77.

ON MECHANICAL INVOLUTION.

[*Johns Hopkins University Circulars*, I. (1882), pp. 242, 243.]

MANY years ago I gave in the *Comptes Rendus* of the Institute of France, one or more geometrical constructions of the problem of Mechanical Involution.

When forces can be introduced along six given lines in space whose statical sum is zero, a certain geometrical condition must be fulfilled by the 6 lines which are then said to be in involution. If two homographic pencils of rays in different planes have two corresponding rays coincident (but their centres apart), any six lines, each of which cuts two corresponding rays, will form an involution system. In the communication to the Society I showed that the analytical condition of involution might be expressed by means of equating to zero a certain compound determinant. I have found since that this determinant is given by Cayley in the *Cam. Phil. Soc. Tr.* 1861, part 2.

Let 1, 2, 3, 4, 5, 6 be the six lines and on each of them let two arbitrary points be taken; let the quadri-planar coordinates of the two arbitrary points on any of the lines, say j, be called j_x, j_y, j_z, j_t; j_x', j_y', j_z', j_t', respectively, the condition of involution referred to will be

$$\begin{vmatrix} & 1.2 & 1.3 & 1.4 & 1.5 & 1.6 \\ 2.1 & & 2.3 & 2.4 & 2.5 & 2.6 \\ 3.1 & 3.2 & & 3.4 & 3.5 & 3.6 \\ 4.1 & 4.2 & 4.3 & & 4.5 & 4.6 \\ 5.1 & 5.2 & 5.3 & 5.4 & & 5.6 \\ 6.1 & 6.2 & 6.3 & 6.4 & 6.5 & \end{vmatrix} = 0$$

where any binary combination $ij = ji$, and where either of them represents the determinant

$$\begin{vmatrix} i_x, & i_y, & i_z, & i_t \\ i_x', & i_y', & i_z', & i_t' \\ j_x, & j_y, & j_z, & j_t \\ j_x', & j_y', & j_z', & j_t' \end{vmatrix}$$

Six lines in involution represent indifferently lines along which forces or axes of couples can be introduced, whose statical sum is zero. Consequently such a system is the analogue in space at one and the same time to three force-lines converging to a point, or to three points in a line regarded as centres of moments, in a plane. But *in plano* the concurrence of right lines is the polar property to the collineation of points. Hence we ought to expect that the polar reciprocal in respect to any quadric of an involution system, should also be an involution system; and such is obviously the case by virtue of the fact that the correspondence of the rays in the two homographic pencils, referred to above, will not be affected when for each ray in either pencil is substituted its polar in respect to any quadric. (A direct proof will be found in the *American Mathematical Journal*, Vol. IV., part 4 *.) I concluded with pointing out the analogy between the problem of Mechanical Involution and what I call Algebraical Involution, which takes place when x, y being each of them matrices of the order ω, a linear equation connects the ω^2 ground-forms represented by the distinct terms of the product

$$(1, x, x^2, \dots x^{\omega-1} \between 1, y, y^2, \dots y^{\omega-1}).$$

Mechanical involution in a plane, in 3-dimensional, in 4-dimensional space, etc., is the analogue of algebraical involution between two matrices of the order 2, 3, 4, etc.; the $\frac{1}{2}(\omega^2+\omega)$ directions in ω-dimensional space being the analogues of the ω^2 ground-forms of matrices of the order ω. Each of the two problems consists of two parts: to obtain the condition of involution being the one part, to assign the relative magnitudes, in the one case, of the forces which cause their statical sum to vanish, and in the other case of the coefficients which enter into the linear function, the other part of the problem. The form of the solution of this second part of the algebraical problem (subject only to a certain ambiguity) has been given in my lectures, and will appear in the Memoir on Multiple Algebra in the *American Journal of Mathematics*; but the former part of the algebraical problem, that is, the determination of the *condition* of Algebraical Involution, except for the case of matrices of the second order, I have not yet succeeded in solving.

[* Cf. p. 560, above.]