63.

NOTE ON THE THEORY OF SIMULTANEOUS LINEAR DIFFERENTIAL OR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS.

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THIS theory is virtually the same for differential as for finite-difference equations. The mere verbal part of the exposition being somewhat easier for the former of the two, I shall prefer in the first instance to deal with them, although the applications are more interesting when made to bear on the latter. Simple to the last degree as are the method of solution and the nature of the result, I do not find the one or the other set out, or even indicated, except in the most perfunctory manner, in the ordinary text-books. This brief notice, designed for the junior readers of the *Journal*, is intended to supply the lacuna.

Let $u_{j,k}$ denote a linear function, with constant coefficients, of ω_k and of its first ϵ_j derivatives in respect to t.

Let

$$\begin{gathered} u_{1,1} + u_{1,2} + \ldots + u_{1,i} = 0, \\ u_{2,1} + u_{2,2} + \ldots + u_{2,i} = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ u_{i,1} + u_{i,2} + \ldots + u_{i,i} = 0, \end{gathered}$$

be the system of differential equations proposed for integration.

Call

$$\epsilon_1 + \epsilon_2 + \ldots + \epsilon_i = \sigma.$$

The process of arriving at the reducing equation for any one of the variables is after the manner of the dialytic method of elimination, namely:

Along with the first equation take each of its $(\sigma - \epsilon_1)$th derivatives, with the second equation each of its $(\sigma - \epsilon_2)$th derivatives, ... and with the ith equation each of its $(\sigma - \epsilon_i)$th derivatives.

There will thus come into existence $(\sigma + 1) i - (\epsilon_1 + \epsilon_2 + \dots + \epsilon_i)$, that is, $i(\sigma + 1) - \sigma$ equations between the $i(\sigma + 1)$ quantities

$$\begin{array}{l} \omega_1, \quad \delta_t \omega_1, \dots \delta_t^{\sigma} \omega_1, \\ \omega_2, \quad \delta_t \omega_2, \dots \delta_t^{\sigma} \omega_2, \\ \dots\dots\dots\dots\dots\dots \\ \omega_i, \quad \delta_t \omega_i, \dots \delta_t^{\sigma} \omega_i. \end{array}$$

If we omit those which appear in any one of the lines above written, there will remain $(\sigma + 1)(i - 1)$ or $i(\sigma + 1) - \sigma - 1$ which might be eliminated between the $i(\sigma + 1) - \sigma$ equations, and there would thus result an equation between the quantities contained in the omitted line. This elimination, it will presently be seen, there is no occasion to perform; the noticeable algebraical fact about it is, that supposing it were performed, the form of the equation resulting between ω_k, $\delta_t \omega_k$, ... $\delta_t^{\sigma} \omega_k$ is invariable, whichever of the numbers 1, 2, 3, ... i be the value assigned to k.

Let the order of the highest derivative of each ω be reduced by one unit below the highest order previously taken, then there will be $i\sigma - \sigma$ or $(i - 1)\sigma$ equations connecting the $i\sigma$ quantities

$$\begin{array}{l} \omega_1, \quad \delta_t \omega_1, \dots \delta_t^{\sigma-1} \omega_1, \\ \omega_2, \quad \delta_t \omega_2, \dots \delta_t^{\sigma-1} \omega_2, \\ \dots\dots\dots\dots\dots\dots \\ \omega_i, \quad \delta_t \omega_i, \dots \delta_t^{\sigma-1} \omega_i, \end{array}$$

and accordingly, if we omit the σ quantities which appear in any one (say the first) of the above lines, the remaining $(i - 1)\sigma$ quantities may each of them be expressed as linear functions of ω_1 and its $(\sigma - 1)$ derivatives: but the elimination previously indicated would lead to a homogeneous linear equation between ω_1 and its σ derivatives, and if in that, each argument $\delta_t^{\lambda} \omega_1$ be replaced by h^{λ} and $\lambda_1, \lambda_2, \dots \lambda_{\sigma}$ be the σ roots of the algebraical equation so formed, it follows from the ordinary theory for a single equation that ω_1 (provided the given equations, and consequently the resulting ones, be left in their general form) will be of the form

$${}^1C_1 e^{h_1 t} + {}^1C_2 e^{h_2 t} + \dots + {}^1C_{\sigma} e^{h_{\sigma} t},$$

and consequently by virtue of the previous remark $\omega_2, \omega_3, \dots \omega_k$ will be of the same form as ω_1 (but, of course, with different coefficients), that is to say, the σ roots $h_1, h_2, \dots h_{\sigma}$ are the same for the equation in σ_k as for the equation in σ_1, so that the coefficients in the equation between ω_k and its σ derivatives are, as premised, independent of the value of k.

Finally, to determine the equation whose roots are $h_1, h_2, \ldots h_\sigma$, let ${}^1Ce^{ht}$, one of the terms in the general value, be taken as a particular value of ω_1, which with corresponding values of the other ω's will serve to satisfy the given equations; $\omega_2, \omega_3, \ldots \omega_i$ being each of them linear functions of ω and derivatives of ω, must be of the forms ${}^2Ce^{ht}$, ${}^3Ce^{ht}, \ldots {}^iCe^{ht}$, so that $\omega_1, \omega_2, \ldots \omega_i$ and the derivatives of each of them will contain the common factor e^{ht}, and by substitution in the original equations we shall obtain a system of simultaneous algebraical equations leading to the equation

$$\begin{vmatrix} R_{1,1}, & R_{1,2} \ldots R_{1,i} \\ R_{2,1}, & R_{2,2} \ldots R_{2,i} \\ \ldots & \ldots \\ R_{i,1}, & R_{i,2} \ldots R_{i,i} \end{vmatrix} = 0,$$

where in general $R_{p,q}$ is what $u_{p,q}$ becomes on writing h^μ in place of $\delta_t{}^\mu \omega_q$.

The above determinant of the ith order will be of degree $\epsilon_1 + \epsilon_2 + \ldots + \epsilon_i$, that is, of the degree σ (for the general case) in h, and the roots of the equation will give the σ values $h_1, h_2, \ldots h_\sigma$.

It follows, therefore, that the result of the hypothetical elimination in the first instance referred to will be a linear function of $\delta_t{}^\sigma \omega_k, \delta_t{}^{\sigma-1}\omega_k, \ldots \delta_t \omega_k, \omega_k$ of which the coefficients will be identical with the coefficients of $h^\sigma, h^{\sigma-1}, \ldots h, 1$ in the above determinant. Hence no matter now what special values may be attributed to the coefficients of the given equations, the result last obtained remains of *universal* validity—without excepting those cases in which the result of the hypothetical elimination would be such that the corresponding algebraical equation possess equal roots, although in those cases the form assumed in the course of the argument for the value of ω_1 (namely, a linear function of exponentials) ceases to hold good. Neither for the same reason need any exception be made for those cases where the number of terms in the equation to ω_k falls below σ on account of one or more of the leading coefficients in the result of the hypothetical elimination becoming zero: the degree to which h rises in the determinant will be in all cases the right degree, whether it reaches the extreme possible limit σ or falls below it.

The result obtained may be briefly summarized as follows.

If
$$(\phi_1 \delta_t)\, x + (\phi_2 \delta_t)\, y + \ldots + (\phi_i \delta_t)\, z = 0,$$
$$(\psi_1 \delta_t)\, x + (\psi_2 \delta_t)\, y + \ldots + (\psi_i \delta_t)\, z = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(\omega_1 \delta_t)\, x + (\omega_2 \delta_t)\, y + \ldots + (\omega_i \delta_t)\, z = 0,$$

(each $\phi, \psi, \ldots, \omega$ standing for a rational-integral functional form) then will

$$(R\delta_t)\, x = 0, \quad (R\delta_t)\, y = 0, \ldots (R\delta_t)\, z = 0,$$

where $R(\delta_t)$ is the resultant in respect to $x, y, \dots z$ of what the above equations become when δ_t is treated as an ordinary algebraical quantity; under which form the proposition (by virtue of Euler's method of multipliers) becomes so nearly intuitive as to abrogate all necessity for any other demonstration*.

To pass to the parallel and more important theory in finite differences, it is only necessary to interpret $u_{j,k}$ to signify a linear function, with constant coefficients, of $(\omega_k)_t, (\omega_k)_{t+1}, \dots (\omega_k)_{t+\epsilon_j}$, where t is the integer independent variable, (*say* $(\omega_k)_t$ and its ϵ_j difference-augmentatives), and instead of taking the differential derivatives of any one of the given equations, to take the corresponding difference-augmentatives. Then by precisely the same reasoning as before we shall have

$$\omega_{t+\sigma} + B\omega_{t+\sigma-1} + \dots + L\omega_t = 0,$$

$B, C, \dots L$ being so taken as that $h^\sigma + Bh^{\sigma-1} + \dots + L$ shall be the determinant represented by the same form of matrix expressed by R's as before, but where $R_{p,q}$ is obtained from $u_{p,q}$ by writing h^θ in lieu of any argument $\omega_t + \theta$ which occurs in it.

The simplest example that can be given is where $i = 2$, $\epsilon_1 = \epsilon_2 = 1$,

$$u_{1,1} = -\eta_{t+1} + a\eta_t, \qquad u_{1,2} = b\theta_t,$$

$$u_{2,1} = c\eta_t \qquad\qquad u_{2,2} = -\theta_{t+1} + d\theta_t;$$

this was the case which occurred in the article on the extension of Tchebycheff's theorem, in the last number of the *Journal* [p. 530, above], leading to the equation

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0,$$

and to expressions for η_t, θ_t as linear functions of λ_1^t, λ_2^t.

It may also be remarked that this same case gives an instantaneous solution of the problem proposed and successfully treated by Babbage in his *Calculus of Functions*, more than half a century ago, and since revived in connection with the theory of substitutions (Serret, *Alg. Sup.* 4 ed., tom. 2, pp. 256—262). The problem is to find $\phi x = \dfrac{ax + \alpha}{\beta x + b}$ so that $\phi^i x$, say $\dfrac{a_i x + \alpha_i}{\beta_i x + b_i}$, shall equal x for a given value of i.

* I regret that this simple reflection did not present itself to my mind before the preceding investigation, the necessity for which it does away with, had been set up in print. It of course applies equally well to the analogous proposition for finite-difference equations ($u_i, v_i, \dots$ being substituted for $x, y \dots$, and $1+\Delta$ for δ_t). This last named proposition, limited to the case of equations of the first order, is the foundation-stone of my new theory of Matrices regarded as Quantities, that is, as subject to every kind of functional operation which ordinary arithmetical or algebraical quantities are or can be subject to: but though so important and so easily established, I know not where it can be found explicitly stated.

To find in general $\phi^i x$ it is only necessary to solve the difference equations

$$u_i = au_{i-1} + \alpha v_{i-1},$$
$$v_i = \beta u_{i-1} + bv_{i-1},$$

and then u_i, v_i will, if $u_0 = 1$, $v_0 = 0$, coincide with a_i, β_i, and if $u_0 = 0$, $v_0 = 1$ with α_i, b_i.

Thus calling ρ_1, ρ_2 the two roots of

$$\begin{vmatrix} -\rho + a & \alpha \\ \beta & -\rho + b \end{vmatrix} = 0,$$

α_i will be of the form $C(\rho_1^i - \rho_2^i)$ and β_i of the same form except as to C, say $\Gamma(\rho_1^i - \rho_2^i)$. Also a_i, b_i will be of the forms $C_1\rho_1^i + C_2\rho_2^i$, $\Gamma_1\rho_1^i + \Gamma_2\rho_2^i$, where $C_1 + C_2 = 1$, $\Gamma_1 + \Gamma_2 = 1$, and the required condition will be fulfilled, provided only that $\rho_1^i = \rho_2^i$, or say

$$\rho_1 = K\left(\cos\frac{\lambda\pi}{i} + \surd(-1)\sin\frac{\lambda\pi}{i}\right)$$
$$\rho_2 = K\left(\cos\frac{\lambda\pi}{i} - \surd(-1)\sin\frac{\lambda\pi}{i}\right)$$

that is, if $(a+b)^2 - 4(ab - \alpha\beta)\left(\cos\frac{\lambda\pi}{i}\right)^2 = 1$, λ having any integer value (which without loss of generality may be taken inferior to i) except zero*.

If $\lambda = 0$, the two roots of the equation in ρ become equal and the form of the solution changes into

$$u_i = (C_1 + C_2 i)\rho^i, \qquad v_i = (C_1' + C_2' i)\rho^i.$$

When $u_0 = 1$ and $v_0 = 0$ then $u_1 = a$, $v_0 = \beta$,

$$C_1 = 1,\ C_1' = 0, \qquad C_2 = \frac{a}{\rho} - 1,\ C_2' = \frac{\beta}{\rho},$$

and when $u_0 = 0$, $v_0 = 1$, $u_1 = \alpha$, $v_0 = b$,

$$C_1 = \frac{\alpha}{\rho},\ C_1' = \frac{b}{\rho} - 1, \qquad C_2 = 0,\ C_2' = 1,$$

and $\phi^i x = \dfrac{[\rho + i(a-\rho)]\,x + i\alpha}{i\beta x + \rho + (b-\rho)\,i}$, which cannot be periodic for any value of i, and when $i = \infty$ becomes

$$\frac{(a-\rho)x + \alpha}{\beta x + b - \rho} = \frac{a-\rho}{\beta} = \frac{\alpha}{b-\rho}, \text{ that is, } = \frac{a-b}{2\beta} \text{ or } \frac{2\alpha}{a-b},$$

so that $\phi^i x$ in this case continually converges to a constant limit.

I may add that $\phi^i x$ converges to a constant limit not merely when the roots ρ_1, ρ_2 of

$$\begin{vmatrix} a - \rho & \alpha \\ \beta & b - \rho \end{vmatrix}$$

* There will thus be $(i-1)$ values of λ which will each give a distinct admissible solution of the problem of periodicity, but of course only those values of λ which are relatively prime to i will give primitive solutions. If $i = i'\delta$ the effect of making $\lambda = \lambda'\delta$ will be to make $\phi^i x = x$ by virtue o its making $\phi^{i'} x = 0$.

are equal, but whenever they are real. For the general form of $\phi^i x$, it may easily be found, is

$$\frac{[(\rho_2 - a)\,\rho_1{}^i + (\rho_1 - a)\,\rho_2{}^i]\,x + \alpha\,(\rho_1{}^i - \rho_2{}^i)}{\beta\,(\rho_1{}^i - \rho_2{}^i)\,x + [(\rho_2 - b)\,\rho_1{}^i + (\rho_1 - b)\,\rho_2{}^i]},$$

which if $\rho_2 > \rho_1$ when $i = \infty$ becomes $\dfrac{(\rho_1 - a)\,x - \alpha}{-\beta x + \rho_1 - b} = \dfrac{a - \rho_1}{\beta}$ or $\dfrac{\alpha}{b - \rho_1}$ where ρ_1 signifies the smaller of the two roots ρ_1, ρ_2; or in other words when $a - b > 2\sqrt{(\alpha\beta)}$, the limiting value to $\phi^i x$, when ϕx represents $\dfrac{ax + \alpha}{\beta x + b}$, is $\dfrac{(a - b) + \sqrt{\{(a - b)^2 - 4\alpha\beta\}}}{2\beta}$, with the understanding that the quantity under the radical sign is to be taken positive.

So, if

$$x_{i+1} : y_{i+1} : z_{i+1} = ax_i + by_i + cz_i : a'x_i + b'y_i + c'z_i : a''x_i + b''y_i + c''z_i,$$

when all the roots of the determinant

$$\begin{vmatrix} a - \lambda & b & c \\ a' & b' - \lambda & c' \\ a'' & b'' & c'' - \lambda \end{vmatrix}$$

are real, the point x_i, y_i, z_i, as i increases, will be found to approach indefinitely near to a fixed straight line; and if all the roots are equal, to a fixed point.

The condition of the system of ratios $x_i : y_i : z_i$ being periodic and having a period m is tantamount to the condition that the mth power of the matrix

$$\begin{matrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{matrix}$$

shall be the matrix

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1. \end{matrix}$$

The complete solution of this problem, and of the more general one of extracting the mth root of any unit-matrix (that is, a matrix in which each element in the principal diagonal is unity, and the rest zero), which constitutes the ultimate generalization of Babbage's problem and is soluble by the same method, will probably appear in a memoir on matrices, in the forthcoming number of the *Journal*.

In general, for a matrix of the order ω, the number of mth roots is m^ω and each of them is perfectly determinate. But when the matrix is a unit-matrix or a zero-matrix (the latter meaning one in which every element is zero) there are distinct genera and species of such roots, and every species contains its own appropriate number of arbitrary constants.