88.

AN INSTANTANEOUS GRAPHICAL PROOF OF EULER'S THEOREM ON THE PARTITIONS OF PENTAGONAL AND NON-PENTAGONAL NUMBERS.

[*Johns Hopkins University Circulars*, II. (1883), p. 71.]

I START with the product

$$(1 + ax)(1 + ax^2)(1 + ax^3)\ldots;$$

the coefficient of $x^n a^j$ in its development in a series according to powers of x and a is the number of partitions of n into j unequal parts; each such partition may be represented by a regular graph and these graphs classified according to the magnitude of the Durfee-square which they contain. Calling the side of any such square θ, two cases arise, namely, the vertical side of the square may either be completely covered or one point in it be left exposed: in the former case any number of the points in the base of the square, in the latter case not more than the first $\theta - 1$ points can be covered.

The first case contributes to the total number of partitions of n into j unequal parts the number of ways of distributing $n - \theta^2$ between two groups, one consisting of θ unequal parts unlimited, the other of j unequal parts not exceeding θ in magnitude.

The second case contributes the number of ways of distributing $n - \theta^2$ between two groups consisting one of $\theta - 1$ unequal parts unlimited, the other of $j - \theta$ unequal parts not exceeding $\theta - 1$ in magnitude.

Hence remembering that the number of ways of partitioning any number ν into θ parts is the coefficient of x^ν in

$$\frac{x^{\frac{\theta^2+\theta}{2}}}{1 - x \,.\, 1 - x^2 \ldots},$$

it is easily seen to follow that

$$(1+ax)(1+ax^2)(1+ax^3)\ldots$$

must be equal to the sum of the two series

$$1+\frac{1+xa}{1-x}x^2a\ldots+\frac{(1+xa)(1+x^2a)\ldots(1+x^\theta a)}{1-x.1-x^2\ldots1-x^\theta}x^{\theta^2+\frac{\theta^2+\theta}{2}}a^\theta+\ldots$$

and $$xa+\ldots+\frac{1+xa.1+x^2a\ldots1+x^{\theta-1}a}{1-x.1-x^2\ldots1-x^{\theta-1}}x^{\theta^2+\frac{\theta^2-\theta}{2}}a^\theta+\ldots;$$

on making $a=-1$ there results

$$(1-x)(1-x^2)(1-x^3)\ldots=1-x-x^2\ldots+(-)^\theta\left(x^{\frac{3\theta^2-\theta}{2}}+x^{\frac{3\theta^2+\theta}{2}}\right)+\ldots$$

which is the theorem to be proved.

In the Appendix or Exodion to a forthcoming paper in the *American Journal of Mathematics* [Vol. IV. of this Reprint] I give a proof by the method of correspondence of Jacobi's generalization of the above theorem, namely:

$$(1\pm x^{n-m})(1\pm x^{n+m})(1-x^{2n})(1\pm x^{3n-m})(1\pm x^{3n+m})(1-x^{4n})\ldots$$
$$=\sum_{-\infty}^{+\infty}(\pm)^i x^{ni^2+mi}.$$