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EQUATIONS FOR THE CHARACTERISTIC FUNCTIONAL AND MOMENTS OF THE COMPLEX STOCHASTIC EVOLUTIONS

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Equations for the characteristic functional and moments of the complex stochastic evolutions.

Abstract

In the paper the method of constructing the equations for characteristic functional and the moments developed in ref. [7] is extended to the case of complex Hilbert space valued stochastic processes.

1. Introduction

In paper [7] the equations for the characteristic functional of the solution of the stochastic evolution equation was derived. Therefrom the complete set of the moment equations was obtained and the appropriate uniqueness and existence theorem was proved. These results were the extension of the ideas formulated in [2] and [3], where an example of some parabolic partial differential equation was analysed.

However, sometimes in applications it is irremissible to consider complex space valued stochastic equations. Restricting our field of interests to the problems of stochastic wave propagation, the example of such an equation could be the stochastic Schrödinger equation or the parabolic equations obtained from the approximation of stochastic Helmholtz equation [6],[9],[10]. In such cases the theory developed in [7] can not be applied directly to the analysis of the systems. However, redefining the characteristic functional and modifying the evolutionary description of the system, it is possible to follow reasonings of the paper [7] and construct the adequate equations.

In this paper we present the extension of the results obtained in [7] to the case of a complex Hilbert space valued evolutions. We define the stochastic evolution equation in such a case along with the characteristic functional of its solution (Section 2). Then we introduce the complex space version of lemmas and theorem proposed previously in [7] for real spaces and concerning the equation for the characteristic functional (Section 3). This equation is the foundation of the construction of the complete set of moment equations for the solution to the stochastic evolution (Section 4). Finally (Section 5), as an illustration of the introduced theory, we consider the example of a parabolic equation studied previously in [10] and in present paper reformulated in the abstract evolutionary form.

2. Formulation

Let $(\Omega,\mathcal{F},\mathcal{P})$ be a complete probabilistic space. Let $(\mathcal{X}, (.,.)_X)$ be a complex and $(\mathcal{Y}, (.,.)_Y)$ a real separable Hilbert space.

Consider the following stochastic multiplicative evolution equation:

$$\text{(2.1)} \qquad \begin{aligned} &dU = AU\,dt + [BU]\,dW(t) \quad , \quad t\in(0,T] \; , \\ &U(0,\omega) = U_0 \quad \mathcal{P}\ a.s. \quad , \quad U_0 \in \mathcal{X} \; , \end{aligned}$$

where:

$U = U(t,\omega)$, $t\in[0,T]$, $\omega\in\Omega$, is an $\mathcal{X}$-valued stochastic process,

$A : \mathcal{D}(A) \longrightarrow \mathcal{X}$ is a linear operator acting from the dense domain $\mathcal{D}(A) \subset \mathcal{X}$ into $\mathcal{X}$ and being the infinitesimal generator of the strongly continuous semigroup of bounded operators $K(t)$, $t\in[0,T]$,

$$K(t) : \mathcal{X} \longrightarrow \mathcal{X} \quad , \quad t\in[0,T] \; ,$$

$$\text{(2.2)} \qquad \| K(t) \| \le M\, e^{\varkappa t} \quad , \quad M \, , \, \varkappa = \text{const.}$$

$B : \mathcal{X} \longrightarrow L(\mathcal{Y},\mathcal{X})$ is a bounded bilinear operator , such that for $U \in \mathcal{X}$, $V \in \mathcal{Y}$

(2.3) $$\|[BU]V\|_X \le C \|U\|_X \|V\|_Y \quad , \quad C = \text{const.}$$

and

W is a $\mathcal{Y}$-valued Wiener process with covariance operator Q (see [4]).

The relevant boundary conditions arising in particular physical problems are included in the definition of $\mathcal{D}(A)$.

Remark

The considerations of this paper can be easily adopted to the case, when in equation (2.1) we have an operator linear with respect to the stochastic disturbance (as an additional term or instead of the one with the bilinear operator).

The operator Q is nuclear, positive definite and self-adjoint, so it can be expanded into the series (see[4]):

(2.4) $$Q = \sum_{k=1}^{\infty} \alpha_k \, e_k \otimes_Y e_k$$

where α_k and e_k for $k=1,2,3, \ldots$, are respectively, eigenvalues and orthonormal eigenvectors of Q ($\{e_k\}_{k=1,2..}$ is the Schauder basis in $\mathcal{Y}$); it is known that $\alpha_k > 0$ for $k=1,2,\ldots$ and $\Sigma\alpha_k < \infty$. Owing to formula (2.4) the Wiener process W has the representation in the form of series:

(2.5) $$W(t) = \sum_{k=1}^{\infty} \sqrt{\alpha_k} \, e_k \beta_k(t) \quad ,$$

where $\beta_k(t)$ for $k=1,2,\ldots$, are real independent Wiener processes with the unit intensities.

Substitution of expression (2.5) into eq. (2.1) leads to the stochastic differential equation with the stochastic differential of a real Hilbert space valued Wiener process being transformed to the series of real stochastic differentials:

$$dU = AU\,dt + \sum_{k=1}^{\infty} \sqrt{\alpha_k}\,[BU]e_k\,d\beta_k(t) ,$$
(2.6)
$$U(0) = U_0 .$$

Equation (2.6) is meant in Stratonovich interpretation; we consider its mild solution, that is the solution of the following integral equation (see [1]):

(2.7) $$U(t) = K(t)U_0 + \sum_{k=1}^{\infty} \sqrt{\alpha_k} \int_0^t K(t-s)[BU]e_k d\beta_k(s) .$$

Remark

In the case when for the description of a physical phenomenon we need a complex space valued Wiener process, the above model can be also applied. For this purpose one should decompose the process into its real and imaginary parts and then redefine the operator B in such a way that it is acting on a real product Hilbert space in which the real and imaginary components take values.

When real-valued processes are considered, for stochastic Stratonovich integral it is known the Furutsu-Novikov formula which, roughly speaking, makes possible to separate the mean values of functionals of Gaussian processes. The essential point of the seperation procedure depends on the differentiating the functional with respect to the process being its argument. In the case of the integral performed in equation (2.7) it is also possible to apply an infinite dimensional counterpart of such a formula. Derivation of it is based on Daletskii-Paramonova theory (see [5]).

The functional differentiation in generalized Furutsu-Novikov formula needs defining the space of measures in which appropriate Gaussian measure (corresponding to the Wiener process in (2.7)) is contained.

Let $\mathfrak{N}$ be the Hilbert space of ℓ^2-valued measures generated by sequences of real functions φ_k, $k=1,2, \dots$, such that:

$$\text{(2.8)} \qquad \sum_{k=1}^{\infty} \alpha_k \int_0^T \varphi_k^2(t)\, dt < \infty .$$

An element of the space has the form:

$$\nu_{\varphi}(.) = (\nu_{\varphi_1}(.), \nu_{\varphi_2}(.), \dots)$$

where

$$\nu_{\varphi_k}(\Delta) = \int_{\Delta} \alpha_k \varphi_k^2(t)\, dt \quad \text{for} \quad \Delta \in \sigma(0,T) , \ k=1,2, \dots .$$

and the inner product in $\mathfrak{N}$ is generated by (2.8).

The stochastic integral in (2.7) can now be regarded as Daletskii - Paramonova integral (see [5]) with respect to the Gaussian element from $\mathfrak{N}$ of the form:

$$\text{(2.9)} \qquad B = (\sqrt{\alpha_1}\, \beta_1, \sqrt{\alpha_2}\, \beta_2, \dots)$$

Let $g(t,s,B)$, $t,s \in [0,T]$, be an $\mathscr{X}$-valued functional on the element $B \in \mathfrak{N}$. The Frechet differential of g on the element $\nu \in \mathfrak{N}$ has the form:

$$\text{(2.10)} \qquad g'(t,s,B)\circ\nu = \sum_{l=1}^{\infty} \int_0^T \frac{\delta g(t,s,B)}{\delta \beta_l(\tau)}\, d\nu(\tau) .$$

The expression (2.10) represents differentiation of g with respect to "deterministic variations of the paths of real white noise processes" corresponding to Wiener processes β_l .

Consider the stochastic integral of the form:

$$\text{(2.11)} \qquad \sum_{k=1}^{\infty} \sqrt{\alpha_k} \int_0^t g_k(t,s,B)\, d\beta_k(s) ,$$

where $g_k(t,s,B)$ for $k=1,2,\dots$, $t,s \in [0,T]$ takes its values in $\mathscr{X}$ and $B \in \mathfrak{N}$ is defined in (2.9). If we assume that g_k , $k=1,2,\dots$, are differentiable and moreover

$$(2.12)\qquad \sum_{k=1}^{\infty} \sqrt{\alpha_k} \int_0^T E\left\{ \left\| \frac{\delta g_k(t,s,\beta)}{\delta\beta_k(s)} \right\|_X \right\} ds < \infty ,$$

then the mean value of integral (2.11) satisfies the ralationship being the infinite dimensional counterpart of Furutsu-Novikov formula:

$$(2.13)\quad E\left\{ \sum_{k=1}^{\infty} \sqrt{\alpha_k} \int_0^t g_k(t,s,\beta) d\beta_k(s) \right\} = \sum_{k=1}^{\infty} \sqrt{\alpha_k} \int_0^t E\left\{ \frac{\delta g_k(t,s,\beta)}{\delta\beta_k(s)} \right\} ds$$

All the formulae written up to now are formally the same as in the case of real Hilbert space $\mathcal{X}$ (see [7]). Difference in notation between real and complex cases starts from definitions of the characteristic functional and moments.

Characteristic functional.

Let $U(t)$, $t\in[0,T]$, be an $\mathcal{X}$-valued stochastic process. Let $\overline{U}(t)$ denotes the complex conjugate of the process $U(t)$ and λ, λ^* be two arbitrary elements of $\mathcal{X}$. The (spatial) characteristic functional of $U(t)$ is defined as:

$$(2.14)\quad F[t,\lambda,\lambda^*] = E\left\{ \exp\left[i\,(U(t),\lambda)_X + i\,(\overline{U(t)},\lambda^*)_X \right] \right\} , \quad t\in [0,T] .$$

The functional defined in such a way gives complete probabilistic characterization of the complex space valued process $U(t)$ at arbitrary fixed time t. In particular, it allows to obtain moments of any order of the process $U(t)$ (at the same t).

Moments.

Let us define:

$$(2.15)\qquad U_1 \overset{\text{def}}{=} U , \qquad U_2 \overset{\text{def}}{=} \overline{U} ,$$

where the bar, as always in this paper, denotes the complex conjugation. Let the moment of l-th order of the stochastic process $U(t)$ be defined as:

$$\Gamma_{k_1 k_2 \ldots k_l}(t) = E\left\{U_{k_1}(t) \otimes U_{k_2}(t) \otimes \ldots \otimes U_{k_l}(t)\right\}, \tag{2.16}$$

for $k_i = 1,2,\quad i=1,2,\ \ldots\ ,l\ ,\quad l=1,2,\ \ldots$.

We identify the moments for which the numbers of symbols "1" in the multiindexes are equal, that is, there are $l+1$ essentially different moments of l-th order; for the sake of simplicity in notation let us write them down as:

$$\Gamma_{l-p,\,p}(t) = E\left\{U_{k_1}(t) \otimes U_{k_2}(t) \otimes \ldots \otimes U_{k_l}(t)\right\}, \tag{2.17}$$

where

$k_i = 1$ for $i=1,2,\ldots,l-p$,

$k_i = 2$ for $i=l-p+1,\ldots,l$,

if $p=1,2,\ldots,l-1$,

(2.18) or

$k_i = 1$ for $i=1,2,\ldots,l$ if $p=0$

$k_i = 2$ for $i=1,2,\ldots,l$ if $p=l$.

Moments (2.17) can be easily obtained from the functional (2.14) by differentiation:

$$\Gamma_{l-p,\,p}(t) = \left.\frac{\delta^l F[t,\lambda,\lambda^*]}{\delta\lambda^{l-p}\,\delta\lambda^{*p}}\right|_{\substack{\lambda=0\\ \lambda^*=0}} \tag{2.19}$$

(the derivatives are in Frechet sense).

The characteristic functional (2.14) and the moments (2.17) are taken on the values of stochastic process $U(t)$ at arbitrarily fixed time t. However, using the governing equation (2.1) it is possible to derive equations which describe the evolution in time of the characteristic functional and the moments of any order. In the following sections we present formulae being the complex space counterparts of the results obtained in [7] for a real Hilbert space valued stochastic processes along with the appropriate lemmas needed for their proof.

3. Equation for the characteristic functional.

The derivation of the equations for the characteristic functional and the moments in the complex space case requires the analysis of two stochastic equations: the equation (2.1) for the process $U(t)$ and the additional one for its complex conjugate $\overline{U(t)}$:

$$d\overline{U} = \overline{AU}\, dt + [\overline{BU}]\, dW(t) \quad , \quad t\in(0,T] \quad , \tag{3.1}$$

$$\overline{U(0,\omega)} = \overline{U}_0 \quad \mathcal{P}\ a.s. \quad , \quad \overline{U}_0 \in \mathcal{X} .$$

The counterparts of all the lemmas needed in proofs of the theorems in the real state space are now extended as they concern two equations mentioned above. We quote them without proofs because the reasonings are quite simmilar to those in [7]).

The first lemma allows us to calculate the Frechet derivatives of the solution processes irremissible to use Furutsu-Novikov formula (2.13) effectively.

Lemma 1

Let $U(t,\beta)$, $\overline{U(t,\beta)}$ be the solutions of equations (2.1) and (3.1) respectively, once Frechet differentiable with respect to β and let

$$U(t,\beta)\ , \quad [\overline{BU(t,\beta)}]e_k\ , \quad \frac{\delta U(t,\beta)}{\delta\beta_k(s)}, \quad \left[\ B\ \frac{\delta U(t,\beta)}{\delta\beta_k(s)}\right]e_k \quad , \quad \text{for}$$

$k=1,2,\dots$ be continuous in t and bounded for $t=0$. Then

$$\frac{\delta U(t,\beta)}{\delta\beta_k(s)} = 0 \quad , \tag{3.2}$$

$$\frac{\delta\overline{U(t,\beta)}}{\delta\beta_k(s)} = 0 \qquad \text{for} \quad s>t$$

and

$$\frac{\delta U(t,\beta)}{\delta\beta_k(s)} = H(t-s)\sqrt{\alpha_k}\ K(t-s)[BU(s,\beta)]e_k \quad +$$

$$+ \sum_{l=1}^{\infty}\sqrt{\alpha_k} \int_s^t K(t-\tau)\ \left[\ B\,\frac{\delta U(\tau,\beta)}{\delta\beta_k(s)}\right]e_l\ d\beta_l(\tau)$$

(3.3) ,

$$\frac{\delta \overline{U(t,B)}}{\delta\beta_k(s)} = H(t-s)\sqrt{\alpha_k}\ \overline{K(t-s)}[\overline{BU(s,B)}]e_k \ +$$

$$+ \sum_{l=1}^{\infty} \sqrt{\alpha_k} \int_s^t \overline{K(t-\tau)}\left[\overline{B}\ \frac{\delta \overline{U(\tau,B)}}{\delta\beta_k(s)}\right]e_l\ d\beta_l(\tau) \quad \text{for} \quad 0 \le s < t\ ,$$

where $H(t)$ is the Heaviside function:

$$H(t) = \begin{cases} 0 & t<0 \\ \frac{1}{2}\ , & t=0\ . \\ 1\ , & t>0 \end{cases}$$

The following lemmas are of technical nature; they constitute essential steps of the proof of the main theorem.

Lemma 2

Let $\gamma : [0,T] \longrightarrow \mathbb{R}$ be an arbitrary smooth function and $\lambda \in \mathcal{D}(A), \lambda^* \in \mathcal{D}(\overline{A})$. Then the characteristic functional of the solution of the equation (2.1) satisfies:

$$iE\left\{ \int_0^T \gamma(t)(U(t,B),A^*\lambda)_X\ e^{i(U(t,B),\lambda)_X + i(\overline{U(t,B)},\lambda^*)_X} dt\right\} \ +$$

$$+ \ iE\left\{ \int_0^T \gamma(t)(\overline{U(t,B)},\overline{A}^*\lambda^*)_X\ e^{i(U(t,B),\lambda)_X + i(\overline{U(t,B)},\lambda^*)_X} dt\right\} =$$

(3.4)

$$= \int_0^T \gamma(t)\left[\frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda},\ A^*\lambda\right]_X dt \ + \ \int_0^T \gamma(t)\left[\frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda^*},\ \overline{A}^*\lambda^*\right]_X dt .$$

where A^* and $\overline{A}^*$ are the operators adjoint to A and $\overline{A}$ in $\mathcal{X}$.

Lemma 3

Under the assumptions of Lemma 1 the following equality holds true:

(3.5)
$$\frac{\delta}{\delta\beta_k(t)} e^{i(U(t,B),\lambda)_X + i(\overline{U(t,B)},\lambda^*)_X} =$$

$$= \frac{1}{2}\sqrt{\alpha_k}\left(\left[B\frac{\delta}{\delta\lambda} e^{i(U(t,B),\lambda)_X+i(\overline{U(t,B)},\lambda^*)_X}\right]e_k,\ \lambda\right)_X +$$

$$+ \frac{1}{2}\sqrt{\alpha_k}\left(\left[\bar{B}\frac{\delta}{\delta\lambda^*} e^{i(U(t,B),\lambda)_X+i(\overline{U(t,B)},\lambda^*)_X}\right]e_k,\ \lambda^*\right)_X$$

for $k=1,2,\ldots$.

Lemma 4

Let the assumptions of Lemmas 1 and 2 be satisfied and let the sequences of functions

(3.6)
$$g_k(s,B) = \left[B\frac{\delta}{\delta\lambda} e^{i(U(t,B),\lambda)_X+i(\overline{U(t,B)},\lambda^*)_X}\right]e_k \quad \text{for}$$

$k=1,2,\ldots$, and

$$g_k(s,B) = \left[\bar{B}\frac{\delta}{\delta\lambda^*} e^{i(U(t,B),\lambda)_X+i(\overline{U(t,B)},\lambda^*)_X}\right]e_k \quad \text{for}$$

$k=1,2,\ldots$, fulfill condition (2.12). Then:

$$E\left\{\int_0^T \gamma(t)\sum_{k=1}^{\infty}\sqrt{\alpha_k}\, i\left(\left[BU(t,B)\right]e_k,\lambda\right)_X e^{i(U(t,B),\lambda)_X+i(\overline{U(t,B)},\lambda^*)_X} d\beta_k(t) +\right.$$

$$+\int_0^T \gamma(t)\sum_{k=1}^{\infty}\sqrt{\alpha_k}\, i\left(\left[\overline{BU(t,B)}\right]e_k,\lambda^*\right)_X e^{i(U(t,B),\lambda)_X+i(\overline{U(t,B)},\lambda^*)_X} d\beta_k(t) =$$

$$+\frac{1}{2}\int_0^T \gamma(t)\left\{\sum_{k=1}^{\infty}\alpha_k\left(\left[B\frac{\delta}{\delta\lambda}\left(\left[B\frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda}\right]e_k,\lambda\right)_X\right]e_k,\lambda\right)_X +\right.$$

(3.7)

$$+ \sum_{k=1}^{\infty} \alpha_k \left(\left[B \frac{\delta}{\delta\lambda} \left(\left[\bar{B} \frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda^*} \right] e_k,\lambda^* \right)_X \right] e_k,\lambda \right)_X +$$

$$+ \sum_{k=1}^{\infty} \alpha_k \left(\left[B \frac{\delta}{\delta\lambda^*} \left(\left[B \frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda} \right] e_k,\lambda \right)_X \right] e_k,\lambda^* \right)_X +$$

$$+ \sum_{k=1}^{\infty} \alpha_k \left(\left[\bar{B} \frac{\delta}{\delta\lambda^*} \left(\left[\bar{B} \frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda^*} \right] e_k,\lambda^* \right)_X \right] e_k,\lambda^* \right)_X \Bigg\} dt.$$

The above lemmas lead to the main theorem displaying the form of differential equation (in Frechet derivatives) satisfied by characteristic functional (2.14) of the solution of equation (2.1).

Theorem 1

Let the solution U of the equation (2.1) fulfill the assumptions proposed in Lemmas 1-4 and let its characteristic functional $F[t,\lambda]$ possess the first order temporal derivative continuous on the interval $[0,T]$. Then $F[t,\lambda]$ satisfies the following differential equation:

$$\frac{\partial}{\partial t} F[t,\lambda,\lambda^*] = \left(\frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda}, A^*\lambda \right)_X + \left(\frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda^*}, \bar{A}^*\lambda^* \right)_X +$$

$$+ \frac{1}{2} \Bigg\{ \sum_{k=1}^{\infty} \alpha_k \left(\left[B \frac{\delta}{\delta\lambda} \left(\left[B \frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda} \right] e_k,\lambda \right)_X \right] e_k,\lambda \right)_X +$$

$$+ \sum_{k=1}^{\infty} \alpha_k \left(\left[B \frac{\delta}{\delta\lambda} \left(\left[\bar{B} \frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda^*} \right] e_k,\lambda^* \right)_X \right] e_k,\lambda \right)_X +$$

(3.8)

$$+ \sum_{k=1}^{\infty} \alpha_k \left(\left[\bar{B} \frac{\delta}{\delta\lambda^*} \left(\left[B \frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda} \right] e_k,\lambda \right)_X \right] e_k,\lambda^* \right)_X +$$

$$+ \sum_{k=1}^{\infty} \alpha_k \left(\left[\bar{B} \frac{\delta}{\delta\lambda^*} \left(\left[\bar{B} \frac{\delta F[t,\lambda,\lambda^*]}{\delta\lambda^*} \right] e_k, \lambda^* \right)_X \right] e_k, \lambda^* \right)_X \Big\} ,$$

where $\lambda \in \mathcal{D}(A)$, $\lambda^* \in \mathcal{D}(\bar{A})$, along with the common initial and normalization conditions:

$$(3.9) \qquad F[0,\lambda,\lambda^*] = e^{i(U_0,\lambda)_X + i(\overline{U_0},\lambda^*)_X}$$

$$(3.10) \qquad F[t,0,0] = 1 .$$

Remark

In paper [8] the method of constructing the characteristic functional of the solution to some particular stochastic differential equation is presented. Following the reasoning of present paper, that is considering the additional equation for the conjugate process, it is also possible to perform such a construction for complex space valued equations.

Equation (3.8) can be the used for derivation of the equations for the moments of the solution of our stochastic evolution equation (2.1).

4. The complete set of moment equations

To obtain the equations for the moments of the solution of eq.(2.1) let us assume that its characteristic functional is represented in the form of series ($\lambda_1 \stackrel{\mathrm{def}}{=} \lambda$, $\lambda_2 \stackrel{\mathrm{def}}{=} \lambda^*$):

$$(4.1) \qquad F[t,\lambda,\lambda^*] = 1 + \sum_{l=1}^{\infty} \sum_{p=0}^{l} \frac{(i)^l}{p!(l-p)!} \Gamma_{k_1 \ldots k_l}(t) \circ \lambda_{k_1} \otimes \ldots \otimes \lambda_{k_l} ,$$

(the dot denotes the inner product of tensors), where the indexes $k_1,\ldots,k_l$ satisfy the condition given in (2.18). After substituting series (4.1) into the equation for the characteristic functional and comparing the coefficients of the like powers of λ and λ^* in the left and right hand sides, one arrives at the following moment equations:

$$\frac{\partial}{\partial t}\Gamma_{k_1..k_l}(t) = \sum_{j=1}^{l} A_j^{k_j}\,\Gamma_{k_1..k_l}(t) + \tag{4.2}$$

$$+ \frac{1}{2}\sum_{k=1}^{\infty}\sum_{i,j=1}^{l}\alpha_k\,[B_j^{k_j}\{[B_i^{k_i}\,\Gamma_{k_1..k_l}(t)]e_k^i\}]e_k^j\,,$$

with the initial conditions:

$$\Gamma_{k_1..k_l}(0) = U_{k_1}(0)\otimes \ldots \otimes U_{k_l}(0)\,, \tag{4.3}$$

where $l=1,2,\ldots$, and the new operators are defined for the simple tensors of the form:

$$\Gamma_{k_1k_2\ldots k_l} = \gamma_{k_1}\otimes\gamma_{k_2}\otimes\ldots\otimes\gamma_{k_l} \tag{4.4}$$

as

$$A_j^{k_j}\,\Gamma_{k_1k_2\ldots k_l} = \gamma_{k_1}\otimes\gamma_{k_2}\otimes\ldots\otimes A^{k_j}\gamma_{k_j}\otimes\ldots\otimes\gamma_{k_l}\,, \tag{4.5}$$

$$[B_j^{k_j}\,\Gamma_{k_1k_2\ldots k_l}]e_k^j = \gamma_{k_1}\otimes\gamma_{k_2}\otimes\ldots\otimes[B^{k_j}\gamma_{k_j}]e_k\otimes\ldots\otimes\gamma_{k_l}\,, \tag{4.6}$$

$k_j=1,2,\quad j=1,2,\ldots$, and, analogously to (2.15):

$$A^1 = A\,,\qquad B^1 = B\,,$$

$$A^2 = \bar{A}\,,\qquad B^2 = \bar{B}\,.$$

It is observed that the equations (4.2) are separated for each l and arbitrary fixed set of indexes $k_1, k_2, \ldots, k_l$. This is the typical property of the linear equations with parametric or external white noise excitations.

The remark concerning symmetrization of the moments in the case of equation (4.2) remains valid but it concerns the couples: the complex operator and the appropriate function. The equations written in the form (4.2) , even if the condition (2.18) is not satisfied, are correct provided the correspondence between indexes k_i in the operators and in the moments holds.

It can be shown that the equations (4.2) for $l=1,2,\dots$, possess unique solutions. The proof is similiar to that in [7] for the real Hilbert space valued equations. This fact preserves also the uniqueness of the analytical solution (that is of the form (4.1)) of the functional equation (3.8). The existence needs some more restrictive assumptions on operator B; they could be explicitely precised for particular examples.

5. Illustrative example

As a simple illustrative example consider the equation obtained from the parabolic approximation of Helmholtz equation (see [10]):

$$2ik_0 \frac{\partial U(\mathbf{r},\omega)}{\partial x} + \Delta_\perp U(\mathbf{r},\omega) + k_0^2\varepsilon(\mathbf{r},\omega)U(\mathbf{r},\omega) = 0, \tag{5.1}$$

$$U(\mathbf{0},\omega) = U_0,$$

where $\mathbf{r}=(x,y,z)$, which can be written down in the abstract evolutionary form (2.1):

$$dU = AU\,dx + [BU]\,dW(x)\ , \tag{5.2}$$

where now spatial variable x corresponds to time t in (2.1), and the operators are defined as:

$$A = \frac{i}{2k_0}\Delta_\perp = \frac{i}{2k_0}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right), \qquad B = \frac{ik_0}{2}. \tag{5.3}$$

The spaces $\mathcal{X}$ and $\mathcal{Y}$ are chosen in this model as:

$$\mathcal{X} = L^2(\mathbb{R}^2,\mathbb{C}), \qquad \mathcal{Y} = L^2(\mathbb{R}^2,\mathbb{R}). \tag{5.4}$$

We assume that random field $\varepsilon(x,y,z)$ is Gaussian with a zero mean and δ-correlated in x:

$$E\left\{ \varepsilon(x,y,z)\varepsilon(x',y',z') \right\} = Q(y-y',z-z')\delta(x-x'). \tag{5.5}$$

The field ε is with respect to y,z sufficiently smooth, such that product εU takes its values in $\mathcal{X}$. The eigenvalues and eigenvectors of operator Q requested in expansion (2.4) are now the solutions of the integral equation:

$$\int\int Q(y-y',z-z')\, e_k(y',z')\, dy'\, dz' = \alpha_k\, e_k(y,z). \tag{5.6}$$

The covariance operator is the integral one with the kernel $Q(y-y',z-z')$ and process W in equation (5.2) is defined as

$$W(x) = \int_0^x \varepsilon(x',y,z)\, dx'. \tag{5.7}$$

The conjugate operators needed in the equations for the functional and the moments are:

$$\bar{A} = -\frac{i}{2k_0}\Delta_\perp = -\frac{i}{2k_0}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right), \tag{5.8}$$

$$\bar{B} = -\frac{ik_0}{2}.$$

The equation for the characteristic functional (3.8) in this particular case of partial differential equation (5.1) is the differential equation with Volterra variational derivatives and has the form:

$$\frac{\partial}{\partial x} F[x,\lambda,\lambda^*] = \frac{i}{2k_0}\int\int d\mathbf{r}_1 \left(\Delta_\perp \frac{\delta}{\delta\lambda(\mathbf{r}_1)} F[x,\lambda,\lambda^*]\, \overline{\lambda(\mathbf{r}_1)} - \right.$$

$$\left. - \Delta_\perp \frac{\delta}{\delta\lambda^*(\mathbf{r}_1)} F[x,\lambda,\lambda^*]\, \overline{\lambda^*(\mathbf{r}_1)} \right) - \tag{5.9}$$

$$- \frac{1}{2}\frac{k_0^2}{4}\int\int d\mathbf{r}_1 \int\int d\mathbf{r}_1'\, Q(\mathbf{r}_1-\mathbf{r}_1') \times$$

$$\times \left\{ \overline{\lambda(\mathbf{r}_1)} \frac{\delta}{\delta\lambda(\mathbf{r}_1)} \left[\overline{\lambda(\mathbf{r}_1')} \frac{\delta}{\delta\lambda(\mathbf{r}_1')} F[x,\lambda,\lambda^*] \right] - \right.$$

$$- \overline{\lambda(\mathbf{r}_1)} \frac{\delta}{\delta\lambda(\mathbf{r}_1)} \left[\overline{\lambda^*(\mathbf{r}_1')} \frac{\delta}{\delta\lambda^*(\mathbf{r}_1')} F[x,\lambda,\lambda^*] \right] -$$

$$- \overline{\lambda^*(\mathbf{r}_1)} \frac{\delta}{\delta\lambda^*(\mathbf{r}_1)^*} \left[\overline{\lambda(\mathbf{r}_1')} \frac{\delta}{\delta\lambda(\mathbf{r}_1')} F[x,\lambda,\lambda^*] \right] +$$

$$\left. + \overline{\lambda^*(\mathbf{r}_1)} \frac{\delta}{\delta\lambda^*(\mathbf{r}_1)} \left[\overline{\lambda^*(\mathbf{r}_1')} \frac{\delta}{\delta\lambda^*(\mathbf{r}_1')} F[x,\lambda,\lambda^*] \right] \right\},$$

where $\mathbf{r}_1=(y\ ,z\ )\ ,\ \mathbf{r}_1'=(y',z') \in \mathbb{R}^2\ ,$

$$\lambda(\mathbf{r}_1),\ \lambda^*(\mathbf{r}_1) \in \mathcal{X} = L^2(\mathbb{R}^2,\mathbb{C})\ ,$$

and the initial conditions (3.9), (3.10) are:

$$F[0,\lambda,\lambda^*] = exp \left\{ i \int\int U_0(y,z)\, \overline{\lambda(y,z)}\, dy\, dz + \right.$$

$$\left. + i \int\int \overline{U_0(y,z)}\, \overline{\lambda^*(y,z)}\, dy\, dz \right\},$$

$$F[x,0,0] = 1\ .$$

The equations for the moments of any order are

$$(5.10) \qquad \frac{\partial}{\partial x} \Gamma_{k_1..k_l}(x,y_1,z_1,\dots,y_l,z_l) =$$

$$= \sum_{i=1}^{l} \frac{i}{2k_0} (-1)^{k_i+1} \left[\frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2} \right] \Gamma_{k_1..k_l}(x,y_1,z_1,\dots,y_l,z_l) -$$

$$- \frac{1}{2} \frac{k_0^2}{4} \sum_{i,j=1}^{l} Q(x_i-x_j,y_i-y_j)\,(-1)^{k_i+k_j}\, \Gamma_{k_1..k_l}(x,y_1,z_1,\dots,y_l,z_l)$$

for $k_1,k_2,\dots,k_l=1,2\ ,\ l=1,2,\dots$ and

$$\Gamma_{k_1 \ldots k_l}(0, y_1, z_1, \ldots, y_l, z_l) = U_0^{k_1}(y_1, z_1) \times \ldots \times U_0^{k_l}(y_l, z_l).$$

As it is known from the general considerations for equations (4.2), the equations (5.10) have unique solutions.

The equations for the characteristic functional and the moments of lover order of the form (5.9) and (5.10) coincide with the analogous equations obtained in [10] with the use of classical Furutsu- Novikov formula in spite of the fact that stochastic integrals used in [10] and in present paper are defined in some other way. The reason of this is the specific form of operator B in equation (5.1) (B does not act on the spatial variables of the Wiener process).

References.

[1] A.Chojnowska-Michalik, *Stochastic differential equations in Hilbert space*, in *Probability Theory*, Banach Center Publications, Vol.5, pp.53-74, Polish Scientific Publishers, Warsaw,1979.

[2] P.L.Chow, *Function-space differential equations associated with a stochastic partial differential equations*, Indiana Univ. Math. Journal, 25(1976), 609-627.

[3] P.L.Chow, *Stochastic partial differential equations in turbulence related problems*, in *Probabilistic Analysis and Related Topics*, Vol.1, A.T.Bharucha-Reid, ed., Academic Press, New York, 1978.

[4] R.Curtain, I.Prichard, *Infinite dimensional linear systems theory*, Lect.Not. in Contr. and Inf. Sc., Vol.8, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[5] Yu.Daletskii, S.N.Paramonova, *Stochastic integrals over normally distributed additive function of sets*, Dokl.Ac.Sc.USSR, 208(1973), 512-515.

[6] D.A.Dawson, G.C.Papanicolaou, *A random wave process*, Appl. Math. Optim. 12(1984), 97-114.

[7] Z.Kotulski, *Equations for the characteristic functional and moments of the stochastic evolutions with an application*, SIAM Journal on Appl. Math., (to appear).

[8] Z.Kotulski, K.Sobczyk, *Characteristic functionals of randomly excited physical syctéms*, Physica 123A(1984), 261-278.

[9] L.C.Lee, *Wave propagation in a random medium. A complete set of the moment equations with different wavenumbers*, J.Math. Phys. 15(1974), 1431-1435.

[10] V.I.Tatarskii, *The light propagation in a medium with random inhomogenities of the refractive index and the Markov process approximation*, J.Exp.Theor.Phys., 56(1969), 2106-2117.