84.

ON THE NUMBER OF FRACTIONS IN THEIR LOWEST TERMS WHOSE NUMERATORS AND DENOMINATORS ARE LIMITED NOT TO EXCEED A CERTAIN NUMBER.

[*Johns Hopkins University Circulars*, II. (1883), pp. 44, 45.]

THE fractions for greater simplicity may be supposed to be proper fractions, except that it is expedient to count in $\frac{1}{1}$ as one of them. To any given limit or argument as it may be called, n, corresponds a finite system of fractions in their lowest terms, which may be arranged in order of magnitude; when so arranged the system will be found to possess some remarkable properties, first apparently noticed by Mr Farey, an English mathematician, in 1816, subsequently made the subject of a proof by Cauchy in the same year (reproduced in his *Exercices de Mathématiques*, t. I. 1826), and again demonstrated and extended by Mr J. W. L. Glaisher in an interesting paper in the *London and Edinburgh Philosophical Magazine* for 1879, the same journal in which the subject was first broached.

I am under the impression that I have seen somewhere the names of one, if not two, English mathematicians who have endeavoured to obtain an empirical law for the number of fractions corresponding to any given limit, but all my endeavours to come upon the traces of those investigations, if such exist, have hitherto proved fruitless. Had anything been done in this direction prior to 1879, there is little doubt that reference would have been made to it by Mr Glaisher, who goes carefully in his paper of that date into the bibliography of the subject*.

This number for the limit or argument j is obviously no other than the sum of the *totients* of all the numbers from unity up to j. I shall use Tx

* Mr Glaisher, however, takes no notice of M. Halphen's important extension of Farey's theory, published in the *Proceedings of the Mathematical Society of France*, and followed by another on the same subject by M. Lucas, nor of Herzer's table in 1864, nor Hrabak's, 1876.

to denote the sum of all the totients of all the integers not exceeding x, where x is any positive quantity whatever, and show how to make Tx the subject of a functional equation, from which limiting functions to its value may be deduced. To this end consider the two sets of terms $1, 2, 3, \ldots i$ and $i+1, i+2, \ldots j$, where $j = 2i$ or $2i+1$ indifferently.

The number of times that an integer r is contained in any given set of quantities, or rather the number of quantities in the set which contain r, I call the *frequency* of r in respect to the set.

Looking to the two sets here in question it is easily seen that the frequency of any integer *quâ* the second set must either be equal to its frequency *quâ* the first set or exceed it by a single unit. The equi-frequent and unequi-frequent integers are determinable by the following theorem which I call the theorem of harmonic division.

Let j_μ in general denote the integer part of j/μ if it is a fraction, or the whole of it if it is an integer.

Write down the successive ranges

$$j, j-1, j-2, \ldots j_2+1;\ j_2, j_2-1, \ldots j_3+1;$$
$$j_3, j_3-1, \ldots j_4+1;\ j_4, j_4-1, \ldots j_5+1;\ \ldots$$

where it will be understood that if $j_k = j_{k+1}$, the range $j_k \ldots$ becomes abortive.

Any number which appears in the first, third, fifth ... range is equi-frequent and any number which appears in the second, fourth ... range is unequi-frequent in respect to the two given series

$$1, 2, \ldots i;\ i+1, i+2, \ldots j.$$

This theorem will be found to be demonstrable without the slightest difficulty.

The second theorem required is one of which a demonstration almost instantaneous and conclusive is given in the Excursus on Rational Fractions and Partitions (*Am. Jour. of Math.*, Vol. v., No. 2*), namely, that the sum of the products formed by multiplying the frequency of any integer in respect to a given set of quantities by its totient is equal to the sum of the quantities contained in the set.

This proposition shows that if fr, $f'r$ be the frequencies of r in respect to the two last-named sets and τr its totient

$$\sum_{r=1}^{r=\infty} (f'r - fr)\,\tau r = (1+2+\ldots+j) - 2(1+2+\ldots+i),$$

and the theorem of harmonic division shows that the left-hand side of this equation is equal to

$$\sum_{\lambda=j_2+1}^{\lambda=j} \tau\lambda + \sum_{\lambda=j_4+1}^{\lambda=j_3} \tau\lambda + \sum_{\lambda=j_6+1}^{\lambda=j_5} \tau\lambda + \ldots$$

[* Above, p. 611.]

because $f'r - fr = 1$ for the odd-ordered, and $= 0$ for the even-ordered harmonic ranges.

The separate sums above written are obviously the same respectively as

$$Tj - T\frac{j}{2},\ T\frac{j}{3} - T\frac{j}{4},\ T\frac{j}{5} - T\frac{j}{6},\ \ldots.$$

Hence, if we write

$$\vartheta j = Tj - T\frac{j}{2} + T\frac{j}{3} - T\frac{j}{4} + T\frac{j}{5} \ldots \textit{ ad inf.}$$

when j is an even integer

$$\vartheta j = (1 + 2 + \ldots + 2i) - 2(1 + 2 + \ldots + i) = i^2 = \frac{j^2}{4},$$

and when j is an odd integer

$$\vartheta j = \{1 + 2 + \ldots + (2i+1)\} - 2(1 + 2 + \ldots + i) = (i+1)^2 = \frac{(j+1)^2}{4}.$$

If now we write for any positive quantity x,

$$\vartheta x = Tx - T\frac{x}{2} + T\frac{x}{3} - T\frac{x}{4} + \ldots,$$

it may be shown by aid of the above results that for all values of x not less than unity,

$$\vartheta x = \text{ or } > \frac{x^2}{4} - \frac{x}{2},\quad \vartheta x = \text{ or } < \frac{x^2}{4} + \frac{x}{2} + \frac{1}{4},$$

and from these two inequalities limiting values to $T(x)$ may be deduced by a process of successive approximation in principle the same as that employed by me in the *Am. Jour. of Math.*, Vol. v., No. 2, pp. 124, 125*, in connexion with Tchebycheff's theory, but differing from it considerably in the mode of application and in the character of the results to which it leads.

The subject has been for so very short a time studied by me that I feel it desirable to reserve this part of its development for a future communication, but I am in a position to state that it is possible to find superior and inferior limits to Tx, say Lx and Λx, such that Lx shall be of the form $Mx^2 + Nx +$ a rational integral function of $\log x$ and Λx of a similar form, $M'x^2 + N'x +$ another rational integral function of x, where M, M' differ by a quantity less than any quantity that may be assigned from one another and from a number λ found from the equation

$$\lambda\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} \ldots\right) = \frac{1}{4},$$

that is, equal to $3/\pi^2$.

Accordingly the ultimate, or so to say asymptotic, value of $\frac{Tx}{x^2}$ is $3/\pi^2$ where Tx is the number of pairs of integers not exceeding x, which are relatively prime to each other; consequently since the total number of such

[* Above, p. 530.]

pairs is ultimately in a ratio of equality to $\frac{x^2+x}{2}$, it will follow, if the above assertion is correct, that the chance of two arbitrary independent integers, being relatively prime to one another, is $\frac{6}{\pi^2}$; the odds in favour of two such numbers being relatively prime, are thus very nearly expressed by the ratio of 60792 to 39268; that is to say, are pretty nearly as 76 to 49 or a trifle better than 3 to 2.

In what precedes I have used the simplest means or formula sufficient for obtaining a functional equation to the sum-totient Tx, but the theorem of harmonic division admits of a very wide generalization, and accordingly the functional equation admits of an indefinite number of distinct presentations.

Thus instead of j belonging to the series $2i$, $2i+1$ it may be considered as belonging to the series ki, $ki+1$, ..., $ki+(k-1)$, and the theorem of harmonic division then is as follows: calling the range commencing with j_λ and ending with $j_{\lambda+1}+1$, range number λ, where λ may be understood to be any integer from 0 upwards (0 itself included) if the residue of λ in respect to k is μ, and if $f'r$ and fr are the frequencies of r in respect to the two series 1, 2, ... j, 1, 2, ... i, then when r belongs to the range whose number is λ, and the residue of λ in respect to k is (λ), it will be found that $f'r-kfr=(\lambda)$.

By way of example suppose $k=3$, then writing

$$\vartheta j=\left(Tj+T\frac{j}{2}-2T\frac{j}{3}\right)+\left(T\frac{j}{4}+T\frac{j}{5}-2T\frac{j}{6}\right)+\ldots$$

it may readily be proved that according as $j=3i$, or $3i+1$, or $3i+2$, ϑj will equal $3i^2$, or $3i^2+3i+1$, or $3i^2+6i+3$, and similarly if

$$\vartheta j=\left(\vartheta j+\vartheta\frac{j}{2}\ldots+\vartheta\frac{j}{k-1}-k\vartheta\frac{j}{k}\right)+\left(\vartheta\frac{j}{k+1}\ldots+\vartheta\frac{j}{2k-1}-k\vartheta\frac{j}{2k}\right)+\ldots$$

then ϑj according as $j=ki$, or $ki+1$, ... or $ki+(k-1)$ will have k distinct and perfectly determinate values of which the first will be $\frac{k-1}{2k}j^2$ and the last $\frac{k-1}{2k}(j+1)^2$.

More general formulæ still may be obtained by supposing

$$j=ki+r, \quad j'=k'i+r',$$

where k, k' are relative prime numbers and r, r' less than k, k' respectively.

Let $P=kk'i+R$, R being less than kk' and congruent to r in respect to modulus k and to r' in respect to modulus k', then if we divide P into harmonic ranges and call fr, $f'r$ the frequencies of r in respect to the two series $1, 2, \ldots j$; $1, 2, \ldots j'$, and call ν the number of the range to which r belongs, and δ, δ' the residues of ν in respect to k, k' respectively, it will be found that $kf'r - k'fr = \delta' - \delta$.

Take as an example $i=20$, $j=41$, $j'=62$, so that $k=2$, $k'=3$, then $P=125$ and for

$$\nu = 0,\ 1,\ 2,\ 3,\ 4,\ 5,$$
$$\delta = 0,\ 1,\ 0,\ 1,\ 0,\ 1,$$
$$\delta' = 0,\ 1,\ 2,\ 0,\ 1,\ 2,$$
$$\delta' - \delta = 0,\ 0,\ 2,\ \bar{1},\ 1,\ 1,$$

the harmonic ranges of P beginning with Range No. 2 will be seen to be

$125-63$, $62-42$, $41-32$, $31-24$, $23-21$, $20-18$, $17-16$, $15-14$, 13, etc.,

and the corresponding frequencies of the numbers in those ranges in regard to the series $1, 2, \ldots 41$, $1, 2, \ldots 62$, will be seen to be

1, 0; 1, 1; 2, 1; 2, 1; 3, 2; 3, 2; 4, 2; 4, 3; ... respectively,

and we have

$$2.1-3.0=2,\ 2.1-3.1=\bar{1},\ 2.2-3.1=1,\ 2.2-3.1=1,\ 2.3-3.2=0,$$
$$2.3-3.2=0,\ 2.4-3.2=2,\ 2.4-3.3=\bar{1}\ \ldots,$$

in which the recurring period is as it ought to be, $2, \bar{1}, 1, 1, 0, 0$.

By means of this division a still wider latitude could be won were it worth while, for the expression of the functional equation to the sum-totient. Another statement and further extensions of the theory are contained in a Note intended for publication in the *Comptes Rendus* of the Institute of France. I may add that I have had a table constructed of the values of Tx for all values of x up to 500 inclusive, and that Tx is always intermediate within this range between $3/\pi^2 x^2$ and $3/\pi^2 (x+1)^2$—a very noteworthy result: and which I have little doubt remains true for all values of x.