86.

ON A NEW THEOREM IN PARTITIONS.

[*Johns Hopkins University Circulars*, II. (1883), p. 70.]

It is a well-known theorem that the number of partitions of n into odd numbers is equal to the number of its partitions into unequal numbers. This equality was seen by Euler to result from the identity

$$(1+x)(1+x^2)(1+x^3)\ldots = \frac{1}{(1-x)(1-x^3)(1-x^5)\ldots}.$$

It may also be proved easily by the method of correspondence. For if we call the partitions of n into odd numbers (repeated or not) the U, and into unequal numbers the V system, any V will be of the form $[V_1, V_2, V_3, \ldots]$, where V_i is of the form

$$q_i \cdot 2^{\alpha_i},\quad q_i \cdot 2^{\alpha_i'},\quad q_i \cdot 2^{\alpha_i''},\quad \ldots,$$

each q being an odd number and all the q's unlike.

Hence writing

$$2^{\alpha_i} + 2^{\alpha_i'} + 2^{\alpha_i''} \ldots = A_i,$$

V is transformable into $A_1q_1, A_2q_2, A_3q_3, \ldots$, which is a member of the U system.

And conversely any U as $A_1q_1, A_2q_2, A_3q_3, \ldots$, will be transformable into a V by decomposing each U into a sum of products of its largest odd divisor into distinct powers of 2 which can be effected in one and only one manner; so that there is a one-to-one correspondence between the U's and V's, and the number of the one set is therefore the same as the number of the other. The theorem which is now to be explained is, so to say, a differentiation (in the Herbert Spencer sense) of this theorem.

It regards the U and V systems each broken up into classes and affirms the equality between the numbers of U's in any class and of the V's in the homonymous class. The proof of this by an analytic identity remains to be

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discovered—it is effected without great difficulty by the method of correspondence: but what is very worthy of notice is that the V which corresponds to a U, in the more refined construction about to be explained, is in general (and it may be universally) different from the V which corresponds to it, when the preceding method of conjugation is adopted.

Every U which contains i distinct parts is said to be a U of the ith class, and every V which contains i distinct sequences (not running together) of consecutive numbers is said to be a V of the ith class—and my theorem may be expressed by saying that there exists a one-to-one correspondence (and therefore equality of content) between the U's of any class and the V's of the same class. I ought perhaps rather to say that a correspondence can be *instituted* than that a correspondence *exists*, for the fact that two absolutely unlike bonds of correspondence connect the totality of the U and that of the V system seems to indicate that such correspondence should rather be regarded as something put into the two systems by the human intelligence than an absolute property inherent in the relation between the two. Kant makes a similar remark upon the elementary conceptions (such as the circle), which form the groundwork of geometry.

As an example of the numerical part of the theorem consider the 3rd class of the U's and V's for $n = 16$.

The U's of this class will be

$$11.3.1^2;\ 9.5.1^2;\ 9.3^2.1;\ 9.3.1^4;\ 7.5.1^4;\ 7.3^2.1^3;$$
$$7.3.1^6;\ 5^2.3.1^3;\ 5.3^3.1^2;\ 5.3^2.1^5;\ 5.3.1^8;$$

and the V's which are somewhat more difficult to calculate by an exhaustive process will be found to be

$$1.6.9;\ 1.2.5.8;\ 2.6.8;\ 1.5.10;\ 1.2.4.9;\ 2.5.9;$$
$$1.4.11;\ 1.3.4.8;\ 3.5.8;\ 2.4.10;\ 1.3.12.$$

So again of the 4th class there is only one U and one V, namely, $1.3.5.7$, which is common to the two systems—and of the first class owing to 16 containing only one odd divisor, namely, unity, there is also but one U and one V, namely, the undivided 16 for each alike. In general for the first class the number of U's is obviously the number of odd divisors of the partible number n and the number of single sequences is easily seen to be the same. Thus, for example, for 15 there exist the sequences $1.2.3.4.5$; $4.5.6$; 7.8; 15; and for 9 the sequences $2.3.4$; 4.5; 9.

I will now indicate the mode of proof, the particulars of which will be found set out in full in the forthcoming number of the *American Journal of Mathematics* [Vol. IV. of this Reprint].

The partible number n being given, I take any U belonging to it and form two graphs, one whose rows represent the major halves of each part

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of U and the other its minor halves [$q+1$ is the major and q the minor half of $2q+1$]. I then dissect each of these graphs into its component angles and take the content of each; it is easily seen that beginning with the major and passing from it to the minor and back again to the major and so on continually in alternate succession, the readings will form a continually decreasing series of numbers whose sum will be the same as of the parts of the U, and thus U will be transformed into V. The number of parts in V, if we agree to consider that number as always *even* by supplying a zero at the end if it should happen to be *actually* odd, will be $2i$ where i is the number of points in the side of the Durfee-square appertaining to the major graph.

Conversely, if any V be given containing or made to contain $2i$ parts, it is easy to construct a system of $2i$ *linear* equations between the contents of the first i lines and the first i columns of an assumed U having a Durfee-square containing i^2 points, which shall transform into the given V, and to prove that these contents will be all of them greater than $2i$: hence *one* and *only one* U corresponds to a V, and consequently there is a one-to-one correspondence between the entire U and entire V systems. It remains to show that any U_i (a U of the ith class) by the prescribed process of transformation becomes a V_i (a V of the ith class).

This is effected as follows: suppose the first exterior angle to be removed simultaneously from a given major graph and its accompanying minor: begin with supposing that U_i becomes V_j: i is the number of unequal lines in either graph and it is easily proved that $i-j$ remains unaltered by the contraction of the graphs in the manner above indicated: that is, it can be shown that the effect of the contractions is to diminish i and j simultaneously each of them by 0, each of them by 1, or each of them by 2.

Continuing this process of *stripping* the graphs of their outside angles we must come at last to a graph consisting of one line and one column or of only one line, or only one column, or only a point. In the first of these four cases i and j are each equal to 2, and in the last three each equal to 1, hence $i-j$ is always zero and every U_i corresponds to a V_i. This establishes the very remarkable theorem that was to be proved.

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