s_{xy}\right]\delta f_i = -\frac{1}{2\pi R^2}\frac{\partial}{\partial f_i}\left[\int\limits_0^l\int\limits_0^{2\pi R}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\,dx\,dy\right]\delta f_i.$$

Substituting this value of $\delta\theta$ into the Eq. (2.8) and taking into account that, according to (2.5), $M = 2\pi R^2 s_{xy}$, we obtain:

$$(2.9)\qquad Q_i = s_{xy}(t)\frac{\partial}{\partial f_i}\left[\int\limits_0^l\int\limits_0^{2\pi R}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\,dx\,dy\right].$$

In the case under consideration, $s_{xy}(t)$ is a given function of the time t.

The kinetic energy of the shell is given by the formula

$$(2.10)\qquad T = \frac{h\varrho}{2}\int\limits_0^l\int\limits_0^{2\pi R}\left(\frac{\partial w}{\partial t}\right)^2dx\,dy = \frac{1}{2}(a_{11}\dot{f}_1^2+a_{22}\dot{f}_2^2+\ldots+2a_{12}\dot{f}_1\dot{f}_2+\ldots)$$

in which

$$(2.11)\qquad a_{jk} = a_{kj} = h\varrho\int\limits_0^l\int\limits_0^{2\pi R}W_jW_k\,dx\,dy.$$

We conclude that the generalized coordinates are not explicitly contained in the expression of kinetic energy T.

In the case of a "closed" cylindrical shell considered, the components of displacements of the middle surface must be periodical functions of coordinate y. It follows from this that for arbitrary x the tangential component v must fulfil the condition

$$v_{(x,y)} = v_{(x,y+2\pi R)}$$

which may be written in the form:

$$\int_0^{2\pi R} \frac{\partial v}{\partial y} dy = 0.$$

Taking into account the formulae $(1.6)_2$, (2.4) and using the generalized Hooke's law, we obtain

$$(2.12)\qquad \int_0^{2\pi R} \frac{\partial v}{\partial y} dy = \int_0^{2\pi R} \left[\frac{1}{Eh}\left(\frac{\partial^2\tilde{\Phi}}{\partial x^2} - \nu\frac{\partial^2\tilde{\Phi}}{\partial y^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 + \frac{w-w_0}{R}\right] dx\,dy = 0.$$

From this equation which must be fulfilled for arbitrary x follows the relation between the coordinate $f_1, f_2, \ldots$.

$$(2.13)\qquad F(f_1, f_2, \ldots) = 0.$$

Because the coordinates $f_1, f_2, \ldots$ are not independent we must use Lagrangian equations with multiplier λ. In the case under consideration $\partial T/\partial f_i = 0$, and these equations have the form:

$$(2.14)\qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{f}_i}\right) + \frac{\partial V}{\partial f_i} + \lambda\frac{\partial F}{\partial f_i} = Q, \quad i = 1, 2, \ldots$$

These differential equations together with the Eq. (2.13) are sufficient to determine the coordinates $f_1, f_2, \ldots$ and the multiplier λ.

3. An application of the proposed method

We take the expression for the deflection of the shell in the form

$$(3.1)\qquad w = f_1 \sin\frac{\pi x}{l}\sin\frac{n(y-kx)}{R} + f_2\sin^2\frac{\pi x}{l},$$

where n is the number of waves in the circumferential direction and k is a constant. The corresponding form of buckling has n circumferential waves which spiral along the cylinder. The expression (3.1) was used by several authors for non-linear analysis of static stability of cylindrical shells subjected to torsion [2, 3].

The expression for the initial deflection w_0 will be taken in analogous form to (3.1),

$$(3.2)\qquad w_0 = f_{01} \sin\frac{\pi x}{l}\sin\frac{n(y-kx)}{R} + f_{02}\sin^2\frac{\pi x}{l},$$

where f_{01} and f_{02} are constant coefficients.

The expression (3.1) fulfils the kinematical boundary conditions for simply supported edges. For $x = 0$ and $x = l$, $w - w_0 = 0$, and $\partial(w-w_0)/\partial x$ is not identically equal to zero.

Introducing (3.1) and (3.2) into the right side of the Eq. (1.2) and integrating this equation, we find the function $\tilde{\Phi}$:

$$(3.3)\quad \tilde{\Phi} = \frac{Eh\vartheta^2}{32}\left[\frac{1}{(1+k^2)^2}\cos\frac{2n(y-kx)}{R}+\frac{1}{\vartheta^4}\cos\frac{2\pi x}{l}\right](f_1^2-f_{01}^2)$$
$$+\frac{Eh\vartheta^2}{2}\left[\frac{1}{(1+b_1^2)^2}\cos\frac{n(y-b_1x)}{R}+\frac{1}{(1+a_3^2)^2}\cos\frac{n(y-a_3x)}{R}\right.$$
$$\left.-\frac{1}{(1+a_1^2)^2}\cos\frac{n(y-a_1x)}{R}-\frac{1}{(1+b_3^2)^2}\cos\frac{n(y-b_3x)}{R}\right](f_1f_2-f_{01}f_{02})$$
$$+\frac{Ehl^2}{2\pi^2R}\vartheta^2\left[\frac{a_1^2}{(1+a_1^2)^2}\cos\frac{n(y-a_1x)}{R}-\frac{b_1^2}{(1+b_1^2)^2}\cos\frac{n(y-nb_1x)}{R}\right](f_1-f_{01})$$
$$-\left[\frac{Ehl^2}{6\pi^2R}\cos\frac{2\pi x}{l}\right](f_2-f_{02}),$$

where

$$(3.4)\quad \vartheta=\frac{\pi R}{nl},\quad a_1=k+\vartheta,\quad b_1=k-\vartheta,\quad a_3=k+3\vartheta,\quad b_3=k-3\vartheta.$$

The function $\tilde{\Phi}$ obtained fulfils the conditions (2.6). We conclude that the static boundary conditions are here fulfilled in an integral manner.

Introducing the expressions (3.1), (3.2), (3.3) into the formula (2.7) and performing the integration, we find the elastic energy of the shell:

$$(3.5)\quad V=\frac{\pi ERlh}{8}\left\{\frac{\pi^4}{16l^4}\left[\frac{1}{(1+k^2)^2}+\frac{1}{\vartheta^4}\right](f_1^2-f_{01}^2)+\frac{\pi^4}{l^4}\left[\frac{1}{(1+a_1^2)^2}+\frac{1}{(1+b_1^2)^2}\right.\right.$$
$$\left.+\frac{1}{(1+a_3^2)^2}+\frac{1}{(1+b_3^2)^2}\right](f_1f_2-f_{01}f_{02})^2-\frac{\pi^2}{2Rl^2}\left[\frac{4a_1^2}{(1+a_1^2)^2}\right.$$
$$\left.+\frac{4b_1^2}{(1+b_1^2)^2}\right](f_1f_2-f_{01}f_{02})(f_1-f_{01})-\frac{\pi^2}{2Rl^2}\frac{1}{\vartheta^2}(f_1^2-f_{01}^2)(f_2-f_{02})+\frac{1}{R^2}\left[\frac{a_1^4}{(1+a_1^2)^2}\right.$$
$$\left.+\frac{b_1^4}{(1+b_1^2)^2}\right](f_1-f_{01})^2+\frac{1}{R^2}(f_2-f_{02})^2+\frac{h^2}{12(1-\nu^2)}\frac{n^4}{R^4}[(1+a_1^2)^2+(1+b_1^2)^2](f_1$$
$$\left.-f_{01})^2+\frac{16h^2}{12(1-\nu^2)}\frac{\pi^4}{l^4}(f_2-f_{02})^2\right\}+2\frac{1+\nu}{Eh}s_{xy}^2\pi Rl.$$

From the Eq. (2.9) we obtain the generalized forces,

$$(3.6)\quad Q_1=-\pi kn^2\frac{l}{R}f_1s_{xy}(t),\quad Q_2=0.$$

Using the Eq. (2.10), we obtain the following expression for kinetic energy:

$$(3.7)\quad T=\frac{h\varrho}{2}\left(\frac{1}{2}\dot{f}_1^2+\frac{3}{4}\dot{f}_2^2\right)\pi Rl.$$

Substituting (3.1), (3.2), (3.3) into the Eq. (2.12), we obtain, after integration, the relation between the coordinates f_1 and f_2:

$$(3.8)\quad F(f_1,f_2)=n^2f_1^2-4Rf_2-n^2f_{01}^2+4Rf_{02}=0.$$

In the case under consideration, we can write two Lagrangean equations of the type (2.14). Taking into account the expressions (3.5)–(3.8), we obtain:

$$
\begin{aligned}
(3.9)\qquad & \frac{d^2 f_1}{dt^2} + \frac{\pi^4}{16}\frac{E}{\varrho l^4}\left[\frac{1}{(1+k^2)^2} + \frac{1}{\vartheta^4}\right](f_1^2 - f_{01}^2) f_1 + \frac{\pi^4}{2}\frac{E}{\varrho l^4}\left[\frac{1}{(1+a_1^2)^2} + \frac{1}{(1+b_1^2)^2}\right. \\
& \left. + \frac{1}{(1+a_3^2)^2} + \frac{1}{(1+b_3^2)^2}\right](f_1 f_2 - f_{01} f_{02}) f_2 - \frac{\pi^2}{2}\frac{E}{\varrho R l^2}\left[\frac{a_1^2}{(1+a_1^2)^2} + \frac{b_1^2}{(1+b_1^2)^2}\right]\times \\
& \times [(f_1 - f_{01}) f_2 + f_1 f_2 - f_{01} f_{02}] - \frac{\pi^2}{4}\frac{E}{\varrho R l^2 \vartheta^2}(f_2 - f_{02}) f_1 + \frac{E}{2\varrho R^2}\left[\frac{a_1^4}{(1+a_1^2)^2}\right. \\
& \left. + \frac{b_1^4}{(1+b_1^2)^2}\right](f_1 - f_{01}) + \frac{Eh^2}{24(1-\nu^2)\varrho}\frac{n^4}{R^4}[(1+a_1^2)^2 + (1+b_1^2)^2](f_1 - f_{01}) \\
& + \lambda \frac{4n^2}{\varrho \pi R l h} f_1 - \frac{2kn^2}{R^2 h \varrho} s_{xy}(t) f_1 = 0, \\
& \frac{d^2 f_2}{dt^2} + \frac{\pi^4 E}{3\varrho l^4}\left[\frac{1}{(1+a_1^2)^2} + \frac{1}{(1+b_1^2)^2} + \frac{1}{(1+a_3^2)^2} + \frac{1}{(1+b_3^2)^2}\right](f_1 f_2 - f_{01} f_{02}) f_1 \\
& - \frac{\pi^2}{3}\frac{E}{\varrho R l^2}\left[\frac{a_1^2}{(1+a_1^2)^2} + \frac{b_1^2}{(1+b_1^2)^2}\right](f_1 - f_{01}) f_1 - \frac{\pi^2}{12}\frac{E}{\varrho R l^2 \vartheta^2}(f_1^2 - f_{01}^2) \\
& + \frac{1}{3}\frac{E}{\varrho R^2}(f_2 - f_{02}) + \frac{4\pi^4 h^2}{9(1-\nu^2)}\frac{E}{\varrho l^4}(f_2 - f_{02}) - \lambda \frac{16}{3\pi l h \varrho} = 0.
\end{aligned}
$$

When $s_{xy}(t)$ is given, the differential Eqs. (3.9) together with the relation (3.8) are sufficient to determine the unknown functions of time $f_1(t)$ and $f_2(t)$. These equations may also be used to obtain solution of the static stability problem. In this case

$$s_{xy} = s = \text{const}$$

and

$$\frac{d^2 f_1}{dt^2} = \frac{d^2 f_2}{dt^2} = 0.$$

Assuming that there is no initial deflection of the middle surface of the shell, and putting in (3.8) $f_{01} = f_{02} = 0$, we have:

$$f_2 = \frac{n^2}{4R} f_1^2. \tag{3.10}$$

Taking this into account and eliminating the multiplier λ from the Eqs. (3.9), we obtain the following relation between s and f_1:

$$
\begin{aligned}
s = & \frac{Eh}{4n^2 k}\left\{\left[\frac{a_1^4}{(1+a_1^2)^2} + \frac{b_1^4}{(1+b_1^2)^2}\right] + \frac{n^4}{12(1-\nu^2)}\left(\frac{h}{R}\right)^2 [(1+a_1^2)^2 + (1+b_1^2)^2]\right\} \\
& - \frac{\pi^2 Eh}{4kl^2}\left\{\frac{a_1^2}{(1+a_1^2)^2} + \frac{b_1^2}{(1+b_1^2)^2} - \frac{\vartheta^2}{8(1+k^2)^2} - \frac{\pi^2 n^2}{6(1-\nu^2)}\left(\frac{h}{l}\right)^2\right\} f_1^2 \\
& + \frac{3\pi^4 Ehn^2}{64\, kl^4}\left[\frac{1}{(1+a_1^2)^2} + \frac{1}{(1+b_1^2)^2} + \frac{1}{(1+a_3^2)^2} + \frac{1}{(1+b_3^2)^2}\right] f_1^4.
\end{aligned}
$$

The parameters a_1, a_3, b_1 and b_3 are expressed in terms of k and $\vartheta = \pi R/nl$ by the formulae (3.6). For thin shells of medium length k and ϑ are small quantities, and the squares of a_1, a_3, b_1 and b_3 can be disregarded in comparison with unity [3]. Taking this into account and introducing the following notations:

$$s^* = \frac{sR}{Eh^2}, \quad \delta = \frac{hR}{l^2}, \tag{3.11}$$

where s^* is the nondimensional tangential sectional force, we obtain the equation describing the postbuckling behaviour of the shell

$$s^* = \frac{1}{2\pi^2\delta^2}\left[\vartheta^2k^3+6k\vartheta^4+\frac{\vartheta^6}{k}+\frac{\pi^4\delta^2}{12(1-\nu^2)}\frac{1}{\vartheta k^2}\right]-\frac{\pi^2\delta}{4k}\left[2k^2+\frac{15}{8}\vartheta^2 - \frac{\pi^4}{6(1-\nu^2)}\frac{\delta^2}{\vartheta^2}\right]\left(\frac{f_1}{h}\right)^2+\frac{3}{16}\frac{\pi^6\delta^3}{k\vartheta^2}\left(\frac{f_1}{h}\right)^4. \tag{3.12}$$

This equation corresponds to the equations obtained by T. GAŁKIEWICZ in his non-linear analysis of the static stability of a cylindrical orthotropic shell subjected to torsion [3].

Putting into the Eq. (3.12) $f_1 = 0$, we obtain the solution for the linear problem:

$$s_0^* = \frac{1}{2\pi^2\delta^2}\left[\vartheta^2k^3+6k\vartheta^4+\frac{\vartheta^6}{k}+\frac{\pi^4\delta^2}{12(1-\nu^2)}\frac{1}{k\vartheta^2}\right]. \tag{3.13}$$

The minimal value of s_0^* is equal to the upper nondimensional critical sectional force; hence

$$s_{cr}^* = \frac{s_{cr}R}{Eh^2} = \min s_0^*. \tag{3.14}$$

The value k_0 of the parameter k, which makes s_0^* minimum must fulfil the condition:

$$\left(\frac{\partial s_0^*}{\partial k}\right)_{k=k_0} = \frac{1}{2\pi^2\delta^2}\left[3\vartheta^2k_0^2+6\vartheta^4-\frac{\vartheta^6}{k_0^2}-\frac{\pi^4\delta^2}{12(1-\nu^2)}\frac{1}{k_0^2\vartheta^2}\right]=0,$$

from which we obtain:

$$k_0 = \sqrt{\sqrt{\frac{4}{3}\vartheta^4+\frac{\pi^4\delta^2}{36(1-\nu^2)\vartheta^4}}-\vartheta^2}. \tag{3.15}$$

After introducing k_0 into the expression (3.13), the numbers of waves $n = \pi R/\vartheta l$ which makes s_0^* minimum can be established and s_{cr}^* can be determined.

Let us return now to the solution of the dynamical problem. We shall consider the case in which the torque applied is proportional to the time. In this case, we have:

$$s_{xy} = hbt, \tag{3.16}$$

where b is the velocity at which the mean tangential stress is increasing.

As regards the initial deflection of the middle surface of the shell, we make the assumption that it is similar to that which occurs in the case of static stability loss. Therefore the constant f_{01} and f_{02} in the expression (3.2) must fulfil and equation analogous to the Eq. (3.10). Hence

$$f_{02} = \frac{n^2}{4R}f_{01}^2. \tag{3.17}$$

In the expression (3.1) for deflection w there enter two parameters: n and k. The number of waves n is treated in our approximate solution of the dynamic problem as independent of time, and as regards the constant k we make the assumption that it has the same value as in the case of static buckling, and is expressed by the formula (3.15). Using the Eqs. (3.8) and (3.17) we can eliminate from the differential Eqs. (3.9) the constant f_{02}, the multiplier λ and the function $f_2(t)$. Treating a_1, a_3, b_1 and b_3 as small quantities, the squares of which can be disregarded in comparison with unity, and introducing the nondimensional time defined as follows

$$t^* = \frac{hb}{s_{cr}} t = \frac{s_{xy}}{s_{cr}} \tag{3.18}$$

and using the notations

$$\zeta(t) = \frac{f_1(t)}{h}, \quad \zeta_0 = \frac{f_{01}}{h}, \qquad \eta^2 = \frac{h}{R} n^2 = \frac{\pi^2 \delta}{\vartheta^2}, \quad \alpha = \frac{\varrho R^4 b^2}{E^3 h^2 s_{cr}^{*2}}, \tag{3.19}$$

we obtain the following non-linear differential equation:

$$\frac{d^2\zeta}{dt^{*2}} + \frac{3}{8}\eta^4\left[\zeta^2 \frac{d^2\zeta}{dt^{*2}} + \zeta\left(\frac{d\zeta}{dt^*}\right)^2\right] + 2\frac{q_0^*}{\alpha}\eta^2 k_0(\zeta - \zeta_0)$$
$$+ \frac{\pi^4\delta^2}{16\alpha}\left[1 + \frac{4\eta^4}{3(1-\nu^2)}\right](\zeta^2 - \zeta_0^2)\zeta - \frac{\pi^2\delta}{4\alpha}(\pi^2\delta + k_0^2\eta^2)(4\zeta^3 - 3\zeta^2\zeta_0 - \zeta_0^3)$$
$$+ \frac{3}{8\alpha}\pi^4\delta^2\eta^4(\zeta^3 - \zeta_0^3)\zeta^2 - 2\frac{s_{cr}^*}{\alpha}k_0\eta^2 t^*\zeta = 0 \tag{3.20}$$

in which k_0 is given by the formula (3.15), s_{cr}^* is the nondimensional upper critical sectional force and q_0^* is the value of s_0^* for $k = k_0$, where s_0^* is defined by the formula (3.13). Hence we have

$$q_0^* = (s_0^*)_{k=k_0} = \frac{1}{2\pi^2\delta^2}\left[\vartheta^2 k_0^3 + 6k_0\vartheta^4 + \frac{\vartheta^6}{k_0} + \frac{\pi^4\delta^2}{12(1-\nu^2)}\frac{1}{k_0\vartheta^2}\right]$$
$$= \frac{1}{2\delta\eta^2}\left[k_0^3 + \frac{6\pi^2\delta}{\eta^2}k_0 + \frac{\pi^4\delta^2}{\eta^4 k_0} + \frac{1}{12(1-\nu^2)}\frac{\eta^4}{k_0}\right]. \tag{3.21}$$

Solving the differential Eq. (3.19), we find the nondimensional generalized coordinate $\zeta = f_1/h$ as the function of time, and we can perform the analysis of the dynamic buckling of the shell.

4. Results of numerical calculations

The differential Eq. (3.20) was solved numerically by the Runge-Kutta method with the following initial conditions for $t^* = 0$:

$$\zeta = \zeta_0, \quad \dot{\zeta} = 0.$$

Some results of these calculations concerning brass cylindrical shells for which:

$$E = 0.981 \cdot 10^5\ \mathrm{MN/m^2}, \quad \varrho = 8.5 \cdot 10^{-3}\ \mathrm{kg/m^3}, \quad R/l = 1/4 \quad \text{and} \quad R/h = 100$$

will now be presented.

The variation of nondimensional deflection ζ of the shell with nondimensional time t^* is represented in Fig. 2, for the case in which the mean tangential stress is increasing at velocity $b = 0.4905\ \mathrm{MN/m^2\ s}$, and the parameter $\zeta_0 = f_{01}/h$ characterising the initial deflection is equal to 0.01. The curves shown in the figure by the full lines are obtained for three different values of the number of waves n in the circumference of the initial deflection surface of the shell ($n = 5$; 6 and 7). We note that the most rapid growth

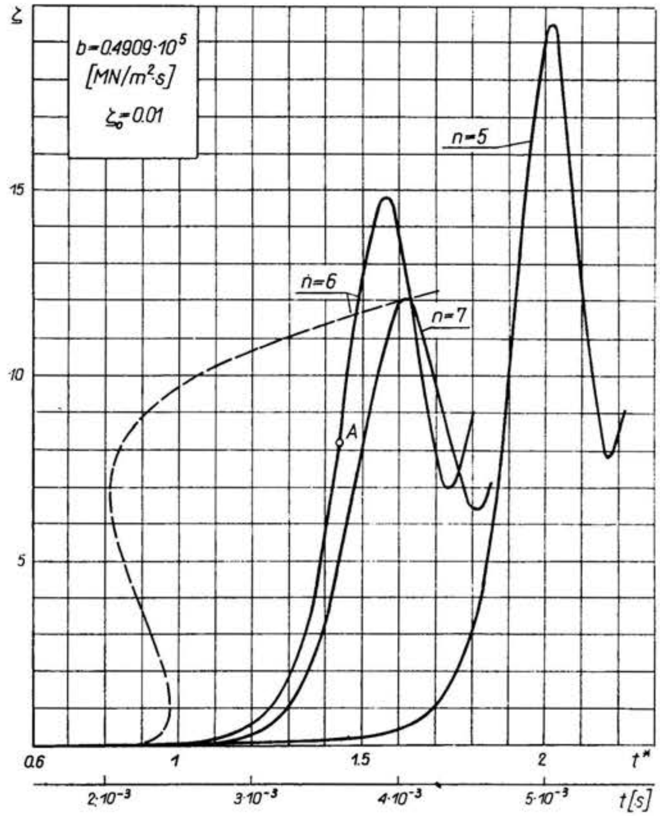

FIG. 2.

of the deflection in time occurs for $n = 6$. This number of waves corresponds also to the static upper critical torque for the shell considered here.

It follows from the relation (3.18) that the mean tangential sectional force $s_{xy}(t)$, and thereby the torque applied, is proportional to the nondimensional time t^*. Hence the curves in Fig. 2 represent also the relation between the dynamic deflection of the shell and the torque applied. The curve shown in the figure by the dotted line corresponds to the solution of the static problem for $n = 6$. This solution was obtained from the Eq. (3.20) in which in this case all terms with derivatives of ζ with respect to t^* were disregarded.

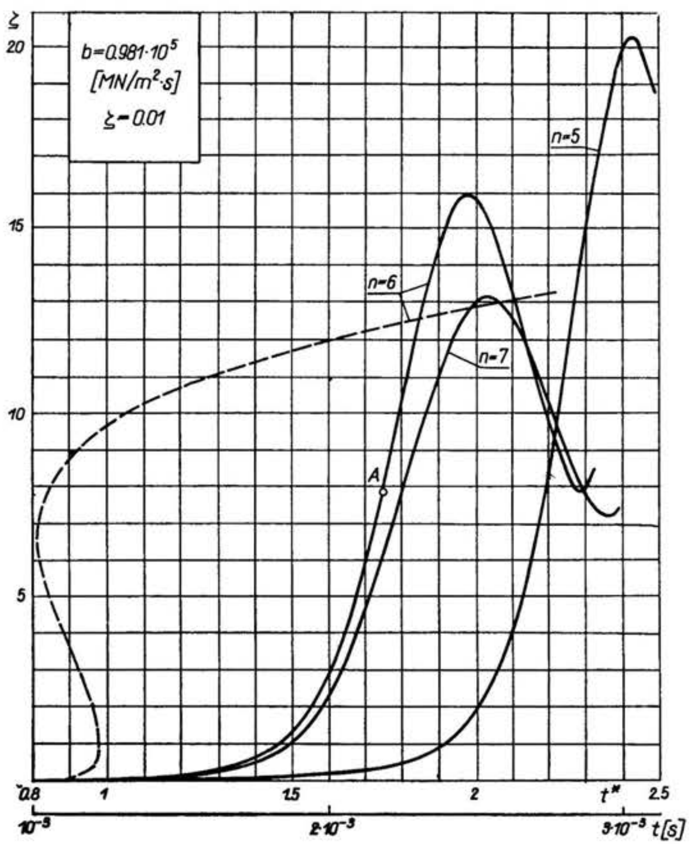
ζ
b=0.981·10^5
[MN/m²·s]
ζ=0.01
20
15
10
5
n=5
n=6
n=7
A
0.8
1
1.5
2
2.5
t*
10^-3
2·10^-3
3·10^-3
t[s]

FIG. 3.

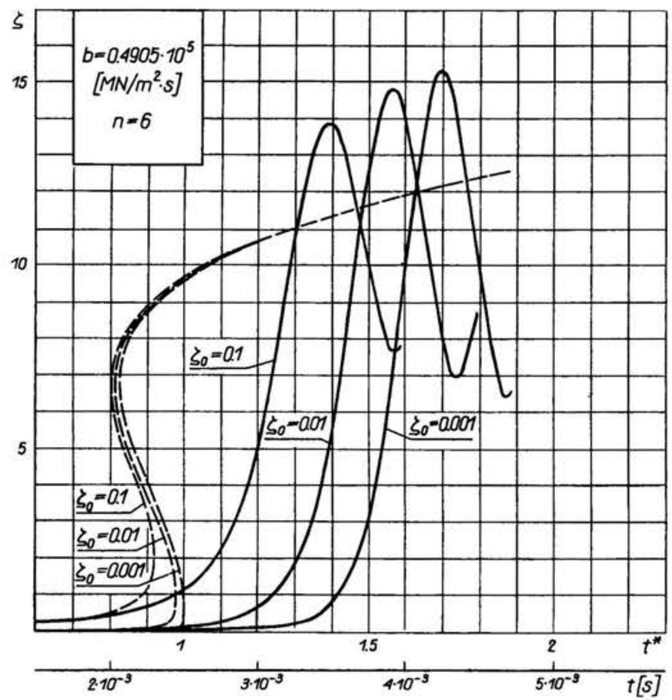
ζ
b=0.4905·10^5
[MN/m²·s]
n=6
15
10
5
ζ0=0.1
ζ0=0.01
ζ0=0.001
1
1.5
2
t*
2·10^-3
3·10^-3
4·10^-3
5·10^-3
t[s]

FIG. 4.

The load corresponding to the first inflexion point of the deflection-time diagram is defined usually as the dynamic critical load [1], because it occurs at the moment when the snap-trough action of the shell attains its greatest velocity. This inflection point of the curve obtained for $n = 6$ is denoted in Fig. 1 by A, and it follows from this figure, that in the case considered, the dynamic critical load is about fifty per cent higher than the upper static critical load.

In Fig. 3 are presented the results of the solution concerning the case in which the mean tangential stress is increasing at velocity $b = 0.981$ MN/m^{2}s (i.e., at double the previous velocity).

In this case, the dynamic critical load (for $n = 6$) is about seventy five per cent higher than the upper static critical load. Also analyzed was the infleunce of the magnitude of initial deflection of the shell on its dynamic stability. The results of the solutions obtained for three different values of the parameter ζ_0 are given in Fig. 4. The curves shown in full lines correspond to the solution of the dynamic problem, and the curves shown in the dotted lines — to the solution of the static problem.

A more detailed analysis of the numerical results obtained and some experimental data will be presented in a separate paper.

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Received May 11, 1973.