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# On discrete dislocations in micropolar elasticity

W. NOWACKI (WARSZAWA)

TWO-DIMENSIONAL problems of dislocations in elastic micropolar media are considered. In the case of a plane state of strain, the compatibility equations may be divided into two mutually independent systems of equations. The first system is connected with the vectors of displacement $\mathbf{u} \equiv (u_1, u_2, 0)$ and rotation $\boldsymbol{\varphi} \equiv (0, 0, \varphi_3)$, the second with vectors $\mathbf{u} = (0, 0, u_3)$ and $\boldsymbol{\varphi} \equiv \equiv (\varphi_1, \varphi_2, 0)$. The equations of compatibility for the both cases are written in terms of stresses. The cases of edge, screw and wedge dislocations are considered in detail.

Rozpatrzono dwuwymiarowe zagadnienia dyslokacji w mikropolarnym ośrodku sprężystym. Dla zagadnienia płaskiego stanu odkształcenia można rozdzielić równania zgodności na dwa niezależne od siebie układy równań. Pierwszy z nich związany jest z wektorem przemieszczenia $\mathbf{u} \equiv (u_1, u_2, 0)$ i obrotu $\boldsymbol{\varphi} \equiv (0, 0, \varphi_3)$, drugi z wektorami $\mathbf{u} \equiv (0, 0, u_3)$ i $\boldsymbol{\varphi} \equiv (\varphi_1, \varphi_2, 0)$. Podano równania zgodności dla obu zagadnień w naprężeniach. Szczegółowo rozpatrzono przypadek dyslokacji krawędziowej, klinowej i śrubowej.

Рассмотрены двумерные задачи дислокации в микрополярной упругой среде. Для задачи плоского деформационного состояния уравнения совместности можно разделить на две независящие друг от друга системы уравнений. Первая из них связана с вектором перемещения $\mathbf{u} \equiv (u_1, u_2, 0)$ и вращения $\boldsymbol{\varphi} \equiv (0, 0, \varphi_3)$, вторая с векторами $\mathbf{u} \equiv (0, 0, u_3)$ и $\boldsymbol{\varphi} \equiv (\varphi_1, \varphi_2, 0)$. Уравнения совместности для обоих задач даются в напряжениях. Подробно рассмотрен случай краевой, клиновой и винтовой дислокаций.

## 1. Introduction

IN THE PRESENT author's paper [1] the state of stress occurring in a micropolar medium containing distortions $\gamma_{ji}^0$, $\chi_{ji}^0$ is discussed. Initial strains of that type may occur in metals deformed beyond the yield limit — such as plastic strains. In this paper, we shall deal with the state of stress produced by discrete dislocations. The action of edge, screw and wedge dislocations will also be discussed.

Returning to distortions let us assume that the total strains provoked by distortions, $\gamma_{ji}^T$, $\chi_{ji}^T$, may be expressed by the formulae

$$\gamma_{ji}^T = \gamma_{ji}^0 + \gamma_{ji}, \quad \chi_{ji}^T = \chi_{ji}^0 + \chi_{ji}. \tag{1.1}$$

Here $\gamma_{ji}$, $\varkappa_{ji}$ are elastic strains. The stresses produced by distortions are [1]

$$\begin{aligned} \sigma_{ji} &= (\mu+\alpha)(\gamma_{ji}^T - \gamma_{ji}^0) + (\mu-\alpha)(\gamma_{ij}^T - \gamma_{ij}^0) + \lambda\delta_{ji}(\gamma_{kk}^T - \gamma_{kk}^0), \\ \mu_{ji} &= (\gamma+\varepsilon)(\varkappa_{ji}^T - \varkappa_{ji}^0) + (\gamma-\varepsilon)(\varkappa_{ij}^T - \varkappa_{ij}^0) + \beta\delta_{ij}(\varkappa_{kk}^T - \varkappa_{kk}^0). \end{aligned} \tag{1.2}$$

Here $\alpha, \beta, \gamma, \varepsilon, \lambda, \mu$ are material constants. In the considerations which follow, the definition of total strains will be needed,

$$\gamma_{ji}^T = u_{i,j}^T - \epsilon_{kji}\varphi_k^T, \quad \varkappa_{ji}^T = \varphi_{i,j}^T. \tag{1.3}$$

Here $u_i^T$ are components of the displacement vector, and $\varphi_i^T$ — of the rotation vector.

On substituting into the equilibrium equations

$$\sigma_{ji,j} = 0, \quad \epsilon_{ijk}\sigma_{jk} + \mu_{ji,j} = 0 \tag{1.4}$$

the constitutive relations (1.2) and using the definitions (1.3), a set of six equations for rotations and displacements is obtained from which the functions $u_i^T$ and $\varphi_i^T$ may be determined:

$$\begin{aligned} &(\mu+\alpha)\nabla^2 u_i^T + (\lambda+\mu-\alpha)u^T_{j,ji} + 2\alpha\epsilon_{ijk}\varphi^T_{k,j} = \sigma^0_{ji,j}, \\ &[(\gamma+\varepsilon)\nabla^2 - 4\alpha]\varphi_i^T + (\beta+\gamma-\varepsilon)\varphi^T_{j,ji} + 2\alpha\epsilon_{ijk}u^T_{k,j} = \epsilon_{ijk}\sigma^0_{jk} + \mu^0_{ji,j}. \end{aligned} \tag{1.5}$$

The following notations are introduced here:

$$\begin{aligned} \sigma^0_{ji} &= (\mu+\alpha)\gamma^0_{ji} + (\mu-\alpha)\gamma^0_{ij} + \lambda\delta_{ij}\gamma^0_{kk}, \\ \mu^0_{ji} &= (\gamma+\varepsilon)\varkappa^0_{ji} + (\gamma-\varepsilon)\varkappa^0_{ij} + \beta\delta_{ij}\varkappa^0_{kk}. \end{aligned} \tag{1.6}$$

Once the functions $u_i^T$, $\varphi_i^T$ are known. we may proceed to determine $\gamma^T_{ji}$, $\varkappa^T_{ji}$ from the Eqs. (1.3), and the force stresses $\sigma_{ji}$ and couple stresses $\mu_{ji}$ from the constitutive relations (1.2).

This is the first method of determining the strains produced in finite or infinite bodies by distortions [1].

On the other hand, the stresses due to distortions may also be determined by the method developed by Beltrami in classical elasticity.

Namely, comparison of the Eqs. (1.3), (1.1) yields:

$$\begin{aligned} u^T_{i,j} - \epsilon_{kji}\varphi^T_k &= \gamma^0_{ji} + \gamma_{ji}, \\ \varphi^T_{i,j} &= \varkappa^0_{ji} + \varkappa_{ji}. \end{aligned} \tag{1.7}$$

Elimination of $u_i^T$, $\varphi_i^T$ from these equations leads to a system of nonhomogeneous equations for the elastic strains $\gamma_{ji}$, $\varkappa_{ji}$ [2, 3],

$$\begin{aligned} \epsilon_{ihl}\gamma_{li,h} - \varkappa_{ij} + \delta_{ij}\varkappa_{kk} &= \alpha_{ji}, \\ \epsilon_{jhl}\varkappa_{li,h} &= \theta_{ji}, \end{aligned} \tag{1.8}$$

where

$$\alpha_{ji} = -\epsilon_{jhl}\gamma^0_{li,h} + \varkappa^0_{ij} - \delta_{ij}\varkappa^0_{kk}, \quad \theta = -\varkappa^0_{li,h}\varepsilon_{jhl}.$$

The above equations may be expressed in terms of stresses in view of the relations:

$$\begin{aligned} \gamma_{ji} &= (\mu'+\alpha')\sigma_{ji} + (\mu'-\alpha')\sigma_{ij} + \lambda'\delta_{ij}\sigma_{kk}, \\ \varkappa_{ji} &= (\gamma'+\varepsilon')\mu_{ji} + (\gamma'-\varepsilon')\mu_{ij} + \beta'\delta_{ij}\mu_{kk}. \end{aligned} \tag{1.9}$$

Here

$$2\mu' = \frac{1}{2\mu}, \quad 2\alpha' = \frac{1}{2\alpha}, \quad 2\gamma' = \frac{1}{2\gamma}, \quad 2\varepsilon' = \frac{1}{2\varepsilon},$$

$$\lambda' = -\frac{\lambda}{2\mu(3\lambda+2\mu)}, \quad \beta' = -\frac{\beta}{2\gamma(3\beta+2\gamma)}.$$

Inserting (1.9) into (1.8) and using the equations of equilibrium, we arrive at an equation in stresses which is analogous to that by Beltrami-Michell.

Let us adopt this manner of solution in further considerations confined, however, to the plane strain state of the body. Assume all the causes and effects to be independent of the variable $x_3$. In this particular case, the set of compatibility equations (1.8) splits into two sets of equations. In the first one, the following strain matrices appear

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{21} & \gamma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \varkappa = \begin{bmatrix} 0 & 0 & \varkappa_{13} \\ 0 & 0 & \varkappa_{23} \\ 0 & 0 & 0 \end{bmatrix}. \tag{1.10}$$

The equations of compatibility take the form

$$\begin{aligned} \partial_1\gamma_{21}-\partial_2\gamma_{11}-\varkappa_{13} &= \alpha_{31}, \\ \partial_1\gamma_{22}-\partial_2\gamma_{12}-\varkappa_{23} &= \alpha_{32}, \\ \partial_1\varkappa_{23}-\partial_2\varkappa_{13} &= \theta_{33}, \end{aligned} \tag{1.11}$$

where

$$\begin{aligned} \alpha_{31} &= -\partial_1\gamma^0_{21}+\partial_2\gamma^0_{11}+\varkappa^0_{13}, \\ \alpha_{32} &= -\partial_1\gamma^0_{22}+\partial_2\gamma^0_{12}+\varkappa^0_{23}, \\ \theta_{33} &= -\partial_1\varkappa^0_{23}+\partial_2\varkappa^0_{13}. \end{aligned} \tag{1.12}$$

The second system of compatibility equations, independent of (1.11), has the form

$$\begin{aligned} &\partial_2\gamma_{31}+\varkappa_{22} = \alpha_{11}, \qquad -\partial_1\gamma_{32}+\varkappa_{11} = \alpha_{22}, \\ &\partial_2\gamma_{32}-\varkappa_{21} = \alpha_{12}, \qquad -\partial_1\gamma_{31}-\varkappa_{12} = \alpha_{21}, \\ &\partial_1\gamma_{23}-\partial_2\gamma_{13}+\varkappa_{11}+\varkappa_{22} = \alpha_{33}, \end{aligned} \tag{1.13}$$

where

$$\begin{aligned} &\alpha_{11} = -\partial_2\gamma^0_{31}-\varkappa^0_{22}, \qquad \alpha_{22} = \partial_1\gamma^0_{32}-\varkappa^0_{11}, \\ &\alpha_{12} = -\partial_2\gamma^0_{32}+\varkappa^0_{21}, \qquad \alpha_{21} = \partial_1\gamma^0_{31}+\varkappa^0_{12}, \\ &\alpha_{33} = -\partial_1\gamma^0_{23}+\partial_2\gamma^0_{13}-\varkappa^0_{11}-\varkappa^0_{22}. \end{aligned} \tag{1.14}$$

## 2. The first problem of plane strain

Let us consider the set of compatibility Eqs. (1.11). Strains appearing in those equations are expressed in terms of stresses by means of the constitutive relations (1.9). First of all, the set of Eqs. (1.11) is transformed to the form:

$$\begin{aligned} \partial_2^2\gamma_{11}+\partial_1^2\gamma_{22}-\partial_1\partial_2(\gamma_{12}+\gamma_{21}) &= A_1, \\ \partial_1\partial_2(\gamma_{11}-\gamma_{22})+\partial_2^2\gamma_{12}-\partial_1^2\gamma_{21}+\partial_1\varkappa_{13}+\partial_2\varkappa_{23} &= A_2, \\ \partial_1\varkappa_{23}-\partial_2\varkappa_{13} &= A_3, \end{aligned} \tag{2.1}$$

where

$$A_1 = \theta_{33}+\partial_1\alpha_{32}-\partial_2\alpha_{31}, \qquad A_2 = -\partial_1\alpha_{31}-\partial_2\alpha_{32}, \qquad A_3 = \theta_{33}.$$

Expressing the strains in terms of stresses, we obtain

$$\begin{aligned} &\partial_2^2\sigma_{11}+\partial_1^2\sigma_{22}-\frac{\lambda}{2(\lambda+\mu)}\nabla_1^2(\sigma_{11}+\sigma_{22})-\partial_1\partial_2(\sigma_{12}+\sigma_{21}) = 2\mu A_1, \\ &(\partial_2^2-\partial_1^2)(\sigma_{12}+\sigma_{21})+\frac{\mu}{\alpha}\nabla_1^2(\sigma_{12}-\sigma_{21})+\frac{4\mu}{\gamma+\varepsilon}(\partial_1\mu_{13}+\partial_2\mu_{23}) \\ &\qquad\qquad +2\partial_1\partial_2(\sigma_{11}-\sigma_{22}) = 4\mu A_2, \\ &\partial_1\mu_{23}-\partial_2\mu_{13} = (\gamma+\varepsilon)A_3. \end{aligned} \tag{2.2}$$

The state of stress is expressed by the following matrices:

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}, \quad u = \begin{bmatrix} 0 & 0 & \mu_{13} \\ 0 & 0 & \mu_{23} \\ \mu_{31} & \mu_{32} & 0 \end{bmatrix}. \tag{2.3}$$

Three of the stress components may be written in terms of the remaining ones. These are

$$\sigma_{33} = -\frac{\lambda}{2(\lambda+\mu)}(\sigma_{11}+\sigma_{22}), \quad \mu_{32} = \frac{\gamma-\varepsilon}{\gamma+\varepsilon}\mu_{23}, \quad \mu_{31} = \frac{\gamma-\varepsilon}{\gamma+\varepsilon}\mu_{13}. \tag{2.4}$$

Six unknown values of stress components appear in the Eqs .(2.2). The equations of compatibility (2.2) are then supplemented by the equation of equilibrium

$$\begin{gathered} \partial_1\sigma_{11}+\partial_2\sigma_{21} = 0, \quad \partial_1\sigma_{12}+\partial_2\sigma_{22} = 0, \\ \partial_1\mu_{13}+\partial_2\mu_{23}+\sigma_{12}-\sigma_{31} = 0, \end{gathered} \tag{2.5}$$

and thus the number of equations equals the number of unknowns.

(a) Let us consider the particular case $A_1 \neq 0$, $A_2 \neq 0$, $A_3 = 0$. The stresses are expressed in terms of the Airy-Mindlin function

$$\begin{gathered} \sigma_{11} = \partial_2^2 F - \partial_1\partial_2\Psi, \qquad \sigma_{22} = \partial_1^2 F + \partial_1\partial_2\Psi, \\ \sigma_{12} = -\partial_1\partial_2 F - \partial_2^2\Psi, \quad \sigma_{21} = -\partial_1\partial_2 F + \partial_1^2\Psi, \\ \mu_3 = \partial_1\Psi, \quad \mu_{23} = \partial_2\Psi. \end{gathered} \tag{2.6}$$

These expressions satisfy the equations equilibrium (2.5) and the last of the Eqs. (2.2). The remaining equations of (2.2) are reduced to simple differential equations

$$\nabla_1^2\nabla_1^2 F = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}A_1, \tag{2.7}$$

$$\nabla_1^2(l^2\nabla_1^2-1)\Psi = -(\gamma+\varepsilon)A_2, \quad l^2 = \frac{(\gamma+\varepsilon)(\mu+\alpha)}{4\mu\alpha}. \tag{2.8}$$

Let us consider the action of a discrete edge dislocation characterized by the Burgers vector $\mathbf{b} \equiv (b_1, 0, 0)$. Assuming the $x_3$-axis for the dislocation line, and passing from distortions to dislocations [4–6], we obtain

$$\gamma_{21}^0 = -b_1 \int_S \delta(\mathbf{x}-\mathbf{x}')dS_2(\mathbf{x}'). \tag{2.9}$$

The remaining magnitudes $\gamma_{ji}^0$ nad $\varkappa_{ji}^0$ vanish. It is assumed that $S_2$ constitutes part of the $x_1x_3$-plane for negative values of $x_1$. The normal to $S_2$ directed towards the negative direction of $x_2$ ($dS_2 = -dx_1'dx_3'$). Hence

$$\gamma_{21}^0 = b_1 \int_{-\infty}^{0} \delta(x_1-x_1')dx_1'\,\delta(x_2) \int_{-\infty}^{\infty} \delta(x_3-x_3')dx_3' = b_1 H(-x_1)\delta(x_2), \tag{2.10}$$

and

$$\alpha_{31} = -\partial_2\gamma_{21}^0 = b_1\delta(x_1)\delta(x_2), \quad A_2 = -\partial_2\alpha_{31}, \quad A_2 = \partial_1\alpha_{31}. \tag{2.11}$$

Here $H(-z)$ denotes the Heaviside function.

The solution of the Eqs. (2.7), (2.8) is found by means of the double integral Fourier transform

$$F = -\frac{B}{4\pi^2}\frac{\partial}{\partial x_2}\int\limits_{-\infty}^{\infty}\!\!\int \frac{e^{-ix_k\xi_k}}{(\xi_1^2+\xi_2)^2}d\xi_1 d\xi_2, \quad B = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}b_1, \tag{2.12}$$

$$\Psi = -\frac{b_1(\gamma+\varepsilon)}{4\pi^2}\frac{\partial}{\partial_1 x}\int\limits_{-\infty}^{\infty}\!\!\int\left(\frac{1}{\xi_1^2+\xi_2^2+\frac{1}{l^2}} - \frac{1}{\xi_1^2+\xi_3^2}\right)e^{-ix_k\xi_k}d\xi_1 d\xi_2. \tag{2.13}$$

The above integrals, being improper, do not exist in the usual sense; nor can we assing to them the Cauchy principal values. We may, however, separate out of the them what are called the "finite parts" [7, 8], which consequently yields the results:

$$\begin{aligned}
I_1 &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\!\!\int \frac{e^{-ix_k\xi_k}}{\xi_1^2+\xi_2^2}d\xi_1 d\xi_2 = -(C+\ln r),\\
I_2 &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\!\!\int \frac{e^{-ix_k\xi_k}}{\xi_1^2+\xi_2^2+\frac{1}{l^2}}d\xi_1 d\xi_2 = K_0\left(\frac{r}{l}\right),\\
I_3 &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\!\!\int \frac{e^{-ix_k\xi_k}}{(\xi_1^2+\xi_2^2)^2}d\xi_1 d\xi_2 = \frac{r^2}{4}(C+\ln r),\\
&\qquad r = (x_1^2+x_2^2)^{12},
\end{aligned} \tag{2.14}$$

Here, $K_0(z)$ denotes the modified Bessel function of the third kind (McDonald) and $C$ is the Euler constant.

Thus we may write

$$F = -\frac{\mu(\lambda+\mu)b_1}{2\pi(\lambda+2\mu)}(2x_2\ln r+x_2), \tag{2.15}$$

$$\Psi = -\frac{(\gamma+\varepsilon)b_1}{2\pi r}\left[\frac{x_1}{r} - \frac{x_1}{l}K_1\left(\frac{r}{l}\right)\right]. \tag{2.16}$$

The stresses are calculated from the Eqs. (2.6)

$$\begin{aligned}
\sigma_{11} &= -\frac{\mu b_1}{2\pi(1-\nu)}\left(\frac{x_2}{r^2}+\frac{2x_1^2x_2}{r^4}\right) - \partial_1\partial_2\Psi,\\
\sigma_{22} &= -\frac{\mu b_1}{2\pi(1-\nu)}\left(\frac{x_2}{r^2}-\frac{2x_1^2x_2}{r^4}\right) + \partial_1\partial_2\Psi,\\
\sigma_{12} &= \frac{\mu b_1}{2\pi(1-\nu)}\left(\frac{x_1}{r^2}-2\frac{x_1x_2^2}{r^4}\right) - \partial_2^2\Psi,\\
\sigma_{21} &= \frac{\mu b_1}{2\pi(1-\nu)}\left(\frac{x_1}{r^2}-\frac{2x_1x_2^2}{r^4}\right) + \partial_1^2\Psi,\\
\mu_{13} &= \partial_1\Psi, \quad \mu_{23} = \partial_2\Psi, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}.
\end{aligned} \tag{2.17}$$

In passing from the Cosserat to Hooke's body, $\Psi$ has to be assumed to vanish, $\Psi = 0$. The known values of stresses in Hookean bodies due to an edge dislocation are obtained as follows:

$$\sigma_{21} = \sigma_{12} \quad \text{and} \quad \mu_{13} = \mu_{23} = 0.$$

(b) Let us consider the case in which $A_3 = \theta_{33}$, and $A_1 = A_2 = 0$. Representation of stresses by the potentials $F$, $\Psi$ does not in such a case yield any result: the equations of equilibrium would be satisfied, but the compatibility condition $(2.2)_3$ could not in this manner be fulfilled.

By consecutive elimination of stresses, from the Eqs. (2.2), (2.5), we obtain

$$\begin{gathered}\nabla_1^2\mu_{13} = -(\gamma+\varepsilon)\partial_2 A_3, \quad \nabla_1^2\mu_{23} = (\gamma+\varepsilon)\partial_1 A_3,\\ \nabla_1^2\nabla_1^2\sigma_{11} = K\partial_2^2 A_3, \qquad \nabla_1^2\nabla_1^2\sigma_{22} = K\partial_1^2 A_3,\\ \nabla_1^2\nabla_1^2\sigma_{12} = -K\partial_1\partial_2 A_3, \quad \sigma_{12} = \sigma_{21}, \quad K = \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}.\end{gathered} \tag{2.18}$$

Consider a discrete wedge dislocation [2]

$$A_3 = \theta_{33}, \quad \theta_{33} = \Omega\delta(x_1)\delta(x_2), \quad \Omega = \text{const}, \quad \alpha_{31} = \alpha_{32} = 0. \tag{2.19}$$

The following functions are solutions of the Eqs. (2.18):

$$\begin{aligned}\mu_{13} &= \frac{(\gamma+\varepsilon)\Omega}{2\pi}\frac{\partial}{\partial x_2}(I_1) = -\frac{(\gamma+\varepsilon)\Omega}{2\pi}\frac{x_2}{r^2},\\ \mu_{23} &= -\frac{(\gamma+\varepsilon)\Omega}{2\pi}\frac{\partial}{\partial x_1}(I_1) = \frac{(\gamma+\varepsilon)\Omega}{2\pi}\frac{x_1}{r^2},\\ \sigma_{12} = \sigma_{21} &= -\frac{K}{2\pi}\frac{\partial^2}{\partial x_1\partial x_2}(I_3) = -\frac{K}{4\pi}\frac{x_1x_2}{r^2},\\ \sigma_{11} &= \frac{K}{2\pi}\frac{\partial^2}{\partial x_2^2}(I_3) = \frac{K}{4\pi}\left(\ln r + \frac{x_2^2}{r^2}\right),\\ \sigma_{22} &= \frac{K}{2\pi}\frac{\partial^2}{\partial x_1^2}(I_3) = \frac{K}{4\pi}\left(\ln r + \frac{x_1^2}{r^2}\right).\end{aligned} \tag{2.20}$$

The above solutions were obtained earlier by K. H. ANTONY by a different method [2].

### 3. The second problem of plane strain

Let us return to the system of Eqs. (1.13) for the second problem of plane state of strain. By eliminating some of the strain components, we obtain five independent compatibility conditions,

$$\begin{aligned}\partial_1^2\varkappa_{22}+\partial_2^2\varkappa_{11}-\partial_1\partial_2(\varkappa_{12}+\varkappa_{21}) &= A_1,\\ \partial_2^2\varkappa_{12}-\partial_1^2\varkappa_{21}+\partial_1\partial_2(\varkappa_{11}-\varkappa_{22}) &= A_2,\\ \partial_1(\gamma_{32}+\gamma_{23})-\partial_2(\gamma_{13}+\gamma_{31}) &= A_3,\\ \varkappa_{21}-\varkappa_{12}-\partial_2\gamma_{32}-\partial_1\gamma_{31} &= A_4,\\ \varkappa_{11}+\varkappa_{22}-\partial_2\gamma_{13}+\partial_1\gamma_{23} &= A_5,\end{aligned} \tag{3.1}$$

where

$$
\begin{aligned}
A_1 &= \partial_1^2\alpha_{11}+\partial_2^2\alpha_{22}+\partial_1\partial_2(\alpha_{12}+\alpha_{21}),\\
A_2 &= \partial_1\partial_2(\alpha_{22}-\alpha_{11})+\partial_2^2\alpha_{21}-\partial_1^2\alpha_{12},\\
A_3 &= \alpha_{33}-\alpha_{11}-\alpha_{22},\quad A_4 = \alpha_{21}-\alpha_{12},\quad A_5 = \alpha_{33}.
\end{aligned}
\tag{3.2}
$$

Let us observe that in the expressions (3.2) the components of disclination density $\theta_{ij}$ do not appear. Using the constitutive relations (1.9) we may express the compatibility conditions (3.1) in terms of stresses. In this manner we obtain

$$
\begin{gathered}
\partial_1^2\mu_{22}+\partial_2^2\mu_{11}-\frac{\beta}{2(\gamma+\beta)}\nabla_1^2(\mu_{11}+\mu_{22})-\partial_1\partial_2(\mu_{12}+\mu_{21}) = 2\gamma A_1,\\
(\partial_2^2-\partial_1^2)(\mu_{12}+\mu_{21})+\frac{\gamma}{\varepsilon}\nabla_1^2(\mu_{12}-\mu_{21})-2\partial_1\partial_2(\mu_{22}-\mu_{11}) = 4\gamma A_2,\\
\partial_1(\sigma_{23}+\sigma_{32})-\partial_2(\sigma_{31}+\sigma_{13}) = 2\mu A_3,\\
\mu_{11}+\mu_{22}+\frac{\gamma+\beta}{\alpha}(\partial_2\sigma_{31}-\partial_1\sigma_{32}) = (\gamma+\beta)\left((2A_5-\frac{\mu+\alpha}{\alpha}A_3\right),\\
\mu_{21}-\mu_{12}-\frac{\varepsilon(\mu+\alpha)}{2\mu\alpha}(\partial_1\sigma_{31}+\partial_2\sigma_{32}) = 2\varepsilon A_4.
\end{gathered}
\tag{3.3}
$$

Observe that the Eq. $(3.3)_4$ may also be written in the form

$$
\mu_{11}+\mu_{22}+\frac{\gamma+\beta}{\alpha}(\partial_1\sigma_{23}-\partial_2\sigma_{13}) = (\gamma+\beta)\left(2A_5+\frac{\mu-\alpha}{\alpha}A_3\right).
$$

Eight unknown stresses $\mu_{11}, \mu_{22}, \mu_{12}, \mu_{21}, \sigma_{13}, \sigma_{31}, \sigma_{23}, \sigma_{32}$ appear in the system of five Eqs. (3.3). Supplementing the system (3.3) by three equations of equilibrium

$$
\begin{gathered}
\partial_1\mu_{11}+\partial_2\mu_{21}+\sigma_{23}-\sigma_{32} = 0,\\
\partial_1\mu_{12}+\partial_2\mu_{22}+\sigma_{31}-\sigma_{13} = 0,\\
\partial_1\sigma_{13}+\partial_2\sigma_{23} = 0,
\end{gathered}
\tag{3.4}
$$

we have at our disposal a number of equations sufficient for the determination of stresses. The system is connected with the vectors $\mathbf{u} \equiv (0, 0, u_3)$, $\boldsymbol{\varphi} \equiv (\varphi_1, \varphi_2, 0)$ and with the matrices

$$
\boldsymbol{\sigma} = \begin{bmatrix} 0 & 0 & \sigma_{13}\\ 0 & 0 & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & 0 \end{bmatrix},\qquad
\boldsymbol{\mu} = \begin{bmatrix} \mu_{11} & \mu_{12} & 0\\ \mu_{21} & \mu_{22} & 0\\ 0 & 0 & \mu_{33} \end{bmatrix}.
\tag{3.5}
$$

The stress $\mu_{33}$ is found from the formula

$$
\mu_{33} = \frac{\beta}{2(\gamma+\beta)}(\mu_{11}+\mu_{22}).
\tag{3.6}
$$

Let us consider the case of a screw dislocation $\mathbf{b} \equiv (0, 0, b_3)$. Assuming the $x_3$-axis for the dislocation line, and the $x_1x_3$-plane $(x_1 < 0)$ for the plane of discontinuity, we arrive at the conclusion that only $\gamma_{23}^0 \neq 0$,

$$
\gamma_{23}^0 = -b_3\int_S \delta(\mathbf{x}-\mathbf{x}')dS_2(\mathbf{x}') = b_3H(-x_1)\delta(x_2)
\tag{3.7}
$$

Now, we have $A_1 = A_2 = A_4 = 0$, and

$$A_3 = A_5 = \alpha_{33} = -\partial_1 \gamma^0_{23} = b_3 \delta(x_1)\delta(x_2). \tag{3.8}$$

Further elimination of stresses from Eqs. (3.3), (3.4) yields (with $A_1 = A_2 = A_4 = 0$, $A_3 = A_5 \neq 0$) the following differential equations

$$H(\mu_{11}+\mu_{22}) = -(\gamma+\beta)A_5 \tag{3.9}$$

$$D(\mu_{12}-\mu_{21}) = 0: \tag{3.10}$$

here

$$H = \nu^2\nabla_1^2-1, \quad D = l^2\nabla_1^2-1, \quad \nu^2 = \frac{2\gamma+\beta}{4\alpha}, \quad l^2 = \frac{(\gamma+\varepsilon)(\mu+\alpha)}{4\mu\alpha}.$$

The force-stresses $\sigma_{ij}$ are determined from the equations

$$\begin{aligned} &H\nabla_1^2\sigma_{13} = -\partial_2 NA_5, \quad H\nabla_1^2\sigma_{23} = \partial_1 NA_5,\\ &H\nabla_1^2\sigma_{31} = -\partial_2 MA_5, \quad H\nabla_1^2\sigma_{32} = \partial_1 MA_5, \end{aligned} \tag{3.11}$$

where

$$N = (\alpha+\mu)H+\alpha, \quad M = (\mu-\alpha)H-\alpha, \quad A_5 = b_3\delta(x_1)\delta(x_2).$$

Application of the exponential Fourier transform leads to the following relations

$$\begin{aligned} &\sigma_{13} = \frac{b_3}{2\pi}\partial_2(\mu I_1+\alpha \mathring{I}_2), \quad \sigma_{31} = \frac{b_3}{2\pi}\partial_2(\mu I_1-\alpha \mathring{I}_2),\\ &\sigma_{23} = -\frac{b_3}{2\pi}\partial_1(\mu I_1-\alpha \mathring{I}_2), \quad \sigma_{32} = -\frac{b_3}{2\pi}\partial_1(\mu I_1+\alpha \mathring{I}_2), \end{aligned} \tag{3.12}$$

where

$$I_1 = -(C+\ln r), \quad \mathring{I}_2 = K_0\left(\frac{r}{\nu}\right).$$

Let us observe that in a Hookean solid, with $\alpha = 0$, the classical, known results are obtained. Force-stresses are then symmetric.

Elimination of stresses from the sets of Eqs. (3.3), (3.4) yields, by means of Eqs. (3.9), (3.10) and (3.11), the following differential equations for couple-stresses:

$$\begin{aligned} &H\nabla_1^2\mu_{22} = -\left(\gamma\partial_2^2+\frac{\beta}{2}\nabla_1^2\right)A_5,\\ &H\nabla_1^2\mu_{11} = -\left(\gamma\partial_1^2+\frac{\beta}{2}\nabla_1^2\right)A_5,\\ &H\nabla_1^2\mu_{12} = H\nabla_2\mu_{21} = -2\gamma\partial_1\partial_2 A_5, \quad A_5 = b_3\delta(x_1)\delta(x_2). \end{aligned} \tag{3.13}$$

which, in turn, yield the results

$$\begin{aligned} &\mu_{11} = \frac{b_3}{2\pi}\left[\frac{\beta}{\nu^2}\mathring{I}_2-2\gamma\partial_1^2(I_1-\mathring{I}_2)\right],\\ &\mu_{22} = \frac{b_3}{2\pi}\left[\frac{\beta}{\nu^2}\mathring{I}_2-2\gamma\partial_2^2(I_2-\mathring{I}_2)\right],\\ &\mu_{12} = \mu_{21} = -\frac{\gamma b_3}{\pi}\partial_1\partial_2(I_1-\mathring{I}_2). \end{aligned} \tag{3.14}$$

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WARSAW UNIVERSITY
INSTITUTE OF MECHANICS

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