64.

NOTE ON MECHANICAL INVOLUTION.

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MECHANICAL involution is the name invented by me to signify the relation between six lines in space, so situated that forces may be made to act along them whose statical sum is zero. The definition may be extended to comprise an indefinite number of lines, any six of which have this property.

I shall use $[p, q]$ for the present to denote the moment of a unit of force acting along the directed line p about the directed line q, taken positive or negative according as to a spectator looking in the given direction (or sense) of q, a force in the given direction (or sense) of p tends to produce a right-handed or a left-handed rotation, which tendency, by a property of our mental constitution, we know is not affected in kind by the lines p and q becoming interchanged—a fact which might also be anticipated with a high degree of probability from the circumstance that the unit-moment is measured by the product of the perpendicular distance from each other, of the two lines, multiplied by the sine of the angle between them, so that *each* factor of this product changes its sign when the relation or aspect of the two lines to each other is reversed. Hence it follows that $[p, q] = [q, p]$.

Three lines in a plane, it may be noticed, are in involution when they intersect in the same point, or, as a particular case, are parallel to each other.

Let a, b, c, d, e, f be any six lines in space, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ six forces capable of balancing when acting along the lines 1, 2, 3, 4, 5, 6 supposed to be in involution.

Then by the equation of moments in regard to each of the first series of lines taken successively as axes of rotation, we must have

$$\lambda_1[1, a] + \lambda_2[2, a] + \lambda_3[3, a] + \lambda_4[4, a] + \lambda_5[5, a] + \lambda_6[6, a] = 0$$
$$\lambda_1[1, b] + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \lambda_6[6, b] = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\lambda_1[1, f] + \lambda_2[2, f] + \lambda_3[3, f] + \lambda_4[4, f] + \lambda_5[5, f] + \lambda_6[6, f] = 0$$

and consequently the determinant

$$\begin{vmatrix} [1, a] & \ldots\ldots\ldots\ldots & [6, a] \\ \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \ldots & \ldots\ldots\ldots\ldots & \ldots \\ \ldots & \ldots\ldots\ldots\ldots & \ldots \\ [1, f] & \ldots\ldots\ldots\ldots & [6, f] \end{vmatrix} = 0.$$

Consequently we may find quantities $\mu_a, \mu_b, \mu_c, \mu_d, \mu_e, \mu_f$ such that

$$\mu_a[a, 1] + \mu_b[b, 1] + \mu_c[c, 1] + \mu_d[d, 1] + \mu_e[e, 1] + \mu_f[f, 1] = 0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mu_a[a, 6] + \mu_b[b, 6] + \mu_c[c, 6] + \mu_d[d, 6] + \mu_e[e, 6] + \mu_f[f, 6] = 0.$$

Thus it becomes evident by regarding μ_a, μ_b, μ_c, μ_d, μ_e, μ_f as the magnitudes of forces acting along the lines a, b, c, d, e, f, that the equations of moments of a given set of forces about six lines which are in general independent, become linearly related when the six axes are in involution—a conclusion which springs also immediately from the consideration that the law of statical composition of directed lengths is the same whether they be regarded as representing forces or as representing the axes of couples. So much by way of introduction.

I now pass to the formation of the intrinsic equation of condition to be satisfied in the case of involution.

To obtain this, let the lines a, b, c, d, e, f be made identical with 1, 2, 3, 4, 5, 6.

In each of these latter lines (say in i) let two points be taken at the distance $\dfrac{1}{l_i}$ apart, whose quadriplanar coordinates are respectively i_x, i_y, i_z, i_t, i_x', i_y', i_z', i_t', and let (i, j)—where j is another of the lines in involution—denote the determinant

$$\begin{vmatrix} i_x & i_y & i_z & i_t \\ i_x' & i_y' & i_z' & i_t' \\ j_x & j_y & j_z & j_t \\ j_x' & j_y' & j_z' & j_t' \end{vmatrix}$$

This determinant will represent (enlarged six-fold) a tetrahedron, two of whose opposite edges are the lengths intercepted between the pairs of points on i, j respectively, and consequently $l_i l_j (i, j)$ will serve to represent (on the same scale) the quantities previously represented by $[i, j]$.

Hence the determinant of the sixth order above written becomes

$$\begin{vmatrix} 0\ , & l_1 l_2(1,2), & l_1 l_3(1,3), & l_1 l_4(1,4), & l_1 l_5(1,5), & l_1 l_6(1,6) \\ l_2 l_1(2,1), & 0\ , & l_2 l_3(2,3), & l_2 l_4(2,4), & l_2 l_5(2,5), & l_2 l_6(2,6) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ l_6 l_1(6,1), & l_6 l_2(6,2), & l_6 l_3(6,3), & l_6 l_4(6,4), & l_6 l_5(6,5), & 0 \end{vmatrix}$$

and this equated to zero gives the intrinsic condition of involution.

Imagining this equation to be formed, the terms in each line and also the terms in each column will have some common factor, removing which, by a two-fold scheme of division, all the quantities l will disappear, so that now regarding each of the pairs of points on the lines 1, 2, 3, 4, 5, 6 respectively as *any two non-coincident points whatever*, the intrinsic condition is represented by the evanescence of the following symmetrical invertebrate (that is, zero-axial) compound determinant

$$\begin{vmatrix} 0 & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & 0 & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & 0 & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & 0 & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & 0 & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & 0 \end{vmatrix}$$

where each pair of numbers within a parenthesis represents a determinant of the fourth order*.

Just as the equations of moments of a system of forces about six lines in space are in general independent, but cease to be so if (and only if) these lines are in involution, so the equations of moments of a system of forces in a plane about three points are in general independent, and only cease to be so when the three points lie in a right line. Thus under the two-fold aspect of a system of force-directions and a system of axes of moments, six lines in involution in space are on the one hand the analogues of three force-directions in a plane in involution, that is, meeting in a point, and on the other hand of three points (centres of moments) lying in a right line; and as *concurrence* is the polar correlative to *collineation* we ought to expect to

* This determinant (which is sufficiently obvious, I have found since going to press) has been given by Professor Cayley in his memoir on line-coordinates, *Camb. Phil. Trans.*, 1861, which is avowedly based upon my constructions connected with the problem of Involution.

find involution in space to be its own polar correlative; that is, that the polar reciprocal of a system of lines in involution in respect to a general quadric should be another such system: and such is the fact: for, as I have shown in the *Comptes Rendus**, the necessary and sufficient condition of six lines being in involution is that they shall respectively intersect pairs of corresponding rays in two homographic pencils lying in two planes whose intersection contains the centres and two corresponding (coincident) rays of the two pencils—a condition which will not be affected by any polar transformation.

This leads to the remark that we may change the signification of the symbol (i, j) in the equation last indicated without destroying its validity as the condition of involution: namely, we may suppose two planes to be drawn through each line instead of two points being fixed upon it: and then if we understand by the determinant of two lines in space the determinant formed by the coefficients of the two pairs of equations which denote the lines, we may interpret (i, j) to mean the determinant of i, j and sum up the result obtained in the following proposition:

The determinants formed by six lines in involution, taken two and two together, are related in precisely the same manner as the squared distances from one another of six points in four-dimensional space.

The legitimacy of the second reading of (i, j) may be proved directly, as follows. For greater clearness let (i, j) when read with reference to pairs of planes through i and j, be called (I, J). Then

$$\begin{matrix} i_x & i_y & i_z & i_t \\ i_x' & i_y' & i_z' & i_t' \\ I_x & I_y & I_z & I_t \\ I_x' & I_y' & I_z' & I_t' \end{matrix}$$

will constitute an example of what in the *Johns Hopkins University Circular* for May, 1882†, I have called a *split matrix*, inasmuch as each of the first two lines multiplied term for term by each of the latter two gives

[* Vol. II. of this Reprint, p. 237.]

† Baltimore: John Murphy & Co.—It is interesting to notice (as there indicated) that the same theory of the split matrix here applied to mechanical involution has an important, although quite a different kind of bearing on the theory of algebraical involution. The two theories of involution have a considerable affinity to each other—groundforms and their coefficients in the equation of linear connection in the one theory, being regarded as the analogues of space-directions and the force-magnitudes acting along them in the other. (See *J. H. U. Circular*, June, 1882.) It was the sense of this connection which caused me to throw a retrospective glance on the theory of mechanical involution, abandoned by me since the remote date of the appearance of my papers on the subject in the *Comptes Rendus*. I ought to mention that I owe the idea of applying the split-matrix theory to the proof of the polar property of an involution-system, to a suggestion of Professor Cayley.

products whose sum is zero. Hence by virtue of the property of such a matrix, each complete minor of the upper pair will bear to the opposite complete minor in the lower pair the ratio of (i) to (I), where

$$(i)^2 = \begin{vmatrix} \Sigma i_x^2 & \Sigma i_x i_x' \\ \Sigma i_x i_x' & \Sigma i_x'^2 \end{vmatrix} \text{ and } (I)^2 = \begin{vmatrix} \Sigma I_x^2 & \Sigma I_x I_x' \\ \Sigma I_x I_x' & \Sigma I_x'^2 \end{vmatrix},$$

and of course the same conclusions apply *mutatis mutandis* when j, J take the place of i, I; from which it immediately follows that

$$(i, j) : (I, J) = (i)(j) : (I)(J).$$

Let now in the (i, j) determinant, which is equated to zero, each element in any θth column be multiplied by $\frac{I_\theta}{i_\theta}$, and then again each element in any θth row by the same; these multiplications will not affect the equality to zero of the determinant so modified, but the effect of the combined multiplications will be to change the element in the ith row and jth column, namely, (i, j), into $\frac{(I)(J)}{(i)(j)}(i, j)$, that is into (I, J). Thus it is proved that we may pass from the first reading of the (i, j) determinant to the second; and this in its turn serves to prove that if six lines are in involution their polars in respect to any quadric must also be in involution.

The theory of involution may of course be extended to a system of $\frac{n(n+1)}{2}$ lines in n-dimensional space.