XXII.

# SIR J. LARMOR'S MECHANICAL MODEL OF THE PRESSURE OF RADIATION

(From a letter of Prof. T. Levi-Civita to Prof. Sir J. Larmor)

« Atti V Congr. Int. dei Matematici », Cambridge, vol. 1, pp. 217-220.

I have perused your beautiful lecture « On the dynamics of radiation » which I was fortunate to hear in Cambridge. Will you allow me to present in a little more general aspect your idea leading to a mechanical model of the pressure of waves? I shall refer myself, as you do, to a vibrating string.

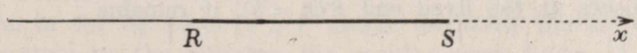
## 1. - Specification of the assumptions. Flow of energy. Pressure on moving end.

The small transverse oscillations of a stretched cord in absence of bodily forces, whatever the initial and boundary conditions may be, follow the equation

$$\varrho \frac{\partial^2 \eta}{\partial t^2} = T \frac{\partial^2 \eta}{\partial x^2}, \tag{1}$$

where $\eta(x, t)$ is the displacement, and $\varrho$ and $T$ are constants of well-known signification. As usual I shall write $c^2$ for $T/\varrho$, $c$ thus designating the velocity of propagation of transverse waves along the string.

Let us suppose that the undulations $\eta$ extend only to a finite (variable) portion




of our string: from a moving end (reflector) $R$, at which $x = vt$, to a fixed $S$, at which $x = b$ ($v$, $b$ positive constants).



www.rcin.org.pl

The condition of (perfect) reflexion at $R$ shall be further introduced. Independently from it, we may specify in usual way the energetic point of view.

The density of energy, both kinetic and potential (at any place $x$, between $R$ and $S$, and time $t$) is

$$e = \tfrac{1}{2}\varrho\dot{\eta}^2 + \tfrac{1}{2}T\left(\frac{\partial\eta}{\partial x}\right)^2 = \tfrac{1}{2}\varrho\left\{\dot{\eta}^2 + c^2\left(\frac{\partial\eta}{\partial x}\right)^2\right\}, \tag{2}$$

where the dot stands for $\partial/\partial t$. Especially, in front of the reflector, it becomes

$$e_R = [e]_{x=vt}. \tag{3}$$

The energy stored in the whole extent of the disturbed string at the time $t$ amounts therefore to

$$E = \int_{vt}^{b} e\,dx, \tag{4}$$

from which we easily get an expression of $dE/dt$ convenient for our aim. It is in fact

$$\frac{dE}{dt} = -e_R v + \int_{vt}^{b}\frac{\partial e}{\partial t}\,dx.$$

But, by (2),

$$\frac{\partial e}{\partial t} = \varrho\left\{\dot{\eta}\,\frac{\partial^2\eta}{\partial t^2} + \frac{\partial\eta}{\partial x}\,\frac{\partial\dot{\eta}}{\partial x}\right\}.$$

Since, by partial integration,

$$\int_{vt}^{b}\frac{\partial\eta}{\partial x}\,\frac{\partial\dot{\eta}}{\partial x}\,dx = \left[\frac{\partial\eta}{\partial x}\,\dot{\eta}\right]_{vt}^{b} - \int_{vt}^{b}\dot{\eta}\,\frac{\partial^2\eta}{\partial x^2}\,dx,$$

and $\dot{\eta}$ vanishes at the fixed end $S\,(x = b)$, it remains

$$\int_{vt}^{b}\frac{\partial e}{\partial t}\,dx = \varrho\int_{vt}^{b}\dot{\eta}\left\{\frac{\partial^2\eta}{\partial t^2} - c^2\,\frac{\partial^2\eta}{\partial x^2}\right\}dx - \varrho c^2\left[\frac{\partial\eta}{\partial x}\,\dot{\eta}\right]_{x=vt}.$$

www.rcin.org.pl

On account of the fundamental equation (1), this reduces to the last term, and gives

$$\frac{dE}{dt} = - e_R v - \varrho c^2 \left[\frac{\partial \eta}{\partial x}\dot{\eta}\right]_{x=vt} .$$

Putting

(5) $$f_R = - \varrho c \left[\frac{\partial \eta}{\partial x}\dot{\eta}\right]_{x=vt} ,$$

the formula may be written

(6) $$\frac{dE}{dt} = - e_R v + f_R c ,$$

or also

(6′) $$\frac{dE}{dt} = Pv ,$$

where

(7) $$P = - e_R + f_R \cdot \frac{c}{v} .$$

The formulae (6) and (6′) are capable of expressive interpretations.

Let us firstly pay attention to the formula (6), supposing $v = 0$ ($R$ fixed). It means that the exchanges of energy between $RS$ and the outside take place as if a flow $f_R$ (directed inward if positive) passed through $R$ with the wave-velocity $c$. We recognize obviously the one-dimensional form of the Poynting-Volterra's investigations.

In the general case where $v$ is not zero, the wave-flow $f_R$ must be increased by the convection-flow $-e_R$, travelling with the velocity $v$, that is - we may say - convected by the moving end $R$.

To get the interpretation of the (equivalent) formula (6′), we have only to recall the principle of conservation of energy in its pure mechanical form. It states that $dE/dt$, for any material system (the string in our case) must be equal to the time-rate of doing work of all external forces. At the present no external forces act on the system, except, at the ends of the disturbed portion, arising from the connections: with the reflector at $R$, with some fixed body at $S$. But the last does not do work because $S$ is at rest.

Hence, in the equation (6′), $P$ means the force exerted *on* the considered system by the reflector, the positive sense being of course that of the increasing $x$. Reversing the positive sense and availing ourselves of the principle of reaction, we may also regard $P$ *as the pressure supported by the advancing reflector* (traction if $P$ should result negative).

www.rcin.org.pl

## 2. - Adiabatic arrangement.

The formula (6) and its consequences have been deduced on the hypothesis that the cord is fixed at $S$, so that $(\partial\eta/\partial x)\dot\eta$ vanishes for $x=b$. If it be not so, we have in the second member of (6) a further term $-cf_S$, where

$$-f_S = \varrho c\left[\frac{\partial\eta}{\partial x}\dot\eta\right]_{x=b}.$$

This would introduce a flow of energy across $S$, to be considered together with the flow across $R$; the preceding argument would therefore be altered.

There is however an obvious arrangement for which the formula equally hold: it consists in admitting a proper supply of energy at $S$, just as it is required to compensate the flow $-cf_S$. The connection at $S$, between $RS$ and the outside, may then be called adiabatic. We shall henceforth adopt this assumption, getting thus free from the more restrictive one of a fixed end.

## 3. - Decomposition of the disturbance in two wave-trains. Perfect reflexion.

If the solution $\eta$ of (1) is a function (any whatever) of $x+ct$, we have the case of waves advancing to the reflector $R$. Then $c(\partial\eta/\partial x)=\dot\eta$, and $f_R$ becomes identical with $-e_R = \varrho\dot\eta^2$, giving to (6) the form

$$\frac{dE}{dt} = -e_R(c+v):$$

the flow of energy occurs as if the waves were carrying their energy with the (absolute) velocity $c$, i.e. $c+v$ relative to the reflector.

For a train $\eta(x-ct)$ (reflected from $R$), we find in analogous way the flow $e_R(c-v)$.

Now any solution of (1) has the form

$$\eta = \eta_1(x+ct)+\eta_2(x-ct) \tag{8}$$

$\eta_1$, $\eta_2$ being arbitrary functions of their respective arguments. We get, accordingly,

$$\dot\eta = \dot\eta_1+\dot\eta_2,$$

$$c\frac{\partial\eta}{\partial x} = c\left(\frac{\partial\eta_1}{\partial x}+\frac{\partial\eta_2}{\partial x}\right) = \dot\eta_1-\dot\eta_2;$$

www.rcin.org.pl

therefore, from (2),

$$e = \varrho(\dot{\eta}_1^2 + \dot{\eta}_2^2) \tag{2'}$$

and, from (5)

$$f_R = \varrho[\dot{\eta}_2^2 - \dot{\eta}_1^2]_{x=vt} \,. \tag{5'}$$

With these values the expression (6) of $dE/dt$ may be written

$$\frac{dE}{dt} = \varrho\dot{\eta}_2^2(c - v) - \varrho\dot{\eta}_1^2(c + v) \,. \tag{6''}$$

Thus the gain of energy appears caused by flows (relative to $R$) of the energies carried by the two opposite wave-trains. It is your favourite point of view.

Now we proceed to the condition of perfect reflexion at $R$. You properly conceive the reflector to be realised by a plate with a hole through which the cord passes. As the plate advances along the cord, it sweeps the waves in front, restoring behind of it the resting straight configuration of the cord. Under these circumstances, the condition at $R$ is obviously that the total displacement shall be annulled, that is

$$\eta_1 + \eta_2 = 0 \quad \text{for} \quad x = vt \,. \tag{9}$$

Having thus achieved the general premisses, a mathematical observation may find place, viz., that it would not be difficult to determine functions $\eta_1$, $\eta_2$ of their respective arguments $x+ct$, $x-ct$, satisfying rigorously to the nodal condition $\eta_1+\eta_2 = 0$ as well for $x = vt$ as for $x = b$. But I propose only to apply the above to your particular solution.

### 4. - Case of simple wave-trains. Mean pressure.

You assume

$$\begin{cases} \eta_1 = \dfrac{A_1}{m_1 c} \sin m_1(x + ct) \,, \\ \eta_2 = -\dfrac{A_2}{m_2 c} \sin m_2(x - ct) \,, \end{cases} \tag{10}$$

and consequently

$$\begin{cases} \dot{\eta}_1 = A_1 \cos m_1(x + ct) \\ \dot{\eta}_2 = A_2 \cos m_2(x - ct) \end{cases} \tag{11}$$

www.rcin.org.pl

$A_1$, $m_1$, $A_2$, $m_2$ being constants to be disposed with regard to (9).
It requires

$$(12) \qquad A_2(c-v) = -A(c+v)$$

$$(13) \qquad m_2(c-v) = \pm m_1(c+v)\,.$$

With the determination (10) of $\dot\eta_1$, $\dot\eta_2$, the expression (2′) of $e$ becomes a sum of two periodic functions of $x$, and of $t$. Its average value $\bar{e}$, with respect to $x$ as well to $t$, is

$$(14) \qquad \bar{e} = \tfrac{1}{2}(A_1^2 + A_2^2)\,.$$

The same value belongs to the time average of $e_R = (e)_{x-vt}$. On the other hand, averaging the expression (5′) of $f_R$, we have

$$(15) \qquad \bar{f}_R = \tfrac{1}{2}\varrho(A_1^2 - A_2^2)\,.$$

But, by (12),

$$A_2^2 = \frac{(c+v)^2}{(c-v)^2} A_1^2;$$

hence, from (14),

$$\bar{e} = \varrho \frac{c^2 + v^2}{(c-v)^2} A_1^2\,,$$

and, from (15),

$$\bar{f}_R = -2\varrho \frac{cv}{(c-v)^2} A_1^2 = -\frac{2cv}{c^2+v^2}\,\bar{e}\,.$$

We finally arrive at the pressure $P$ defined by (7). Its mean value ($e_R$ being identical with $e$) becomes

$$\bar{P} = -\bar{e} - \frac{c}{v} f_R = \frac{c^2 - v^2}{c^2 + v^2}\,\bar{e}\,,$$

which is your result.

*University of Padua, Oct. 9.*

www.rcin.org.pl