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ON THE FUNDAMENTAL THEOREM IN THE NEW METHOD OF PARTITIONS.

[*Johns Hopkins University Circulars*, II. (1883), p. 22.]

THE new method of partitions which I gave to the world more than a quarter of a century ago is an application of a theorem which, I think it must be conceded, is, after Newton's Binomial Theorem, the most important organic theorem which exists in the whole range of the Old Algebra. What Newton's theorem effects for the development of *radical*—that theorem accomplishes for the development of *fractional* forms of algebraical functions.

One (but not the most perfect) form in which it can be presented is the following. If Fx be any proper algebraical fraction in x, whose infinity roots (that is, the values of x which make Fx infinite) are $a, b, \dots l$, quantities all supposed to differ from zero, then the coefficient of x^n for any value of n will be the residue, that is, the coefficient of $\frac{1}{x}$ in

$$\Sigma\,(\lambda^{-n}e^{nx})\,F(\lambda e^{-x}) \quad [\lambda = a, b, \dots, l].$$

By supposing Fx broken up into proper simple fractions of the form $\Sigma\,\frac{fx}{(a-x)^i}$ it is very easy to see that the theorem will be true in general if true for $\frac{fx}{(a-x)^i}$, and from this it is but a step to see that the theorem will be true in general if true for the simplest form of rational function, that is, $\frac{1}{(1-x)^i}$.

All then that remains to do is to show that the coefficient of x^n in this fraction is the same as the coefficient of $\frac{1}{x}$ in $\frac{e^{nx}}{(1-e^{-x})^i}$ which may be done as follows:

$$\frac{1}{(1-e^{-x})^i} = \left(1 - \frac{\delta_x}{1}\right)\left(1 - \frac{\delta_x}{2}\right)\left(1 - \frac{\delta_x}{3}\right) \dots \left(1 - \frac{\delta_x}{(i-1)}\right)\left(\frac{1}{1-e^{-x}}\right)$$

$$= (1 - A\delta_x + B\delta_x^2 - C\delta_x^3 \dots)\left(\frac{1}{x} + \dots\right)$$

$$= \left(\frac{1}{x} + \frac{A}{x^2} + \frac{1 \,.\, 2B}{x^3} + \frac{1 \,.\, 2 \,.\, 3C}{x^4} + \dots\right) + \text{positive powers of } x.$$

Therefore the coefficient of $\frac{1}{x}$ in

$$\frac{1 + nx + \frac{n^2}{1 \,.\, 2} x^2 + \frac{n^3}{1 \,.\, 2 \,.\, 3} x^3 + \dots}{(1-e^{-x})^i}$$

$$= (1 + An + Bn^2 + Cn^3 + \dots)$$

$$= \left(1 + \frac{n}{1}\right)\left(1 + \frac{n}{2}\right)\left(1 + \frac{n}{3}\right) \dots \left(1 + \frac{n}{i-1}\right)$$

$$= \frac{(n+1)(n+2) \dots (n+i-1)}{1,\, 2,\, \dots\, (i-1)} = \text{coefficient of } x^n \text{ in } \frac{1}{(1-x)^i}. \quad \text{Q.E.D.}$$

This method of proof, however, is not the simplest or best; as soon as we mould the theorem into a form most easily admitting of being expressed in general terms that very form itself suggests a simpler (nay, so to say, an instantaneous) proof, and moreover relaxes an unnecessarily stringent condition in the previous statement of the theorem.

Of course by a finite infinity root of a function no one can fail to understand a value of the variable differing from zero which makes the function infinite. This then is the true statement of the theorem in general terms.

In any proper-fractional function developed in ascending powers of a variable, the constant term is equal to the Residue (with its sign changed) of a sum of functions obtained by substituting in the given function in place of the variable the product of each, in succession, of its finite infinity roots into the exponential of the variable.

That is to say, if we take the proper-fraction

$$Fx = \frac{\phi x}{x^i (x-a)^j (x-b)^k \dots (x-l)^\omega},$$

the constant term (with its sign changed) in this fraction developed in ascending powers of x is the same as the Residue of $\Sigma F(\lambda e^x)\,[\lambda = a, b, \dots\, l]$.

To prove this it is only necessary to suppose the fraction Fx separated into simple partial fractions with constant numerators and the theorem becomes self-evident*.

* It must, however, previously be shown that the residue of $\frac{1}{(1-e^x)^i}$, where i is a positive

It follows, therefore, writing n in place of i that the coefficient of x^n in ascending-power series for the fraction

$$Gx = \left(\frac{\phi x}{(x-a)^j \ldots (x-l)^\omega}\right)$$

will be the Residue with its sign changed of $\Sigma\,(a^{-n}e^{-nx})\,G\,(ae^x)$, or which is the same thing is the Residue of $\Sigma a^{-n}e^{nx}\,G\,(ae^{-x})$, which theorem we now see is true not merely for the case where G is a proper-fraction, that is, is a function of x whose degree is a negative integer, but remains true when the degree of G is any number inferior to n, for when that condition is satisfied $\frac{G}{x^n}$ is a proper fraction, which is all that is required in order for the parent theorem to apply.

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integer, is the same as that of $\frac{1}{1-e^x}$, that is, is -1; this becomes obvious from the consideration that the change of i into $i+1$ alters the quantity to be residuated by $\frac{e^x}{(1-e^x)^{i+1}}$, that is, by the differential derivative of $\frac{1}{(1-e^x)^i}$ divided by i, of which the residue is necessarily zero—that being true for the differential derivative of any series of powers of a variable.