

CHAPTER XXII.

SURFACES AND VOLUMES IN GENERAL, AND THEIR CENTROIDS, ETC. DOUBLE AND TRIPLE INTEGRATION.

760. Let the equation of a surface be $\phi(x, y, z)=0$ referred to three mutually perpendicular coordinate axes Ox, Oy, Oz . Let us discuss the volume contained between the boundaries

$$z=0, \phi(x, y, z)=0; \quad y=0, y=F(x); \quad x=0, x=a.$$

Let planes $X=x, X=x+\delta x,$
 $Y=y, Y=y+\delta y,$
 $Z=z, Z=z+\delta z,$

be drawn.

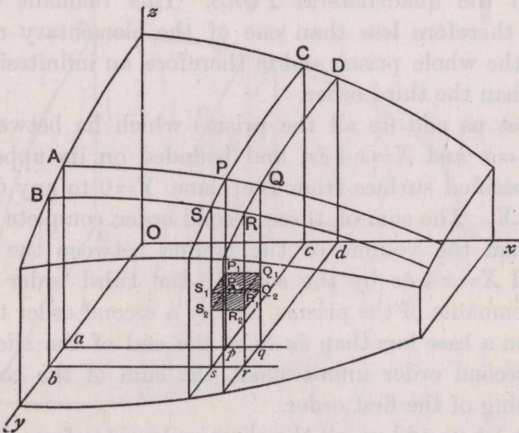


Fig. 262.

Planes $X=x, X=x+\delta x$ intercept between them a thin slice or lamina of thickness δx .

Planes $Y=y$, $Y=y+\delta y$ cut from this lamina a prism or tube on rectangular base $\delta x \delta y$.

Planes $Z=z$, $Z=z+\delta z$ cut from this prism an elementary rectangular box or "cuboid" of volume $\delta x \delta y \delta z$, represented in the figure as $P_1Q_1R_1S_1P_2Q_2R_2S_2$. Regarding δx , δy , δz as infinitesimals of the first order, the volume of the slice is a first order infinitesimal, the volume of the prism is a second order infinitesimal, and the volume of the cuboid is a third order infinitesimal. Let the prism intercept on the surface a curvilinear quadrilateral figure $PQRS$, and on the plane $x-y$ the elementary rectangle $pqrs$, viz. $\delta x \delta y$. These areas are both infinitesimals of the second order.

If we add up all the complete cuboids on base $\delta x \delta y$ from $z=0$ to z =the smallest of the values of z of the surface within the quadrilateral $PQRS$, we get the volume of the prism, less by a third order infinitesimal, viz. the portion of a cuboid bounded by a base $\delta x \delta y$ for its lower surface, by the curvilinear quadrilateral $PQRS$ for its upper surface, and by four plane faces parallel to the $y-z$ or $z-x$ planes. We may regard the infinitesimal δz as having been taken not less than the difference of the greatest and the least values of z for points on the quadrilateral $PQRS$. This remnant of the prism is therefore less than one of the elementary cuboids forming the whole prism, and is therefore an infinitesimal of not less than the third order.

Next let us add up all the prisms which lie between the planes $X=x$ and $X=x+\delta x$, and bounded on its upper side by the specified surface from the plane $Y=0$ to any definite value of Y . The sum of these second order complete prisms differs from the volume of the lamina between the planes $X=x$ and $X=x+\delta x$ by the sum of the third order infinitesimal remnants of the prisms, and by a second order tubular element on a base less than $\delta x \delta y$ at the end of the slice, that is by a second order infinitesimal, the sum of the complete prisms being of the first order.

Finally, let us add up all the slices or laminae from $X=0$ to any definite value of X . The sum of the portions of these laminae made up of complete prisms is a finite quantity. The sum of the remnants of the laminae is the sum of a set

of second order infinitesimals, and forms a first order infinitesimal. Hence it appears that the sum of all the complete cuboids within the figure bounded by the coordinate planes, the planes $X=x_1$, $Y=y_1$, say, and the surface, differs from the whole volume of that figure by a first order infinitesimal at most, and in the limit when δx , δy , δz are diminished without limit, we have the volume given by

$$V = \iiint dx dy dz.$$

The limits for z are from $z=0$ to z =the value found from $\phi(x, y, z)=0$ in terms of x and y , say $z=f(x, y)$.

The limits for y will be from $y=0$ to the value of y specified in any particular manner, say $y=F(x)$.

The limits for x will be such as to go from $x=0$ to $x=a$.

761. Ex. Consider the volume of an octant of an ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Here the limits for z are $z=0$ to $z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$ for the elementary prism, to add up all the cuboids in the prism.

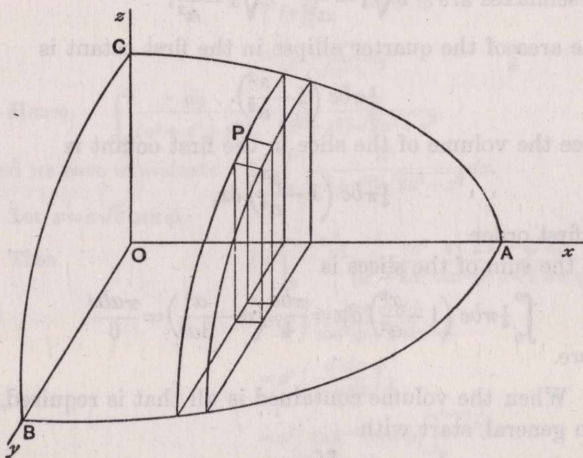


Fig. 263.

For y ; $y=0$ to $y=b\sqrt{1-\frac{x^2}{a^2}}$ for the slice, to add up all the prisms in the slice.

For x ; from $x=0$ to $x=a$, to add up all the slices.

$$\text{And } V = \int_0^a \int_0^b \sqrt{1 - \frac{x^2}{a^2}} \int_0^c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dx dy dz = \int_0^a \int_0^b \sqrt{1 - \frac{x^2}{a^2}} [z] dx dy,$$

and taking $[z]$ between its limits, this integral

$$= c \int_0^a \int_0^b \sqrt{1 - \frac{x^2}{a^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dx dy. \text{ Write } \frac{\eta^2}{b^2} \text{ for } 1 - \frac{x^2}{a^2}.$$

$$\text{Now } \int_0^\eta \sqrt{\frac{\eta^2}{b^2} - \frac{y^2}{b^2}} dy = \frac{1}{b} \left[\eta \frac{\sqrt{\eta^2 - y^2}}{2} + \frac{\eta^2}{2} \sin^{-1} \frac{y}{\eta} \right]_0^\eta = \frac{\eta^2}{2b} \cdot \frac{\pi}{2};$$

$$\begin{aligned} \therefore V &= \frac{c}{b} \frac{\pi}{4} \int_0^a b^2 \left(1 - \frac{x^2}{a^2}\right) dx \\ &= \frac{c}{b} \cdot \frac{\pi}{4} b^2 \left(a - \frac{a^3}{3a^2}\right) = \frac{c}{b} \cdot \frac{\pi}{4} \cdot b^2 \frac{2a}{3} = \frac{\pi abc}{6}. \end{aligned}$$

And the volume of the whole ellipsoid is $8V = \frac{4}{3}\pi abc$.

762. Obviously in cases where the volume of a slice can be written down at once, the labour of computation may be saved.

In the case just considered, for instance, the section at distance $X=x$ from the plane of yz is an ellipse, viz.

$$\frac{y^2}{b^2 \left(1 - \frac{x^2}{a^2}\right)} + \frac{z^2}{c^2 \left(1 - \frac{x^2}{a^2}\right)} = 1,$$

whose semiaxes are $b\sqrt{1 - \frac{x^2}{a^2}}$, $c\sqrt{1 - \frac{x^2}{a^2}}$;

and the area of the quarter ellipse in the first octant is

$$\frac{1}{4}\pi bc \left(1 - \frac{x^2}{a^2}\right).$$

Hence the volume of the slice in the first octant is

$$\frac{1}{4}\pi bc \left(1 - \frac{x^2}{a^2}\right) \delta x,$$

to the first order.

And the sum of the slices is

$$\int_0^a \frac{1}{4}\pi bc \left(1 - \frac{x^2}{a^2}\right) dx = \frac{\pi bc}{4} \left(a - \frac{a^3}{3a^2}\right) = \frac{\pi abc}{6},$$

as before.

763. When the volume contained is all that is required, we may, in general, start with

$$V = \iiint z dx dy,$$

i.e. we may use the elementary prism on $\delta x \delta y$ for base as our element of volume. This amounts of course to integrating with

regard to z in the triple integral formula $\iiint dx dy dz$ between limits $z=0$ and z =the ordinate of the surface under consideration.

If the upper surface of the region whose volume is required is $z=f_1(x, y)$, and the lower surface be $z=f_2(x, y)$, instead of $z=0$, as taken in Art. 760, we have

$$V = \iiint \{f_1(x, y) - f_2(x, y)\} dx dy.$$

764. Illustrative Examples.

1. The curve $z(a^2+x^2)^{\frac{3}{2}}=a^4$ lying in the plane $z-x$ revolves about the axis of z . Find the volume in the positive octant included between this surface and the planes $x=0, x=a, y=0, y=a$. [COLLEGES ϵ , 1883.]

The equation of the surface generated is

$$z = \frac{a^4}{(a^2+x^2+y^2)^{\frac{3}{2}}},$$

and $V = \int \int z dx dy = a^4 \int_0^a \int_0^a \frac{dx dy}{(a^2+x^2+y^2)^{\frac{3}{2}}}$. Write b^2 for a^2+x^2 .

Then
$$\int_0^a \frac{dy}{(b^2+y^2)^{\frac{3}{2}}} = \int_0^{\tan^{-1} \frac{a}{b}} \frac{b \sec^2 \theta d\theta}{b^3 \sec^3 \theta}, \text{ where } y = b \tan \theta,$$

$$= \frac{1}{b^2} \int_0^{\tan^{-1} \frac{a}{b}} \cos \theta d\theta$$

$$= \frac{1}{b^2} \cdot \frac{a}{\sqrt{a^2+b^2}}.$$

Hence
$$\int_0^a \frac{dy}{(a^2+x^2+y^2)^{\frac{3}{2}}} = \frac{a}{(a^2+x^2)\sqrt{2a^2+x^2}},$$

and we have to evaluate $I = \int_0^a \frac{a^5}{(a^2+x^2)\sqrt{2a^2+x^2}} dx$.

Let $x = a\sqrt{2} \tan \phi$.

Then
$$I = \int_0^{\tan^{-1} \frac{1}{\sqrt{2}}} \frac{a^5 \cdot a\sqrt{2} \sec^2 \phi d\phi}{(a^2+2a^2 \tan^2 \phi) a\sqrt{2} \sec \phi d\phi}$$

$$= a^3 \int \frac{\cos \phi d\phi}{\cos^2 \phi + 2 \sin^2 \phi}$$

$$= a^3 \int \frac{d \sin \phi}{1 + \sin^2 \phi}$$

$$= a^3 \left[\tan^{-1}(\sin \phi) \right]_0^{\tan^{-1} \frac{1}{\sqrt{2}}}$$

$$= a^3 \tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi a^3}{6};$$

$\therefore V = \frac{\pi a^3}{6}$.

2. Express the volume contained between the surfaces whose equations are $x^2 + y^2 + z^2 = a^2$, $x^2 + y^2 = a^2$, $z = a$ and the coordinate planes in the forms $V = \iint z \, dx \, dy$, $V = \iint x \, dz \, dy$; investigating the limits of the integrations and determining the value of V .

(i) For the portion of the elementary prism on $\delta x \delta y$ for base lying between the sphere and the plane $z = a$, the length is

$$a - \sqrt{a^2 - x^2 - y^2}.$$

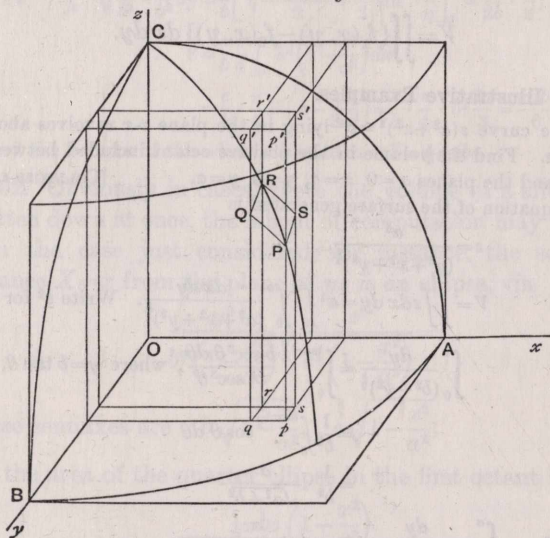


Fig. 264.

This is to be multiplied by $\delta x \delta y$ and summed for values of y from $y=0$ to $y=\sqrt{a^2-x^2}$, and afterwards the result is to be summed from $x=0$ to $x=a$.

$$\begin{aligned} \text{Then, } V &= \int_0^a \int_0^{\sqrt{a^2-x^2}} (a - \sqrt{a^2-x^2-y^2}) \, dx \, dy \\ &= \int_0^a \left[ay - \left(\frac{y\sqrt{a^2-x^2-y^2}}{2} + \frac{a^2-x^2}{2} \sin^{-1} \frac{y}{\sqrt{a^2-x^2}} \right) \right]_0^{\sqrt{a^2-x^2}} dx \\ &= \int_0^a \left\{ a\sqrt{a^2-x^2} - \frac{\pi}{4}(a^2-x^2) \right\} dx \\ &= \left[a \left\{ x \frac{\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right\} - \frac{\pi}{4} \left(a^2 x - \frac{x^3}{3} \right) \right]_0^a \\ &= \frac{a^3}{2} \cdot \frac{\pi}{2} - \frac{\pi}{4} \cdot \frac{2a^3}{3} \\ &= \frac{\pi a^3}{12}. \end{aligned}$$

(ii) If we use the formula $V = \iint x \, dz \, dy$, integrating with regard to y first, we have for the length of the prism on base $\delta y \, \delta z$ intercepted between the cylinder and the sphere $\sqrt{a^2 - y^2} - \sqrt{a^2 - z^2 - y^2}$, until the prism ceases to cut the sphere, *i.e.* from $y=0$ to $y=\sqrt{a^2 - z^2}$, and afterwards the length of this prism is $\sqrt{a^2 - y^2}$ from $y=\sqrt{a^2 - z^2}$ to $y=a$, and the limits for z are from 0 to a .

Hence

$$\begin{aligned} V &= \int_0^a \int_0^{\sqrt{a^2 - z^2}} (\sqrt{a^2 - y^2} - \sqrt{a^2 - z^2 - y^2}) \, dz \, dy + \int_0^a \int_{\sqrt{a^2 - z^2}}^a \sqrt{a^2 - y^2} \, dz \, dy \\ &= \int_0^a \left[\int_0^a \sqrt{a^2 - y^2} \, dy - \int_0^{\sqrt{a^2 - z^2}} \sqrt{a^2 - z^2 - y^2} \, dy \right] dz \\ &= \int_0^a \left\{ \left[y \frac{\sqrt{a^2 - y^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{y}{a} \right]_0^a - \left[y \frac{\sqrt{a^2 - z^2 - y^2}}{2} \right. \right. \\ &\qquad \qquad \qquad \left. \left. + \frac{a^2 - z^2}{2} \sin^{-1} \frac{y}{\sqrt{a^2 - z^2}} \right]_0^{\sqrt{a^2 - z^2}} \right\} dz \\ &= \int_0^a \left\{ \frac{a^2}{2} \cdot \frac{\pi}{2} - \frac{\pi}{4} (a^2 - z^2) \right\} dz \\ &= \frac{\pi}{4} \int_0^a z^2 \, dz = \frac{\pi}{4} \frac{a^3}{3} = \frac{\pi a^3}{12}. \end{aligned}$$

(iii) If we use the formula $V = \iint x \, dy \, dz$, integrating with regard to x before we integrate with regard to y , we have the same peculiarity as before, *viz.* that the prism is of length $\sqrt{a^2 - y^2} - \sqrt{a^2 - y^2 - z^2}$ from $z=0$ to $z=\sqrt{a^2 - y^2}$, and of length $\sqrt{a^2 - y^2}$ from $z=\sqrt{a^2 - y^2}$ to $z=a$, and

$$V = \int_0^a \int_0^{\sqrt{a^2 - y^2}} (\sqrt{a^2 - y^2} - \sqrt{a^2 - y^2 - z^2}) \, dy \, dz + \int_0^a \int_{\sqrt{a^2 - y^2}}^a \sqrt{a^2 - y^2} \, dy \, dz,$$

which, as before, $= \frac{\pi a^3}{12}$.

765. Mass, Moment, Centroid, etc.

If we regard the space bounded as described in Art. 760 to be filled with matter of specific density ρ at each point, the **Mass** of the elementary cuboid $\delta x \, \delta y \, \delta z$ is $\rho \, \delta x \, \delta y \, \delta z$, where ρ may be either a constant or a variable. And following the same argument as in finding the volume, we have for the mass of the body thus enclosed,

$$M = \iiint \rho \, dx \, dy \, dz.$$

766. In the same way, if the **Moment** of this mass be required about any line whose equations are known, say

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n},$$

l, m, n , being direction cosines; then, if p be the perpendicular from x, y, z upon this line, viz.

$$p^2 = (x-a)^2 + (y-b)^2 + (z-c)^2 - [l(x-a) + m(y-b) + n(z-c)]^2,$$

the moment of the solid about this line is

$$\iiint \rho p \, dx \, dy \, dz.$$

767. To determine the coordinates of the **Centroid**, we have only to translate the expressions

$$\bar{x} = \frac{\Sigma mx}{\Sigma m}, \quad \bar{y} = \frac{\Sigma my}{\Sigma m}, \quad \bar{z} = \frac{\Sigma mz}{\Sigma m}$$

into the language of the *Integral Calculus*. And m , being $\rho \, \delta x \, \delta y \, \delta z$, we have

$$\bar{x} = \frac{\iiint \rho x \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}, \quad \bar{y} = \frac{\iiint \rho y \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}, \quad \bar{z} = \frac{\iiint \rho z \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}.$$

768. If the **Moment of Inertia** about a straight line be required, and if p be the perpendicular from (x, y, z) upon the line, we have **Moment of inertia** = Σmp^2 ,

i.e. in the language of the *Calculus*,

$$\iiint \rho p^2 \, dx \, dy \, dz.$$

Thus, if A, B, C be the moments of inertia about the coordinate axes Ox, Oy, Oz respectively,

$$A = \iiint \rho (y^2 + z^2) \, dx \, dy \, dz,$$

$$B = \iiint \rho (z^2 + x^2) \, dx \, dy \, dz,$$

$$C = \iiint \rho (x^2 + y^2) \, dx \, dy \, dz.$$

769. Similarly for "**Products of Inertia**," *i.e.* for quantities such as

$$D = \Sigma myz, \quad E = \Sigma mzx, \quad F = \Sigma mxy,$$

we have

$$D = \iiint \rho yz \, dx \, dy \, dz, \quad E = \iiint \rho zx \, dx \, dy \, dz, \quad F = \iiint \rho xy \, dx \, dy \, dz.$$

770. The integration in all such cases takes the same course as in the finding of a volume, first as regards the proper assignment of limits, and second as regards the successive integrations (1) with regard to z , (2) with regard to y , (3) with regard to x .

The order of integration may be changed to suit circumstances, the several limits being suitably changed to ensure that the elementary cuboids into which the specified region is divided are thereby all added up.

As in the case of finding a volume, in some cases one, or perhaps two, of the integrations may be avoided by taking the elementary prism, or the elementary lamina described above, as the primary element, as was done in Art. 762 in the evaluation of the volume of the octant of an ellipsoid.

771. Ex. In the case of a sphere, viz. $x^2 + y^2 + z^2 = a^2$, let us find the mass of an octant of the sphere, the density at any point being $\rho = kxyz$.

Here

$$M = k \iiint xyz \, dx \, dy \, dz.$$

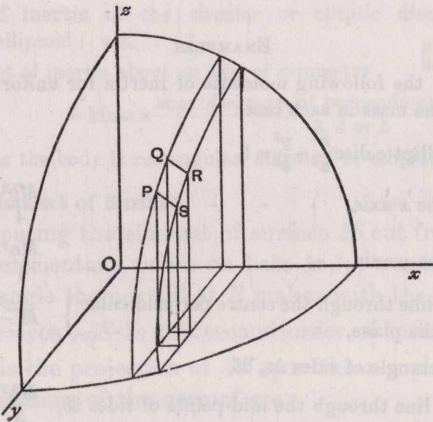


Fig. 265.

The limits for z in the positive octant are

$$z=0 \quad \text{to} \quad z = \sqrt{a^2 - x^2 - y^2};$$

for y ,

$$\text{from } y=0 \quad \text{to} \quad y = \sqrt{a^2 - x^2};$$

for x ,

$$\text{from } x=0 \quad \text{to} \quad x = a.$$

Hence

$$\begin{aligned}
 M &= k \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dx \, dy \, dz \\
 &= k \int_0^a \int_0^{\sqrt{a^2-x^2}} xy \left[\frac{z^2}{2} \right]_0^{\sqrt{a^2-x^2-y^2}} dx \, dy \\
 &= \frac{k}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} xy(a^2-x^2-y^2) dx \, dy \\
 &= \frac{k}{2} \int_0^a x \left[(a^2-x^2) \frac{y^2}{2} - \frac{y^4}{4} \right]_0^{\sqrt{a^2-x^2}} dx \\
 &= \frac{k}{2} \int_0^a x \left\{ \frac{(a^2-x^2)^2}{2} - \frac{(a^2-x^2)^2}{4} \right\} dx \\
 &= \frac{k}{8} \int_0^a x(a^2-x^2)^2 dx \\
 &= \frac{k}{8} \left[\frac{a^4 x^2}{2} - \frac{2a^2 x^4}{4} + \frac{x^6}{6} \right]_0^a \\
 &= \frac{ka^6}{48}.
 \end{aligned}$$

If D be the density at a specific point, say the centre of the surface of the octant, i.e. where $x=y=z=\frac{a}{\sqrt{3}}$, we have

$$D = k \frac{a^3}{3\sqrt{3}} \quad \text{and} \quad M = \frac{1}{18} Da^3 \sqrt{3}.$$

EXAMPLES.

1. Establish the following moments of inertia for uniform density, M representing the mass in each case :

(1) For an elliptic disc $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

about the x axis, - - - - - $\frac{Mb^2}{4}$;

about the y axis, - - - - - $\frac{Ma^2}{4}$;

about a line through the centre perpendicular } $M \frac{a^2+b^2}{4}$.
to the plane, - - - - - }

(2) For a rectangle of sides $2a, 2b$,

about a line through the mid-points of sides $2b$, $\frac{Mb^2}{3}$;

about a line through the mid-points of sides $2a$, $\frac{Ma^2}{3}$;

about a line through the centre perpendicular } $M \frac{a^2+b^2}{3}$.
to the plane, - - - - - }

(3) For a sphere about any diameter $\frac{2Ma^2}{5}$, a being the radius.

(4) For an ellipsoid of semi-axes a, b, c , viz.

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1,$$

about the axis of length $2a$, - - - - $M \frac{b^2 + c^2}{5}$;

about the axis of length $2b$, - - - - $M \frac{c^2 + a^2}{5}$;

about the axis of length $2c$, - - - - $M \frac{a^2 + b^2}{5}$.

2. Obtain the position of the centroid of

(1) the quadrant of an ellipse,

$$x^2/a^2 + y^2/b^2 = 1 ; \quad \bar{x} = \frac{4a}{3\pi} ; \quad \bar{y} = \frac{4b}{3\pi} ;$$

(2) the positive octant of the sphere,

$$x^2 + y^2 + z^2 = a^2 ; \quad \bar{x} = \bar{y} = \bar{z} = \frac{3a}{8} ;$$

(3) the positive octant of the ellipsoid,

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 ; \quad \bar{x} = \frac{3a}{8} ; \quad \bar{y} = \frac{3b}{8} ; \quad \bar{z} = \frac{3c}{8} .$$

3. Show that in all the above cases for the whole elliptic disc, rectangle, sphere or ellipsoid, the products of inertia with regard to two axes of symmetry are zero.

Dr. Routh gave the following useful **mnemonic rule for the moment of inertia** of the circular or elliptic disc, rectangle and sphere or ellipsoid ; viz.

Moment of inertia about an axis of symmetry

$$= \text{Mass} \times \frac{\text{sum of squares of perpendicular semi-axes}}{3, 4 \text{ or } 5},$$

according as the body is rectangular, elliptical or ellipsoidal.

772. **Element of Surface.**

In estimating the element of surface δS cut from the surface S by the elementary prism on base $\delta x \delta y$, we may note that if γ be the angle the normal at P makes with the z -axis,

$$\delta x \delta y = \cos \gamma \delta S \text{ to the second order of infinitesimals,}$$

for $\delta x \delta y$ is the projection of δS upon the x - y plane.

The equations of the normal are :

$$\frac{X-x}{\phi_x} = \frac{Y-y}{\phi_y} = \frac{Z-z}{\phi_z},$$

where $\phi_x \equiv \frac{\partial \phi}{\partial x}$, etc.

Hence
$$\cos \gamma = \frac{\phi_z}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} .$$

Then

$$S = \iint \frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{\phi_z} dx dy,$$

when we proceed to the limit and sum the elements by integration.

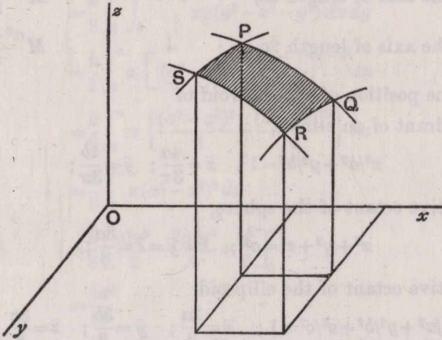


Fig. 266.

If the equation of the surface be thrown into the form

$$z = f(x, y),$$

and if we use the ordinary notation

$$p \equiv \frac{\partial z}{\partial x}, \quad q \equiv \frac{\partial z}{\partial y},$$

this equation becomes $S = \iint \sqrt{1 + p^2 + q^2} dx dy.$

We may note in passing that the equation $\delta x \delta y = \delta S \cos \gamma$ also gives another expression for the volume, viz.

$$V = \iint z dx dy = \int z \cos \gamma dS.$$

We have taken, as is ordinarily the case, x, y as the independent variables.

If this be inconvenient, we should have

$$S = \iint \sqrt{1 + \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2} dy dz,$$

or

$$S = \iint \sqrt{1 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2} dz dx,$$

according as y, z or z, x be chosen as the independent variables.

773. We may note that the coordinates of P, Q, S and R , the coordinates of the curvilinear "parallelogram" bounding δS are:

$$\left. \begin{array}{l} \text{for } P, \quad x, \quad y, \quad z; \\ \text{for } Q, \quad x + \delta x, \quad y, \quad z + \frac{\partial z}{\partial x} \delta x; \\ \text{for } S, \quad x, \quad y + \delta y, \quad z + \frac{\partial z}{\partial y} \delta y; \\ \text{for } R, \quad x + \delta x, \quad y + \delta y, \quad z + \frac{\partial z}{\partial x} \delta x + \frac{\partial z}{\partial y} \delta y, \end{array} \right\} \text{to the first order;}$$

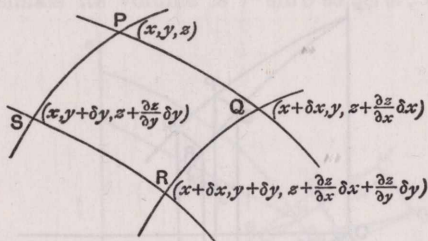


Fig. 267.

and the projections of this curvilinear parallelogram upon the coordinate planes are parallelograms of areas:

(1) upon the x - y plane, $\delta x \delta y$;

(2) upon the y - z plane,

$$\pm \begin{vmatrix} y, & z, & 1 \\ y, & z + \frac{\partial z}{\partial x} \delta x, & 1 \\ y + \delta y, & z + \frac{\partial z}{\partial y} \delta y, & 1 \end{vmatrix} = \pm \begin{vmatrix} y, & z, & 1 \\ 0, & \frac{\partial z}{\partial x} \delta x, & 0 \\ \delta y, & \frac{\partial z}{\partial y} \delta y, & 0 \end{vmatrix} = \frac{\partial z}{\partial x} \delta x \delta y;$$

(3) upon the z - x plane,

$$\pm \begin{vmatrix} x, & z, & 1 \\ x + \delta x, & z + \frac{\partial z}{\partial x} \delta x, & 1 \\ x, & z + \frac{\partial z}{\partial y} \delta y, & 1 \end{vmatrix} = \pm \begin{vmatrix} x, & z, & 1 \\ \delta x, & \frac{\partial z}{\partial x} \delta x, & 0 \\ 0, & \frac{\partial z}{\partial y} \delta y, & 0 \end{vmatrix} = \frac{\partial z}{\partial y} \delta x \delta y;$$

and the area δS is the square root of the sum of the squares of its projections upon any three mutually perpendicular planes (C. Smith, *Solid Geom.*, Art. 33).

Hence
$$\delta S^2 = \delta x^2 \delta y^2 \left[1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \right],$$

giving
$$S = \iint \sqrt{1 + p^2 + q^2} dx dy, \text{ as before.}$$

774. **Element of Volume for Cylindrical Coordinates.**

Instead of taking as our elementary volume one defined as bounded by planes parallel to three coordinate planes, other choices may be made. In some investigations it may be desirable to employ cylindrical coordinates, viz. ordinary polar coordinates r, θ in the x - y plane, retaining the Cartesian

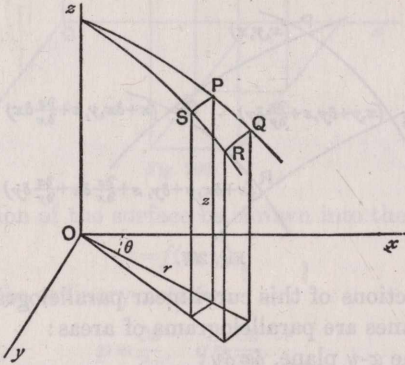


Fig. 268.

z -coordinate. An elementary prism, with this system, will be on a base $r \delta \theta \delta r$ with a height z , and to the second order its volume is $r \delta \theta \delta r \times z$, and the volume will be $\iint r z d\theta dr$, taken between suitable limits. If for any reason it be desirable to subdivide this elementary prism by planes perpendicular to the z -axis, our expression for the volume will be

$$\iiint r d\theta dr dz.$$

Such a necessity would arise, for instance, if the mass of the solid be required and the density be not a constant, but a known function of r, θ, z , when the mass of the elementary prism is $r \delta \theta \delta r \int \rho dz$, r and θ being regarded as constants during this integration, so as to add up all the elements of varying density through the elementary prism before summing the

masses of the several prisms themselves. We should then write the integral as

$$\text{Mass} = \iiint \rho r \, d\theta \, dr \, dz.$$

775. Spherical Polar Element of Volume.

Again, a spherical polar element of volume may be employed, using r the radius vector, θ the co-latitude and ϕ the azimuthal angle as coordinates.

Here the element of volume has three of its edges, mutually at right angles, δr , $r \delta\theta$ and $r \sin \theta \delta\phi$, and to the third order of infinitesimals its volume is $r^2 \sin \theta \delta\theta \delta\phi \delta r$, the difference

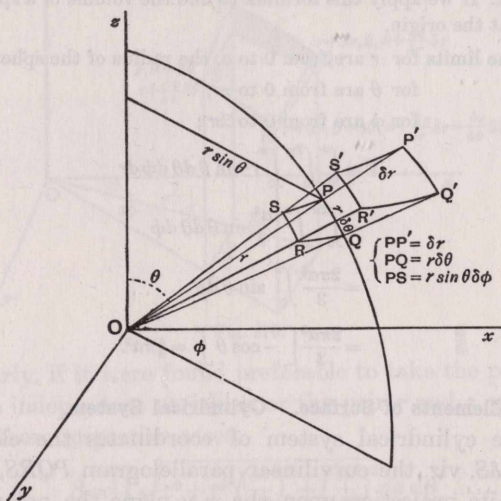


Fig. 269.

between this and the actual volume being at least of the fourth order of infinitesimals.

Upon integrating successively with regard to r , θ and ϕ in any order, the accumulated difference after the three integrations between the volume of any space required and the sum of these elements will be a first order infinitesimal at most, and therefore vanishes when the limit is taken.

Hence we have for the volume required

$$\iiint r^2 \sin \theta \, d\theta \, d\phi \, dr.$$

Further, if it be required to integrate any function of (r, θ, ϕ) throughout the volume, say $f(r, \theta, \phi)$, that is to add up all such elements as $f(r, \theta, \phi) r^2 \sin \theta \delta \theta \delta \phi \delta r$, the expression for the result will be

$$\iiint f(r, \theta, \phi) r^2 \sin \theta \, d\theta \, d\phi \, dr,$$

the limits being such as to include in the summation all the elements

$$f(r, \theta, \phi) r^2 \sin \theta \delta \theta \delta \phi \delta r,$$

which are included in the region under discussion, and no more.

776. Ex. If we apply this formula to find the volume of a sphere whose centre is at the origin,

the limits for r are from 0 to a , the radius of the sphere ;

for θ are from 0 to π ;

for ϕ are from 0 to 2π ;

and

$$\begin{aligned} V &= \int_0^a \int_0^{2\pi} \int_0^\pi r^2 \sin \theta \, d\theta \, d\phi \, dr \\ &= \int_0^{2\pi} \int_0^\pi \frac{a^3}{3} \sin \theta \, d\theta \, d\phi \\ &= \frac{2\pi a^3}{3} \int_0^\pi \sin \theta \, d\theta \\ &= \frac{2\pi a^3}{3} \left[-\cos \theta \right]_0^\pi = \frac{4}{3}\pi a^3. \end{aligned}$$

777. Elements of Surface. Cylindrical System.

In the cylindrical system of coordinates the element of surface δS , viz. the curvilinear parallelogram $PQRS$, Fig. 270, has for its projection upon the x - y plane the polar element $r \delta \theta \delta r$. Its projection upon the meridian plane through P is to the first order, an oblique parallelogram of area $\delta r \cdot \frac{\partial z}{\partial \theta} \delta \theta$, for one of its sides is the change in z due to increase of $\delta \theta$ in the independent variable θ , i.e. $\frac{\partial z}{\partial \theta} \delta \theta$, and the perpendicular between this side and the parallel side is δr .

And the projection upon a plane through P parallel to the z -axis and at right angles to the meridian plane, is similarly $r \delta \theta \frac{\partial z}{\partial r} \delta r$, for $r \delta \theta$ is the height of this parallelogram, and

$\frac{\partial z}{\partial r} \delta r$ is the change in z due to an increase δr in r , keeping θ constant, viz. the difference of the ordinates parallel to the z -axis of the points P and Q .

Hence

$$\delta S^2 = \delta r^2 (r \delta \theta)^2 + \left(\delta r \cdot \frac{\partial z}{\partial r} \delta \theta \right)^2 + (r \delta \theta \delta r)^2 \left(\frac{\partial z}{\partial r} \right)^2,$$

and taking the square root, proceeding to the limit and integrating,

$$S = \iint \sqrt{r^2 + r^2 \left(\frac{\partial z}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2} d\theta dr. \dots\dots\dots(1)$$

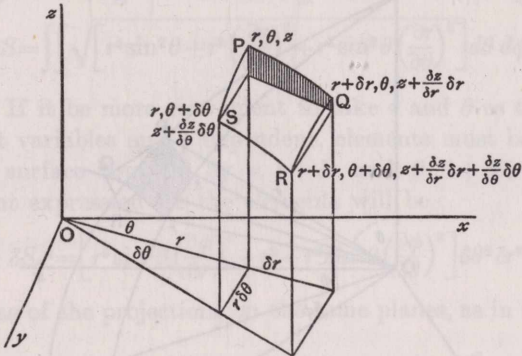


Fig. 270.

Similarly, if it were found preferable to take the pair z and θ for the independent variables, or the pair r and z , we should have in these respective cases,

$$S = \iint \sqrt{r^2 + r^2 \left(\frac{\partial r}{\partial z} \right)^2 + \left(\frac{\partial r}{\partial \theta} \right)^2} dz d\theta \dots\dots\dots(2)$$

and
$$S = \iint \sqrt{1 + r^2 \left(\frac{\partial \theta}{\partial z} \right)^2 + r^2 \left(\frac{\partial \theta}{\partial r} \right)^2} dr dz. \dots\dots\dots(3)$$

To establish (2) an element is taken on the surface bounded by lines on the surface along which z is constant and θ const., viz. $z, z + \delta z, \theta, \theta + \delta \theta$, and projected upon the same planes as in Case (1), the areas of the projections being

$$r \delta \theta \delta z, \quad r \delta \theta \left(\frac{\partial r}{\partial z} \delta z \right) \quad \text{and} \quad \delta z \left(\frac{\partial r}{\partial \theta} \delta \theta \right).$$

And to establish (3) an element is taken on the surface bounded by lines on the surface along which $r = \text{const.}$ and

$z = \text{const.}$, viz. $r, r + \delta r, z, z + \delta z$, and projection is made upon the same planes as in Case (1), the areas of the projections being

$$\delta r \delta z, \left(r \frac{\partial \theta}{\partial z} \delta z \right) \delta r \quad \text{and} \quad \left(r \frac{\partial \theta}{\partial r} \delta r \right) \delta z.$$

The figures are, however, somewhat troublesome, and we shall deduce these formulae from a more general result later.

778. In the spherical polar system of coordinates let the meridian planes ϕ and $\phi + \delta\phi$ cut the surface in the curves

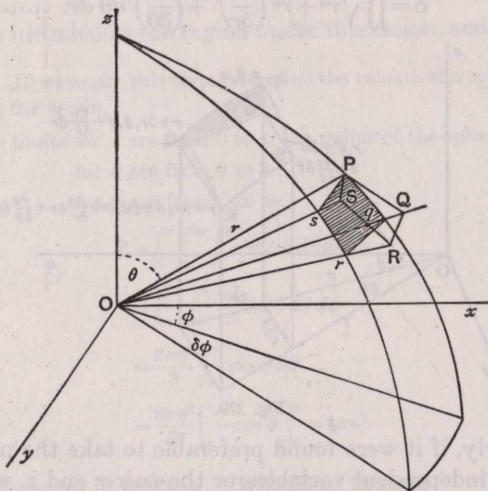


Fig. 271.

PQ, SR , and let the cones $\theta, \theta + \delta\theta$ cut the surface in curves PS, QR . Then $PQRS$ is our element of surface. Let the coordinates of the points P, Q, R, S be respectively :

for $P,$	$r,$	$\theta,$	$\phi,$
for $Q,$	$r + \frac{\partial r}{\partial \theta} \delta\theta,$	$\theta + \delta\theta,$	$\phi,$
for $S,$	$r + \frac{\partial r}{\partial \phi} \delta\phi,$	$\theta,$	$\phi + \delta\phi,$
for $R,$	$r + \frac{\partial r}{\partial \theta} \delta\theta + \frac{\partial r}{\partial \phi} \delta\phi,$	$\theta + \delta\theta,$	$\phi + \delta\phi.$

The projections of this elementary area upon

(1) a plane through P at right angles to the radius vector ;

(2) the meridian plane through P ;

(3) a plane through P perpendicular to these two planes are respectively, to the second order,

$$r \delta\theta . r \sin \theta \delta\phi, \quad r \delta\theta . \left(\frac{\partial r}{\partial \phi} \delta\phi\right) \quad \text{and} \quad r \sin \theta \delta\phi . \left(\frac{\partial r}{\partial \theta} \delta\theta\right),$$

and to the fourth order we have for δS_r , the element of area

$$\delta S_r^2 = \left[r^4 \sin^2 \theta + r^2 \left(\frac{\partial r}{\partial \phi}\right)^2 + r^2 \sin^2 \theta \left(\frac{\partial r}{\partial \theta}\right)^2 \right] \delta\theta^2 \delta\phi^2 ;$$

whence, extracting the root, taking the limit and integrating,

$$S = \iint \sqrt{\left[r^4 \sin^2 \theta + r^2 \left(\frac{\partial r}{\partial \phi}\right)^2 + r^2 \sin^2 \theta \left(\frac{\partial r}{\partial \theta}\right)^2 \right]} d\theta d\phi . \dots\dots(1)$$

779. If it be more convenient to take r and θ as the independent variables and ϕ dependent, elements must be chosen on the surface bounded by $r, r + \delta r$ and $\theta, \theta + \delta\theta$, and the resultant expression for the elements will be

$$\delta S_\phi^2 = \left[r^4 \sin^2 \theta \left(\frac{\partial \phi}{\partial r}\right)^2 + r^2 + r^2 \sin^2 \theta \left(\frac{\partial \phi}{\partial \theta}\right)^2 \right] \delta\theta^2 \delta r^2,$$

the areas of the projections on the same planes, as in Case (1), being

$$r \delta\theta . \delta r, \quad \left(r \sin \theta \frac{\partial \phi}{\partial r} \delta r \right) . r \delta\theta \quad \text{and} \quad \left(r \sin \theta \frac{\partial \phi}{\partial \theta} \delta\theta \right) . \delta r,$$

and the formula for S being

$$S = \iint \sqrt{\left[r^4 \sin^2 \theta \left(\frac{\partial \phi}{\partial r}\right)^2 + r^2 + r^2 \sin^2 \theta \left(\frac{\partial \phi}{\partial \theta}\right)^2 \right]} d\theta dr . \dots\dots(2)$$

And in the same way, if we wish to regard r and ϕ as the independent variables and θ dependent, an element of surface is to be chosen bounded by $r, r + \delta r, \phi, \phi + \delta\phi$, and its projections upon the same planes, as in Case (1), being

$$(r \sin \theta \delta\phi) . \left(r \frac{\partial \theta}{\partial r} \delta r \right), \quad \left(r \frac{\partial \theta}{\partial \phi} \delta\phi \right) . \delta r, \quad (r \sin \theta \delta\phi) . \delta r,$$

we have

$$\delta S_\theta^2 = \left[r^4 \sin^2 \theta \left(\frac{\partial \theta}{\partial r}\right)^2 + r^2 \left(\frac{\partial \theta}{\partial \phi}\right)^2 + r^2 \sin^2 \theta \right] \delta\phi^2 \delta r^2$$

and
$$S = \iint \sqrt{\left[r^4 \sin^2 \theta \left(\frac{\partial \theta}{\partial r}\right)^2 + r^2 \left(\frac{\partial \theta}{\partial \phi}\right)^2 + r^2 \sin^2 \theta \right]} d\phi dr . \dots\dots(3)$$

But the figures required are, as in the Cases (2) and (3), for cylindrical coordinates somewhat troublesome, and we propose to deduce these formulae from the more general result of Art. 790.

780. Areas on a Spherical Surface, the Origin being at the Centre.

Let a be the radius of the sphere. Then, putting $r=a$, the general formula

$$S = \iint \sqrt{r^4 \sin^2 \theta + r^2 \sin^2 \theta \left(\frac{\partial r}{\partial \theta}\right)^2 + r^2 \left(\frac{\partial r}{\partial \phi}\right)^2} d\theta d\phi$$

reduces to

$$S = a^2 \iint \sin \theta d\theta d\phi$$

$$= a^2 \int [-\cos \theta] d\phi.$$

If we apply the result to find the area bounded by two meridian arcs and some specified curve, $\theta=f(\phi)$, the limits for θ are from $\theta=0$ to $\theta=f(\phi)$, and

$$S = a^2 \int [1 - \cos f(\phi)] d\phi,$$

the result of Art. 734.

COR. For the whole sphere $f(\phi)=\pi$, and

$$S = 2a^2 \int_0^{2\pi} d\phi = 4\pi a^2.$$

781. Spherical Triangle.

Ex. Let us apply the formula obtained to the case of the area bounded by a great circle and two meridian arcs, the radius of the sphere being a .

Take as the plane of xz that through the centre which cuts the great circle perpendicularly, and let p be the spherical perpendicular from the pole upon the great circle arc. The equation of the great circle is then

$$\cos \phi = \frac{\cot \theta}{\cot p}.$$

Then

$$\sin \phi d\phi = \frac{\operatorname{cosec}^2 \theta d\theta}{\cot p},$$

and

$$\text{Area} = a^2 \int (1 - \cos \theta) d\phi = a^2 \int \frac{(1 - \cos \theta) \operatorname{cosec}^2 \theta}{\sqrt{\cot^2 p - \cot^2 \theta}} d\theta;$$

$$\therefore \frac{\text{Area}}{a^2} = \cos^{-1} \frac{\cot \theta}{\cot p} + \sin^{-1} \frac{\sin \theta}{\sin p}$$

$$= [\phi] + [\chi],$$

where χ is the angle a meridian makes with the great circle and ϕ is the azimuthal angle.

If we take limits $\phi = a$ to $\phi = a + A$, the limits for χ will be $\pi - C$ to B where ABC is the spherical triangle formed by the meridians AB , AC and the arc BC .

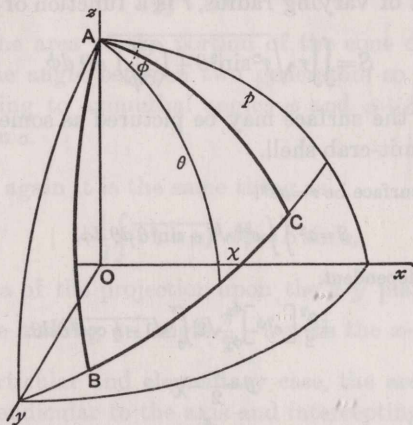


Fig. 272.

$$\begin{aligned} \text{This area is therefore } a^2[A + B - (\pi - C)] \\ = a^2[A + B + C - \pi] \\ = a^2 E, \end{aligned}$$

where E is the spherical excess, a result readily established in an elementary manner. (GIRARD'S THEOREM. See Todhunter and Leathem, *Sph. Trig.*, Art. 127.) Other illustrations have been given earlier. (See Art. 734.)

782. Case of a Solid of Revolution.

In the case of any solid of revolution about the z -axis ϕ varies, but r is independent of ϕ and depends only upon the revolving curve generating the solid.

The general formula

$$S = \iint \sqrt{\left\{ r^4 \sin^2 \theta + r^2 \sin^2 \theta \left(\frac{\partial r}{\partial \theta} \right)^2 + r^2 \left(\frac{\partial r}{\partial \phi} \right)^2 \right\}} d\theta d\phi$$

now reduces to

$$\begin{aligned} S &= \iint r \sin \theta \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta} \right)^2} d\theta d\phi \\ &= 2\pi \int r \sin \theta \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} d\theta = 2\pi \int r \sin \theta ds, \end{aligned}$$

in conformity with the result of Art. 748.

783. In the case of solids formed by the revolution about the z -axis of circles whose planes pass through the z -axis, centred at the origin, but of varying radius, r is a function of ϕ alone, and

$$S = \iint r \sqrt{r^2 \sin^2 \theta + \left(\frac{\partial r}{\partial \phi}\right)^2} d\theta d\phi.$$

The shape of the surface may be pictured as somewhat resembling the hermit-crab shell.

Ex. Let the surface be $r = ae^\phi$.

$$S = a^2 \iint e^{2\phi} \sqrt{1 + \sin^2 \theta} d\theta d\phi,$$

and θ, ϕ are independent,

$$= \frac{a^2}{2} \left[e^{2\phi} \right]_{\phi_1}^{\phi_2} \sqrt{2} \int_0^\pi \sqrt{1 - \frac{1}{2} \cos^2 \theta} d\theta.$$

Let

$$\theta = \frac{\pi}{2} - \chi.$$

$$S = \frac{a^2}{\sqrt{2}} (e^{2\phi_2} - e^{2\phi_1}) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{2} \sin^2 \chi} d\chi$$

$$= a^2 \sqrt{2} (e^{2\phi_1} - e^{2\phi_2}) \int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{2} \sin^2 \chi} d\chi$$

$$= a^2 \sqrt{2} (e^{2\phi_1} - e^{2\phi_2}) E_1; \text{ mod. } \frac{1}{\sqrt{2}};$$

and if the area be taken from $r=0$, i.e. $\phi_2 = -\infty$ to any value of r ,

$$S = r^2 \sqrt{2} E_1; \text{ mod. } \frac{1}{\sqrt{2}}.$$

784. In the case of an area of a portion of a right circular cone, vertex at the origin, axis the z -axis and semivertical angle α , the general formula

$$S = \iint \sqrt{\left\{ r^4 \sin^2 \theta \left(\frac{\partial \theta}{\partial r}\right)^2 + r^2 \left(\frac{\partial \theta}{\partial \phi}\right)^2 + r^2 \sin^2 \theta \right\}} d\phi dr$$

reduces to $\iint r \sin \alpha d\phi dr = \frac{\sin \alpha}{2} \int [r^2] d\phi$.

And supposing the area in question to be bounded by some curve drawn upon the cone, say $r=f(\phi)$, and two generators, we have $[r^2] = \{f(\phi)\}^2$, the lower limit being $r=0$, and

$$\therefore S = \frac{\sin \alpha}{2} \int \{f(\phi)\}^2 d\phi.$$

785. The formula is obviously the same thing as

$$\frac{1}{2} \int r^2 d(\phi \sin \alpha),$$

which is the area of the portion of the cone developed upon a plane, the angle between two generators so developed and corresponding to azimuthal angles ϕ and $\phi + \delta\phi$ on the cone, being $\delta\phi \sin \alpha$.

786. Or again it is the same thing as

$$\frac{1}{2} \int (r \sin \alpha)^2 d\phi = S \sin \alpha,$$

i.e. the area of the projection upon the x - y plane, all elements of the cone making an angle $\frac{\pi}{2} - \alpha$ with the x - y plane.

As a particular and elementary case, the area cut off by a plane perpendicular to the axis and intercepting generators of length l is

$$S = \frac{\sin \alpha}{2} l^2 \int_0^{2\pi} d\phi = \pi a l,$$

where a is the radius of the base $= l \sin \alpha$ and l the "slant height," the ordinary mensuration formula.

787. In the case of any cone with vertex at the origin, the equation is of the form $\phi = f(\theta)$, r being absent from the equation. Hence $\frac{\partial \phi}{\partial r} = 0$. The general expression

$$S = \iint \sqrt{\left\{ r^4 \sin^2 \theta \left(\frac{\partial \phi}{\partial r} \right)^2 + r^2 + r^2 \sin^2 \theta \left(\frac{\partial \phi}{\partial \theta} \right)^2 \right\}} d\theta dr$$

in this case reduces to

$$\iint r \sqrt{1 + \sin^2 \theta \left(\frac{\partial \phi}{\partial \theta} \right)^2} d\theta dr,$$

i.e.
$$S = \frac{1}{2} \int [r^2] \sqrt{1 + \sin^2 \theta \left(\frac{\partial \phi}{\partial \theta} \right)^2} d\theta.$$

Hence, if a surface cut a cone whose vertex is the origin, *viz.* $\phi = f(\theta)$, the area of the cone between two of its generators and the curve in which it meets the surface is

$$\frac{1}{2} \int r^2 \{ (1 + \sin^2 \theta f'^2(\theta)) \}^{\frac{1}{2}} d\theta.$$

788. Ex. The equations of a cylinder and a cone are

$$r \sin \theta = a \quad \text{and} \quad \cot \theta = \sinh \phi.$$

If A_1, A_2, A_3 be the areas of the cone from $\phi=0$ to $\phi=\beta-a, \beta$ and $\beta+a$ respectively, then will

$$A_1 + A_3 = 2A_2 \cosh a. \quad [\text{MATH. TRIPOS, 1875.}]$$

In this case $-\operatorname{cosec}^2 \theta = \cosh \phi \frac{d\phi}{d\theta}$.

$$\begin{aligned} \text{Hence } r^2 \sqrt{1 + \sin^2 \theta \left(\frac{\partial \phi}{\partial \theta}\right)^2} d\theta &= \frac{a^2}{\sin^2 \theta} \sqrt{1 + \frac{\operatorname{cosec}^2 \theta}{\cosh^2 \phi}} d\theta \\ &= -a^2 \cosh \phi d\phi \sqrt{1 + \frac{1 + \sinh^2 \phi}{\cosh^2 \phi}} \\ &= -a^2 \sqrt{2} \cosh \phi d\phi, \end{aligned}$$

and

$$S = -a^2 \sqrt{2} [\sinh \phi].$$

$$\text{Hence } \frac{A_1 + A_3}{A_2} = \frac{\sinh \beta - a + \sinh \beta + a}{\sinh \beta} = 2 \cosh a.$$

789. Generalised Results. Orthogonal Coordinates.

If $f(x, y, z) = \lambda$ be any surface, it is required to find the normal distance between the surface and the contiguous sur-

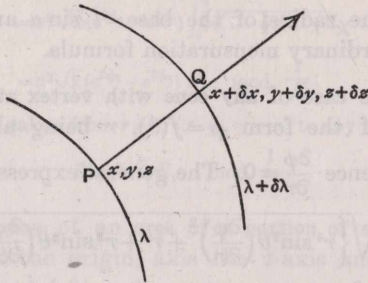


Fig. 273.

face $\lambda + \delta\lambda$ at the point (x, y, z) . Let the normal at P to the surface λ cut the surface $\lambda + \delta\lambda$ at Q , whose coordinates are $x + \delta x, y + \delta y, z + \delta z$.

The direction cosines of the normal are $\frac{\lambda_x}{h}, \frac{\lambda_y}{h}, \frac{\lambda_z}{h}$, where suffixes represent partial differentiations and $h^2 = \lambda_x^2 + \lambda_y^2 + \lambda_z^2$.

Then projecting the broken line $\delta x, \delta y, \delta z$ upon PQ , we have

$$PQ = \delta x \frac{\lambda_x}{h} + \delta y \frac{\lambda_y}{h} + \delta z \frac{\lambda_z}{h} = \frac{\delta \lambda}{h}.$$

Let $f_1(x, y, z) = \lambda$, $f_2(x, y, z) = \mu$, $f_3(x, y, z) = \nu$ be three mutually orthogonal surfaces. Consider the small element of space whose faces are the three surfaces λ , μ , ν and the contiguous surfaces $\lambda + \delta\lambda$, $\mu + \delta\mu$, $\nu + \delta\nu$.

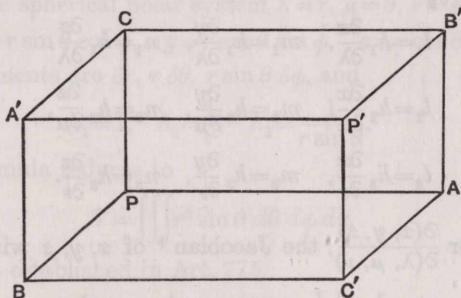


Fig. 274.

Let P be the point (λ, μ, ν) , PP' the diagonal through P of the element and $\lambda + \delta\lambda$, $\mu + \delta\mu$, $\nu + \delta\nu$ the coordinates of P' . Let the edges of this element be PA , PB , PC , $P'A'$, $P'B'$, $P'C'$ etc., PA being an element of the normal to λ , etc. This elementary space is ultimately an infinitesimal rectangular

parallelepiped or 'cuboid.' Its edges are $\frac{\delta\lambda}{h_1}$, $\frac{\delta\mu}{h_2}$, $\frac{\delta\nu}{h_3}$, where $h_1^2 = \lambda_x^2 + \lambda_y^2 + \lambda_z^2$, $h_2^2 = \mu_x^2 + \mu_y^2 + \mu_z^2$, $h_3^2 = \nu_x^2 + \nu_y^2 + \nu_z^2$.

Its volume is $\frac{\delta\lambda \delta\mu \delta\nu}{h_1 h_2 h_3}$.

Moreover, if (l_1, m_1, n_1) , (l_2, m_2, n_2) , (l_3, m_3, n_3) , be the direction cosines of the elements

$$\frac{\delta\lambda}{h_1}, \frac{\delta\mu}{h_2}, \frac{\delta\nu}{h_3},$$

$$\begin{aligned} \frac{\delta\lambda}{h_1} l_1 &= \text{the projection of } PA \text{ upon the } x\text{-axis} \\ &= \text{the small change in } x \text{ due to increase of } \lambda \text{ to } \\ &\quad \lambda + \delta\lambda, \mu \text{ and } \nu \text{ remaining unaltered,} \\ &= \frac{\partial x}{\partial \lambda} \delta\lambda; \end{aligned}$$

hence $l_1 = h_1 \frac{\partial x}{\partial \lambda}$.

Similarly $\frac{\delta\lambda}{h_1} m_1 = \frac{\partial y}{\partial\lambda} \delta\lambda, \quad \frac{\delta\lambda}{h_1} n_1 = \frac{\partial z}{\partial\lambda} \delta\lambda,$

$$\frac{\delta\mu}{h_2} l_2 = \frac{\partial x}{\partial\mu} \delta\mu, \text{ etc.};$$

hence we have

$$l_1 = h_1 \frac{\partial x}{\partial\lambda}, \quad m_1 = h_1 \frac{\partial y}{\partial\lambda}, \quad n_1 = h_1 \frac{\partial z}{\partial\lambda},$$

$$l_2 = h_2 \frac{\partial x}{\partial\mu}, \quad m_2 = h_2 \frac{\partial y}{\partial\mu}, \quad n_2 = h_2 \frac{\partial z}{\partial\mu},$$

$$l_3 = h_3 \frac{\partial x}{\partial\nu}, \quad m_3 = h_3 \frac{\partial y}{\partial\nu}, \quad n_3 = h_3 \frac{\partial z}{\partial\nu}.$$

Thus J or $\frac{\partial(x, y, z)}{\partial(\lambda, \mu, \nu)}$, the Jacobian* of x, y, z with regard to λ, μ, ν ,

$$= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm \frac{1}{h_1 h_2 h_3}.$$

(See C. Smith, *Solid Geometry*, Art. 46.)

Thus the volume of the elementary cuboid is $\pm J \delta\lambda \delta\mu \delta\nu$, and V , the volume of any region which is divided up into elements by this system, is given by

$$V = \iiint J d\lambda d\mu d\nu.$$

The ambiguity of sign disappears when the limits have been suitably assigned for the evaluation of the whole volume under consideration.

COR. (1). In the Cartesian system

$$\lambda = x, \quad \mu = y, \quad \nu = z, \quad h_1 = h_2 = h_3 = 1,$$

and the formula reduces to

$$V = \iiint dx dy dz;$$

the formula of Art. 760.

(2) In the cylindrical system $\lambda = r, \mu = \theta, \nu = z, x = r \cos \theta, y = r \sin \theta, z = z$, and the elements are $\delta r, r \delta \theta, \delta z$,

$$h_1 = 1, \quad h_2 = \frac{1}{r}, \quad h_3 = 1,$$

* See *Diff. Calc.*, Art. 534.

and the formula reduces to

$$V = \iiint r \, d\theta \, dr \, dz;$$

the formula of Art. 774.

(3) In the spherical polar system $\lambda = r, \mu = \theta, \nu = \phi,$

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta,$$

and the elements are $\delta r, r \delta \theta, r \sin \theta \delta \phi,$ and

$$h_1 = 1, \quad h_2 = \frac{1}{r}, \quad h_3 = \frac{1}{r \sin \theta},$$

and the formula reduces to

$$V = \iiint r^2 \sin \theta \, d\theta \, d\phi \, dr,$$

the formula established in Art. 775.

790. Element of Surface.

Suppose the region bounded by any surface S to have been divided up in the manner described by three families of ortho-

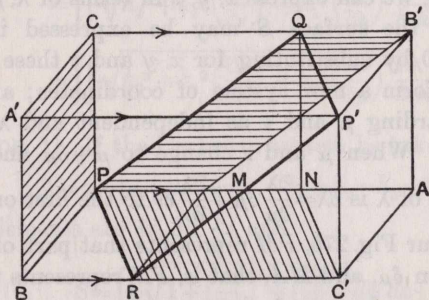


Fig. 275.

gonal surfaces whose distinctive parameters are $\lambda, \mu, \nu;$ any pair, say $\mu, \nu,$ with their contiguous surfaces $\mu + \delta\mu, \nu + \delta\nu,$ form a tubular region within $S.$ Suppose this tube to cut the tangent plane at P to the surface in the plane $P'RPQ,$ which may in the limit be regarded as an indefinitely small parallelogram element of the surface. Its area is an infinitesimal of the second order. We may take it as axiomatic that the difference between the area of the intercepted portion δS_λ of the surface, and the area of this parallelogram is at least of the third order, on the supposition that the curvature is finite and

continuous over the portion considered. The area of the parallelogram $PP'Q$ is readily found from the fact that the square of any plane area is the sum of the squares of its projections upon any three mutually perpendicular planes (C. Smith, *Solid Geom.*, Art. 33). Let the cuboid element of the μ - ν tube, for which PP' is a diagonal, be constructed as in Art. 789, with PA , PB , PC for adjacent edges through P and $P'A'$, $P'B'$, $P'C'$ for opposite edges through P' (Fig. 275). Let QN and RM be drawn at right angles to PA . Join $C'N$ and $B'M$. Thus the parallelograms $PBA'C$, $PQB'M$, $PRC'N$ are the projections of $PP'Q$ upon three mutually perpendicular planes. The areas of these figures are respectively

$$PB \cdot PC, \quad PC \cdot PM, \quad PB \cdot PN,$$

and it will be observed that $PN = RC' = MA$, *i.e.*

$$PM + PN = PA.$$

Now, as we have taken $f_1(x, y, z) = \lambda$, $f_2(x, y, z) = \mu$ and $f_3(x, y, z) = \nu$, we can express x, y, z in terms of λ, μ, ν , and the equation of the surface S may be expressed in the form $F(\lambda, \mu, \nu) = 0$ by substituting for x, y and z these values. In fact λ, μ, ν form a new system of coordinates; and of these we are regarding μ and ν as independent and λ depending upon them. When μ and ν change to $\mu + \delta\mu$ and $\nu + \delta\nu$, the total change of λ is $\delta\lambda = \frac{\partial\lambda}{\partial\mu} \delta\mu + \frac{\partial\lambda}{\partial\nu} \delta\nu$ to the first order.

Now, in our Fig. 275, PM represents that part of PA which depends upon $\delta\mu$, and MA , that is, PN represents that part of PA which depends upon $\delta\nu$, *i.e.*

$$PM = \frac{1}{h_1} \frac{\partial\lambda}{\partial\mu} \delta\mu \quad \text{and} \quad PN = \frac{1}{h_1} \frac{\partial\lambda}{\partial\nu} \delta\nu,$$

the two making up the total length of PA , *i.e.* $\frac{\delta\lambda}{h_1}$.

We thus have, to the fourth order,

$$\begin{aligned} \delta S_\lambda^2 &= (PB \cdot PC)^2 + (PC \cdot PM)^2 + (PB \cdot PN)^2 \\ &= \left(\frac{\delta\mu}{h_2} \cdot \frac{\delta\nu}{h_3}\right)^2 + \left(\frac{\delta\nu}{h_3} \cdot \frac{1}{h_1} \frac{\partial\lambda}{\partial\mu} \delta\mu\right)^2 + \left(\frac{\delta\mu}{h_2} \cdot \frac{1}{h_1} \frac{\partial\lambda}{\partial\nu} \delta\nu\right)^2 \end{aligned}$$

$$\text{or} \quad \delta S_\lambda^2 = \left[h_1^2 + h_2^2 \left(\frac{\partial\lambda}{\partial\mu}\right)^2 + h_3^2 \left(\frac{\partial\lambda}{\partial\nu}\right)^2 \right] \frac{\delta\mu^2 \delta\nu^2}{h_1^2 h_2^2 h_3^2}.$$

Similarly, if we had taken ν, λ or λ, μ as the independent pair of parameters and constructed the corresponding tubes, we should have had

$$\delta S_{\mu}^2 = \left[h_2^2 + h_3^2 \left(\frac{\partial \mu}{\partial \nu} \right)^2 + h_1^2 \left(\frac{\partial \mu}{\partial \lambda} \right)^2 \right] \frac{\delta \nu^2 \delta \lambda^2}{h_1^2 h_2^2 h_3^2},$$

$$\delta S_{\nu}^2 = \left[h_3^2 + h_1^2 \left(\frac{\partial \nu}{\partial \lambda} \right)^2 + h_2^2 \left(\frac{\partial \nu}{\partial \mu} \right)^2 \right] \frac{\delta \lambda^2 \delta \mu^2}{h_1^2 h_2^2 h_3^2},$$

and any of the three surface elements δS_{λ} , δS_{μ} , δS_{ν} intercepted by μ - ν tubes, ν - λ tubes or λ - μ tubes respectively, may be taken as an element of the surface for integration for the whole.

Thus we obtain, when we proceed to take the square root and integrate,

$$\begin{aligned} S &= \iint \sqrt{h_1^2 + h_2^2 \left(\frac{\partial \lambda}{\partial \mu} \right)^2 + h_3^2 \left(\frac{\partial \lambda}{\partial \nu} \right)^2} \frac{d\mu d\nu}{h_1 h_2 h_3} \\ &= \iint \sqrt{h_2^2 + h_3^2 \left(\frac{\partial \mu}{\partial \nu} \right)^2 + h_1^2 \left(\frac{\partial \mu}{\partial \lambda} \right)^2} \frac{d\nu d\lambda}{h_1 h_2 h_3} \\ &= \iint \sqrt{h_3^2 + h_1^2 \left(\frac{\partial \nu}{\partial \lambda} \right)^2 + h_2^2 \left(\frac{\partial \nu}{\partial \mu} \right)^2} \frac{d\lambda d\mu}{h_1 h_2 h_3}. \end{aligned}$$

791. COR. 1. If the Cartesian system be taken,

$$\lambda = x, \quad \mu = y, \quad \nu = z, \quad h_1 = h_2 = h_3 = 1,$$

and the elements are δx , δy , δz , and

$$\begin{aligned} S &= \iint \sqrt{1 + \left(\frac{\partial x}{\partial y} \right)^2 + \left(\frac{\partial x}{\partial z} \right)^2} dy dz \\ &= \iint \sqrt{1 + \left(\frac{\partial y}{\partial z} \right)^2 + \left(\frac{\partial y}{\partial x} \right)^2} dz dx \\ &= \iint \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} dx dy, \end{aligned}$$

viz. the formulae of Art. 772.

COR. 2. If the cylindrical system be taken,

$$\begin{aligned} \lambda &= r, \quad \mu = \theta, \quad \nu = z, \\ x &= r \cos \theta, \quad y = r \sin \theta, \quad z = z, \end{aligned}$$

and r, θ, z form an orthogonal system, the elements being

$$\delta r, \quad r \delta \theta, \quad \delta z \quad \text{and} \quad h_1=1, \quad h_2=\frac{1}{r}, \quad h_3=1;$$

$$\delta S_r^2 = (r \delta \theta \cdot \delta z)^2 + (\delta z)^2 \left(\frac{\partial r}{\partial \theta} \delta \theta \right)^2 + (r \delta \theta)^2 \left(\frac{\partial r}{\partial z} \delta z \right)^2,$$

$$\delta S_\theta^2 = (\delta z \cdot \delta r)^2 + (\delta r)^2 \left(r \frac{\partial \theta}{\partial z} \delta z \right)^2 + (\delta z)^2 \left(r \frac{\partial \theta}{\partial r} \delta r \right)^2,$$

$$\delta S_z^2 = (\delta r \cdot r \delta \theta)^2 + (r \delta \theta)^2 \left(\frac{\partial z}{\partial r} \delta r \right)^2 + (\delta r)^2 \left(\frac{\partial z}{\partial \theta} \delta \theta \right)^2,$$

according as r, θ or z is the dependent variable, giving the formulae

$$S = \iint \sqrt{r^2 + \left(\frac{\partial r}{\partial \theta} \right)^2 + r^2 \left(\frac{\partial r}{\partial z} \right)^2} d\theta dz$$

$$= \iint \sqrt{1 + r^2 \left(\frac{\partial \theta}{\partial z} \right)^2 + r^2 \left(\frac{\partial \theta}{\partial r} \right)^2} dz dr$$

$$= \iint \sqrt{r^2 + r^2 \left(\frac{\partial z}{\partial r} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2} dr d\theta,$$

which are in agreement with those of Art. 777.

COR. 3. In the spherical polar system,

$$\lambda = r, \quad \mu = \theta, \quad \nu = \phi$$

and $x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta,$

and r, θ, ϕ form an orthogonal system, the elements being

$$\delta r, \quad r \delta \theta, \quad r \sin \theta \delta \phi \quad \text{and} \quad h_1=1, \quad h_2=\frac{1}{r}, \quad h_3=\frac{1}{r \sin \theta};$$

whence

$$\delta S_r^2 = r^2 \delta \theta^2 \cdot r^2 \sin^2 \theta \delta \phi^2 + r^2 \sin^2 \theta \delta \phi^2 \left(\frac{\partial r}{\partial \theta} \delta \theta \right)^2$$

$$+ r^2 \delta \theta^2 \left(\frac{\partial r}{\partial \phi} \delta \phi \right)^2,$$

$$\delta S_\theta^2 = r^2 \sin^2 \theta \delta \phi^2 \cdot \delta r^2 + \delta r^2 \left(r \frac{\partial \theta}{\partial \phi} \delta \phi \right)^2$$

$$+ r^2 \sin^2 \theta \delta \phi^2 \left(r \frac{\partial \theta}{\partial r} \delta r \right)^2,$$

$$\delta S_\phi^2 = \delta r^2 r^2 \delta \theta^2 + r^2 \delta \theta^2 \left(r \sin \theta \frac{\partial \phi}{\partial r} \delta r \right)^2$$

$$+ \delta r^2 \left(r \sin \theta \frac{\partial \phi}{\partial \theta} \delta \theta \right)^2,$$

giving the formulae

$$\begin{aligned} S &= \iint \sqrt{r^4 \sin^2 \theta + r^2 \sin^2 \theta \left(\frac{\partial r}{\partial \theta}\right)^2 + r^2 \left(\frac{\partial r}{\partial \phi}\right)^2} d\theta d\phi \\ &= \iint \sqrt{r^2 \sin^2 \theta + r^2 \left(\frac{\partial \theta}{\partial \phi}\right)^2 + r^4 \sin^2 \theta \left(\frac{\partial \theta}{\partial r}\right)^2} d\phi dr \\ &= \iint \sqrt{r^2 + r^4 \sin^2 \theta \left(\frac{\partial \phi}{\partial r}\right)^2 + r^2 \sin^2 \theta \left(\frac{\partial \phi}{\partial \theta}\right)^2} dr d\theta, \end{aligned}$$

according as r , θ or ϕ is taken as the dependent variable, formulae in agreement with those of Arts. 778 and 779.

792. CHANGE OF THE VARIABLES. Form of Element of Area.

Supposing the coordinates x , y of any point in the plane of x - y to be expressed in terms of two new variables u , v , let us consider the nature of the figure bounded by the four curves obtained by assigned values of u , v , viz.

$$u, u + \delta u, v, v + \delta v.$$

Let the figure thus bounded be $PQRS$,

δu being zero along PS ,

δv being zero along PQ .

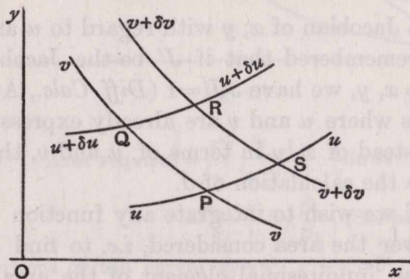


Fig. 276.

The several Cartesian coordinates of the four corners are, to the first order,

for P , x , y ;

for Q , $x + \frac{\partial x}{\partial u} \delta u$, $y + \frac{\partial y}{\partial u} \delta u$;

for S , $x + \frac{\partial x}{\partial v} \delta v$, $y + \frac{\partial y}{\partial v} \delta v$;

for R , $x + \frac{\partial x}{\partial u} \delta u + \frac{\partial x}{\partial v} \delta v$, $y + \frac{\partial y}{\partial u} \delta u + \frac{\partial y}{\partial v} \delta v$.

The direction ratios of PQ and SR are $\frac{\partial x}{\partial u} \delta u, \frac{\partial y}{\partial u} \delta u,$

and of PS and QR $\frac{\partial x}{\partial v} \delta v, \frac{\partial y}{\partial v} \delta v.$

Hence the chords joining the corresponding points are such as, to the first order, to form the four sides of a parallelogram whose area is

$$\left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{array} \right| \delta u \delta v \text{ or } \frac{\partial(x, y)}{\partial(u, v)} \delta u \delta v.$$

This then is, to the second order, the area of the elementary curvilinear "parallelogram" PR , the difference between this area and that of the chordal parallelogram being at least of the third order of infinitesimals. Hence, taking the limit and integrating between any assigned limits, for u and v , we have

$$\text{Area} = \iint \frac{\partial(x, y)}{\partial(u, v)} du dv = \iint J du dv,$$

where J is the Jacobian of x, y with regard to u and v .

It will be remembered that if J' be the Jacobian of u, v with regard to x, y , we have $JJ' = 1$ (*Diff. Calc.*, Art. 540).

And in cases where u and v are already expressed in terms of x and y , instead of x, y in terms of u and v , this rule will often facilitate the calculation of J .

Similarly, if we wish to integrate any function of x and y , say $f(x, y)$, over the area considered, *i.e.* to find $\Sigma f(x, y) \delta A$ where δA is an infinitesimal element of the area, it is only necessary to express x and y in terms of u and v , and then to transform the function $f(x, y)$ so as to express it as a function of u and v , say $F(u, v)$, then to multiply it by $J \delta u \delta v$, and integrate, the result being

$$\iint F(u, v) J du dv.$$

793. Illustrative Examples.

1. Find the area of the Carnot's cycle bounded by the isothermals $xy = a_1, xy = a_2$, and the adiabatics $xy^\gamma = \beta_1, xy^\gamma = \beta_2$.

Putting $xy = u$, $xy^\gamma = v$, take an element of the area bounded by the curves u , v , $u + \delta u$, $v + \delta v$.

Here

$$J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & y^\gamma \\ x & \gamma xy^{\gamma-1} \end{vmatrix}$$

$$= (\gamma - 1) xy^\gamma = (\gamma - 1) v;$$

$$\therefore J = \frac{1}{\gamma - 1} \cdot \frac{1}{v};$$

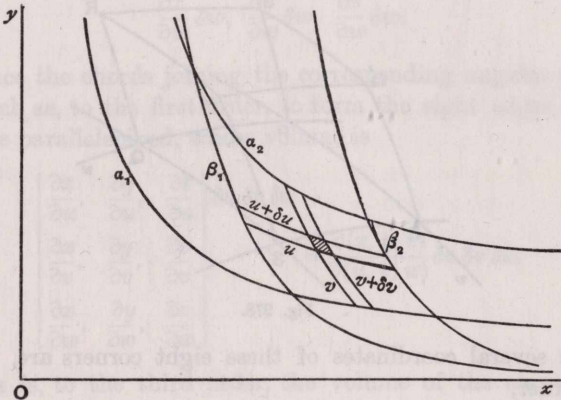


Fig. 277.

and

$$\text{Area required} = \int_{a_1}^{a_2} \int_{\beta_1}^{\beta_2} \frac{1}{\gamma - 1} \cdot \frac{1}{v} du dv$$

$$= \frac{a_2 - a_1}{\gamma - 1} \log \frac{\beta_2}{\beta_1}. \quad (\text{See page 63, Ex. 28.})$$

2. The portions of the curves $xy = a^2$, $x^2 - y^2 = b^2$, which lie in the positive quadrant, are drawn intersecting at B . The former intersects the asymptote of the latter in C , and the latter meets OX in A . If every element of the area $OABC$ be multiplied by the square of its distance from the origin O , the sum will be equal to $\frac{1}{2}a^2b^2$. [COLLEGES a , 1884.]

794. CHANGE OF THE VARIABLES. Form of Element of Volume.

Again, let the coordinates x , y , z of any point in space be expressed in terms of three new independent variables u , v , w , the surfaces $u = \text{const.}$, $v = \text{const.}$, $w = \text{const.}$, not necessarily as in Art. 789, forming an orthogonal system.

Let us consider the nature of the figure bounded by the six surfaces obtained by assigned values of u, v, w , viz.

$$u, u + \delta u, v, v + \delta v, w, w + \delta w.$$

Let the figure thus bounded be $PQS'RP'Q'SR'$,
 δu being zero over the surface $PRQ'S$,
 δv being zero over the surface $PQR'S$,
 δw being zero over the surface $PQS'R$, } Fig. 278.

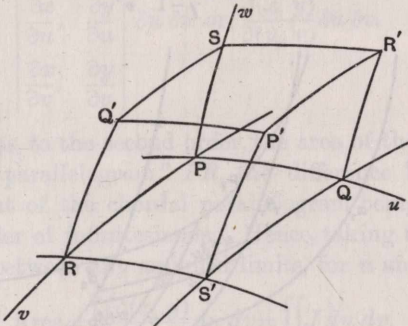


Fig. 278.

The several coordinates of these eight corners are, to the first order,

$$\text{for } P, \quad x, \quad y, \quad z,$$

$$\text{for } Q, \quad x + \frac{\partial x}{\partial u} \delta u, \quad y + \frac{\partial y}{\partial u} \delta u, \quad z + \frac{\partial z}{\partial u} \delta u,$$

$$\text{for } R, \quad x + \frac{\partial x}{\partial v} \delta v, \quad y + \frac{\partial y}{\partial v} \delta v, \quad z + \frac{\partial z}{\partial v} \delta v,$$

$$\text{for } S, \quad x + \frac{\partial x}{\partial w} \delta w, \quad y + \frac{\partial y}{\partial w} \delta w, \quad z + \frac{\partial z}{\partial w} \delta w,$$

$$\text{for } Q', \quad x + \frac{\partial x}{\partial v} \delta v + \frac{\partial x}{\partial w} \delta w, \text{ etc.},$$

$$\text{for } R', \quad x + \frac{\partial x}{\partial w} \delta w + \frac{\partial x}{\partial u} \delta u, \text{ etc.},$$

$$\text{for } S', \quad x + \frac{\partial x}{\partial u} \delta u + \frac{\partial x}{\partial v} \delta v, \text{ etc.},$$

$$\text{for } P', \quad x + \frac{\partial x}{\partial u} \delta u + \frac{\partial x}{\partial v} \delta v + \frac{\partial x}{\partial w} \delta w, \text{ etc.}$$

The direction ratios of PQ , RS' , $Q'P'$, SR' are

$$\frac{\partial x}{\partial u} \delta u, \quad \frac{\partial y}{\partial u} \delta u, \quad \frac{\partial z}{\partial u} \delta u;$$

those of PR , QS' , $R'P'$, SQ' are

$$\frac{\partial x}{\partial v} \delta v, \quad \frac{\partial y}{\partial v} \delta v, \quad \frac{\partial z}{\partial v} \delta v,$$

and those of PS , RQ' , $S'P'$, QR' are

$$\frac{\partial x}{\partial w} \delta w, \quad \frac{\partial y}{\partial w} \delta w, \quad \frac{\partial z}{\partial w} \delta w.$$

Hence the chords joining the corresponding angular points are such as, to the first order, to form the eight edges of an oblique parallelepiped, whose volume is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \delta u \delta v \delta w = \frac{\partial(x, y, z)}{\partial(u, v, w)} \delta u \delta v \delta w.$$

This is, to the third order, the volume of the elementary solid PP' , the difference between this volume and that of the oblique parallelepiped being at least of the fourth order of infinitesimals. Hence, taking the limit and integrating between any assigned limits for u , v , w , we have

$$V = \iiint \frac{\partial(x, y, z)}{\partial(u, v, w)} du dv dw = \iiint J du dv dw,$$

where J is the Jacobian of x , y , z with regard to u , v , w ; and, as noted in Art. 792, it is to be remembered that if J' be the Jacobian of u , v , w with regard to x , y , z , we have $JJ' = 1$ (*Diff. Calc.*, Art. 540). And for cases where u , v , w are expressed as functions of x , y , z , instead of x , y , z , in terms of u , v , w , this rule will facilitate the calculation of J .

795. Ex. Find the volume enclosed by the six hyperbolic cylinders

$$\begin{aligned} yz &= a_1^2, & yz &= a_2^2, \\ zx &= b_1^2, & zx &= b_2^2, \\ xy &= c_1^2, & xy &= c_2^2. \end{aligned}$$

Putting

$$yz = u, \quad zx = v, \quad xy = w,$$

$$J' = \begin{vmatrix} 0, & z, & y \\ z, & 0, & x \\ y, & x, & 0 \end{vmatrix} = zxy + yzx = 2\sqrt{uvw};$$

$$\therefore V = \frac{1}{2} \int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} \frac{du \, dv \, dw}{\sqrt{uvw}} = 4(a_2 - a_1)(b_2 - b_1)(c_2 - c_1).$$

796. It follows that if we wish to integrate the function $f(x, y, z)$ throughout the volume bounded by surfaces specified by two specific values of u , two specific values of v and two specific values of w , *i.e.* to add up all quantities of the form

$$f(x, y, z) \times \text{an element of volume at } x, y, z,$$

we have only to express x, y, z in terms of u, v, w , and substitute these values for x, y, z in $f(x, y, z)$, obtaining, say $F(u, v, w)$, as the transformed function. Then taking, as before, the same element of volume, *viz.* $J \, \delta u \, \delta v \, \delta w$, the integral required will be

$$\iiint F(u, v, w) J \, du \, dv \, dw.$$

797. Thus, if we wished to obtain the product of inertia with regard to the y, z axes in the above example (of Art. 795), each element of mass $\rho J \, \delta u \, \delta v \, \delta w$ is to be multiplied by yz , *i.e.* u , and assuming a uniform volume density ρ , the product of inertia required is $\iiint \rho u J \, du \, dv \, dw$, or

$$\begin{aligned} \frac{\rho}{2} \iiint \frac{\sqrt{u}}{\sqrt{vw}} \, du \, dv \, dw &= \frac{4}{3} \rho (a_2^3 - a_1^3) (b_2 - b_1) (c_2 - c_1) \\ &= \frac{M}{3} (a_2^2 + a_1 a_2 + a_1^2), \end{aligned}$$

where M is the mass of the solid in question.

798. If we wish for the x -coordinate of the centroid of the solid,

$$\begin{aligned} \bar{x} &= \frac{\sum m x}{\sum m} = \frac{\iiint \rho x J \, du \, dv \, dw}{\iiint \rho J \, du \, dv \, dw} = \frac{\iiint \sqrt{\frac{vw}{u}} \cdot \frac{du \, dv \, dw}{2\sqrt{uvw}}}{\iiint \frac{du \, dv \, dw}{2\sqrt{uvw}}} \\ &= \frac{\iiint \frac{du \, dv \, dw}{2u}}{\iiint \frac{du \, dv \, dw}{2\sqrt{uvw}}} = \frac{(\log a_2 - \log a_1)(b_2^2 - b_1^2)(c_2^2 - c_1^2)}{8(a_2 - a_1)(b_2 - b_1)(c_2 - c_1)} \\ &= \frac{1}{8} \frac{(b_2 + b_1)(c_2 + c_1)}{a_2 - a_1} \log \left(\frac{a_2}{a_1} \right), \end{aligned}$$

and similarly for other integrals.

799. We consider next the case in which the three coordinates x, y, z are expressed, or expressible, in terms of two independent parameters u and v , and therefore the point travels upon a definite surface. Consider the four points P, Q, S, R on the surface defined by the values

$$(u, v), (u + \delta u, v), (u, v + \delta v), (u + \delta u, v + \delta v),$$

i.e. $x, y, z;$

$$x + \frac{\partial x}{\partial u} \delta u, \quad y + \frac{\partial y}{\partial u} \delta u, \quad z + \frac{\partial z}{\partial u} \delta u;$$

$$x + \frac{\partial x}{\partial v} \delta v, \quad y + \frac{\partial y}{\partial v} \delta v, \quad z + \frac{\partial z}{\partial v} \delta v;$$

$$x + \frac{\partial x}{\partial u} \delta u + \frac{\partial x}{\partial v} \delta v, \quad y + \frac{\partial y}{\partial u} \delta u + \frac{\partial y}{\partial v} \delta v, \quad z + \frac{\partial z}{\partial u} \delta u + \frac{\partial z}{\partial v} \delta v.$$

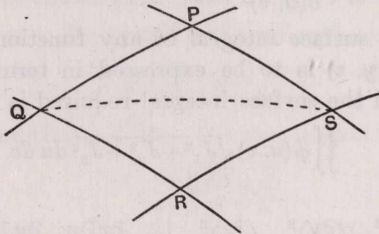


Fig. 279.

The direction ratios of PQ and SR are each .

$$\frac{\partial x}{\partial u} \delta u, \quad \frac{\partial y}{\partial u} \delta u, \quad \frac{\partial z}{\partial u} \delta u,$$

and those of PS and QR are each

$$\frac{\partial x}{\partial v} \delta v, \quad \frac{\partial y}{\partial v} \delta v, \quad \frac{\partial z}{\partial v} \delta v,$$

and to the first order $PQRS$ is a parallelogram. Let its area be δS .

The coordinates of the projections of P, Q, S, R on the plane of $x-y$ are

$$(x, y), \quad \left(x + \frac{\partial x}{\partial u} \delta u, y + \frac{\partial y}{\partial u} \delta u\right), \text{ etc.,}$$

and the area of this projection is

$$\begin{vmatrix} x + \frac{\partial x}{\partial u} \delta u, & y + \frac{\partial y}{\partial u} \delta u, & 1 \\ x + \frac{\partial x}{\partial v} \delta v, & y + \frac{\partial y}{\partial v} \delta v, & 1 \\ x, & y, & 1 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u}, & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v}, & \frac{\partial y}{\partial v} \end{vmatrix} \delta u \delta v = \frac{\partial(x, y)}{\partial(u, v)} \delta u \delta v,$$

and similarly its projections upon the other coordinate planes are

$$\frac{\partial(y, z)}{\partial(u, v)} \delta u \delta v; \quad \frac{\partial(z, x)}{\partial(u, v)} \delta u \delta v,$$

whence its area δS is given by

$$\delta S^2 = \left[\frac{\partial(y, z)}{\partial(u, v)} \right]^2 \delta u^2 \delta v^2 + \left[\frac{\partial(z, x)}{\partial(u, v)} \right]^2 \delta u^2 \delta v^2 + \left[\frac{\partial(x, y)}{\partial(u, v)} \right]^2 \delta u^2 \delta v^2.$$

Hence, proceeding to the limit and integrating,

$$S = \iint \sqrt{\left[\left\{ \frac{\partial(y, z)}{\partial(u, v)} \right\}^2 + \left\{ \frac{\partial(z, x)}{\partial(u, v)} \right\}^2 + \left\{ \frac{\partial(x, y)}{\partial(u, v)} \right\}^2 \right]} du dv$$

i.e.
$$S = \iint \sqrt{J_1^2 + J_2^2 + J_3^2} du dv,$$

where $J_1 = \frac{\partial(y, z)}{\partial(u, v)}, \quad J_2 = \text{etc.}, \quad J_3 = \text{etc.}$

Also if the surface integral of any function $f(x, y, z)$ be required, $f(x, y, z)$ is to be expressed in terms of u and v , as $\phi(u, v)$, and the surface integral required is

$$\iint \phi(u, v) \sqrt{J_1^2 + J_2^2 + J_3^2} du dv.$$

If we write

$$E \equiv \left(\frac{\partial x}{\partial u} \right)^2 + \left(\frac{\partial y}{\partial u} \right)^2 + \left(\frac{\partial z}{\partial u} \right)^2, \quad F \equiv \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v},$$

$$G \equiv \left(\frac{\partial x}{\partial v} \right)^2 + \left(\frac{\partial y}{\partial v} \right)^2 + \left(\frac{\partial z}{\partial v} \right)^2;$$

we have, from the algebraic identity,

$$(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2 + (ll' + mm' + nn')^2 \\ = (l^2 + m^2 + n^2)(l'^2 + m'^2 + n'^2),$$

$$J_1^2 + J_2^2 + J_3^2 = EG - F^2;$$

\therefore the surface integral may be written

$$\iint \phi(u, v) \sqrt{EG - F^2} du dv,$$

as shown otherwise in Art. 744.

800. Results connecting δV and δS .

If δS be an element of the area S of a surface, and P be the perpendicular from the origin on the corresponding tangent plane, we have for the volume of the cone whose vertex is at the origin and base δS , $\frac{1}{3}P \delta S$.

Hence the volume of any region bounded by a given surface and a cone with vertex at the origin, and generators passing through the perimeter of any closed curve drawn upon the surface, is

$$V = \frac{1}{3} \int P \, dS;$$

or, which is the same thing, if l, m, n be the direction cosines of the normal to the element δS , so that

$$P = lx + my + nz,$$

is the equation of the tangent plane, we have

$$V = \frac{1}{3} \int (lx + my + nz) \, dS.$$

801. If the equation of the surface be written as $z = f(x, y)$, the equation of the tangent plane at x, y, z is

$$Z - z = p(X - x) + q(Y - y),$$

where

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y},$$

and the perpendicular P from the origin upon it is

$$P = \frac{px + qy - z}{\sqrt{1 + p^2 + q^2}}.$$

Hence the formula for the volume, viz. $\frac{1}{3} \int P \, dS$, becomes

$$\frac{1}{3} \iint (px + qy - z) \, dx \, dy,$$

for

$$\delta x \, \delta y = \delta S \cos \gamma = \frac{\delta S}{\sqrt{1 + p^2 + q^2}},$$

where $\cos \alpha, \cos \beta, \cos \gamma$ are the direction cosines of the normal,

i.e.
$$V = \frac{1}{3} \iint [px + qy - f(x, y)] \, dx \, dy.$$

802. Let the inward drawn normal at a point P on a surface make an angle χ with the radius vector from the origin, and let p be the perpendicular from the origin upon the tangent plane at P , r the radius vector from the origin to P , and δS an element of the surface about P

Then $\frac{p}{r} = \cos \chi$, and the formula for an element of volume forming an elementary cone with vertex O and base δS , viz. $\frac{1}{3} p \, \delta S$, becomes $\frac{1}{3} r \cos \chi \, \delta S$.

Hence we have another expression for the volume bounded by any curved surface and a cone whose vertex is the origin and passing through the perimeter of the region defined by a given closed curve drawn upon the surface, viz.

$$V = \frac{1}{3} \int r \cos \chi \, dS;$$

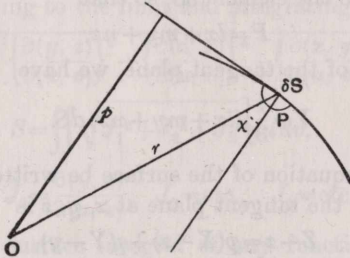


Fig. 280.

or again, seeing that this element of volume is

$$\frac{r^3}{3} \sin \theta \, \delta \theta \, \delta \phi,$$

we have

$$\delta S = \frac{r^3}{p} \sin \theta \, \delta \theta \, \delta \phi$$

and

$$S = \iint \frac{r^3}{p} \sin \theta \, d\theta \, d\phi.$$

803. Ex. Find the surface and the volume of the solid formed by the revolution of the cardioid $r = a(1 + \cos \theta)$ about the initial line.

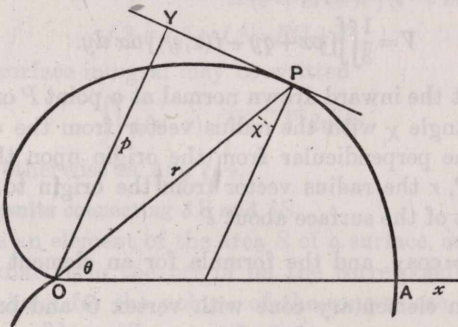


Fig. 281.

Here $\chi = \frac{\theta}{2}$, $p = r \cos \chi = 2a \cos^3 \frac{\theta}{2}$.

$$\begin{aligned} S &= \int_0^\pi \int_0^{2\pi} \frac{r^3}{p} \sin \theta \, d\theta \, d\phi \\ &= 2\pi \cdot \frac{(2a)^3}{(2a)} \int_0^\pi \cos^3 \frac{\theta}{2} \cdot 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \, d\theta \\ &= 16\pi a^2 \int_0^\pi \cos^4 \frac{\theta}{2} \sin \frac{\theta}{2} \, d\theta \\ &= 16\pi a^2 \left[-\frac{2}{5} \cos^5 \frac{\theta}{2} \right]_0^\pi = \frac{32}{5} \pi a^2. \end{aligned}$$

Also $V = \iiint r^2 \sin \theta \, d\theta \, d\phi \, dr$,

the limits for r being 0 to $a(1 + \cos \theta)$,

ϕ from 0 to 2π ,

θ from 0 to π .

Hence
$$\begin{aligned} \bar{V} &= \frac{2\pi a^3}{3} \int_0^\pi (1 + \cos \theta)^3 \sin \theta \, d\theta \\ &= \frac{2\pi a^3}{3} \left[-\frac{(1 + \cos \theta)^4}{4} \right]_0^\pi = \frac{8}{3} \pi a^3. \quad (\text{See Art. 751, Ex. 3.}) \end{aligned}$$

804. Tetrahedral Volume.

An expression for the evaluation of a volume for a surface given by a tetrahedral equation may be obtained in the same

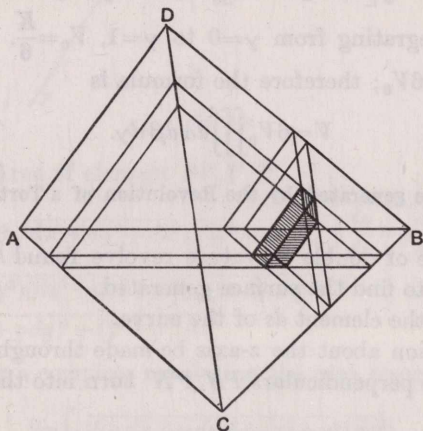


Fig. 282.

way as that adopted for an area in areal coordinates (Art. 461).

For let V_0 be the volume of the tetrahedron of reference, and let $\alpha, \beta, \gamma, \delta$ be the tetrahedral coordinates of a point P ,

and x, y, z be their Cartesian equivalents with reference to some given rectangular system of axes; then x, y and z are linear functions of α, β and γ , for we have $\alpha + \beta + \gamma + \delta = 1$.

$$\text{Hence} \quad V = \iiint dx dy dz = K \iiint d\alpha d\beta d\gamma,$$

where K is some determinate constant (Art. 794).

To determine K , apply the formula to the fundamental tetrahedron itself. If we integrate first with regard to α for the tube bounded by two given planes β and $\beta + \delta\beta$, and two planes γ and $\gamma + \delta\gamma$, keeping β and γ constant, the limits for α will be from the point at which this tube cuts the plane $\alpha = 0$ to the point in which it cuts $\delta = 0$, *i.e.* from $\alpha = 0$ to $\alpha = 1 - \beta - \gamma$. Then we have

$$V_0 = K \iint (1 - \beta - \gamma) d\beta d\gamma.$$

Next, integrating this with respect to β , keeping γ constant, the limits for β will be from $\beta = 0$ to the point where $\alpha = 0$ and $\delta = 0$, *i.e.* where $\beta = 1 - \gamma$, and

$$V_0 = K \int \left[\beta - \frac{\beta^2}{2} - \gamma\beta \right]_0^{1-\gamma} d\gamma = K \int \frac{(1-\gamma)^2}{2} d\gamma.$$

Lastly, integrating from $\gamma = 0$ to $\gamma = 1$, $V_0 = \frac{K}{6}$.

Hence $K = 6V_0$; therefore the formula is

$$V = 6V_0 \iiint d\alpha d\beta d\gamma.$$

805. Surface generated by the Revolution of a Tortuous Curve about an Axis.

Let a curve of double curvature revolve round the z -axis; it is required to find the surface generated.

Let PP' be the element ds of the curve.

Let revolution about the z -axis be made through the angle $d\theta$, and let the perpendiculars $PN, P'N'$ turn into the positions $P_1N, P_1'N'$.

Then

$$PP_1 = NP d\theta,$$

$$P'P_1' = N'P' d\theta = NP d\theta$$

to the first order, and $NP = \sqrt{x^2 + y^2}$, and the area of the element $PP_1P_1'P'$ is $NP d\theta \cdot ds \sin \chi$ to the second order, where χ

is the angle between P_1P and P_1P_1' , i.e. between directions whose direction cosines are

$$\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \quad \text{and} \quad \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0.$$

Hence $\cos \chi = \left(x \frac{dy}{ds} - y \frac{dx}{ds} \right) / \sqrt{x^2+y^2}$

and $\sin \chi = \sqrt{(x^2+y^2) - \left(x \frac{dy}{ds} - y \frac{dx}{ds} \right)^2} / \sqrt{x^2+y^2}.$

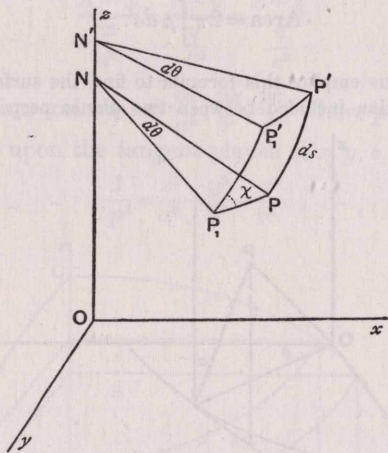


Fig. 283.

Hence Area of element $PP_1P_1'P'$

$$\begin{aligned} &= \sqrt{x^2+y^2} d\theta \sqrt{dx^2+dy^2+dz^2} \sqrt{(x^2+y^2) - \left(x \frac{dy}{ds} - y \frac{dx}{ds} \right)^2} / \sqrt{x^2+y^2} \\ &= d\theta \sqrt{(x^2+y^2)(dx^2+dy^2+dz^2) - (x dy - y dx)^2} \\ &= d\theta \sqrt{(x dx + y dy)^2 + (x^2+y^2) dz^2}. \end{aligned}$$

Hence, for a complete revolution the area traced out is

$$2\pi \int \sqrt{\{(x dx + y dy)^2 + (x^2+y^2) dz^2\}},$$

or in cylindricals, $(\rho, \phi, z),$

$$= 2\pi \int \rho \sqrt{d\rho^2 + dz^2}.$$

That is the area of the surface described is the same as would be traced out by a rotation about the z -axis through the same angle, of a new plane curve constructed by first swinging back each point of the tortuous curve from its actual position without alteration of its distance from the axis of rotation into a corresponding position upon the initial plane.

And if ds' be an elementary arc of this new curve,

$$ds'^2 = d\rho^2 + dz^2,$$

and therefore

$$\text{Area} = 2\pi \int \rho ds'.$$

806. Ex. Let us employ this formula to find the surface of a hyperboloid of revolution included between two planes perpendicular to the

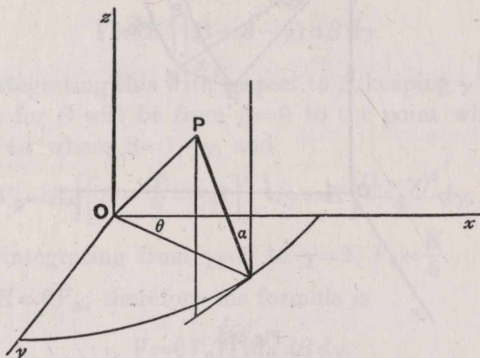


Fig. 284.

axis, the surface being regarded as generated by the revolution of a straight line about the axis, which we take as the z -axis, the line making a constant angle with the z -axis and not cutting it. The equations of the line are

$$x = a \cos \theta - z \tan a \sin \theta,$$

$$y = a \sin \theta + z \tan a \cos \theta.$$

Hence

$$x^2 + y^2 = a^2 + z^2 \tan^2 a$$

and

$$x dx + y dy = z dz \tan^2 a;$$

$$\therefore S = 2\pi \int \sqrt{z^2 dz^2 \tan^4 a + (a^2 + z^2 \tan^2 a) dz^2}$$

$$= 2\pi \int \sqrt{a^2 + z^2 \tan^2 a} \sec^2 a dz$$

$$= 2\pi \tan a \sec a \int \sqrt{z^2 + a^2 \frac{\cos^4 a}{\sin^2 a}} dz.$$

Hence

$$S = \pi \tan \alpha \sec \alpha \left[z \sqrt{z^2 + a^2 \frac{\cos^4 \alpha}{\sin^2 \alpha}} + \frac{a^2 \cos^4 \alpha}{\sin^2 \alpha} \sinh^{-1} \frac{z \sin \alpha}{a \cos^2 \alpha} \right]_{z_1}^{z_2}$$

807. Case of an Annular Element of Surface. Surface of the Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad (a > b > c).$$

Legendre's Formula.

The equations of the normal at x, y, z are

$$\frac{X-x}{\frac{x}{a^2}} = \frac{Y-y}{\frac{y}{b^2}} = \frac{Z-z}{\frac{z}{c^2}},$$

and its direction cosines are $\frac{px}{a^2}, \frac{py}{b^2}, \frac{pz}{c^2}$, where p is the central perpendicular upon the tangent planes at x, y, z , viz. such that

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}.$$

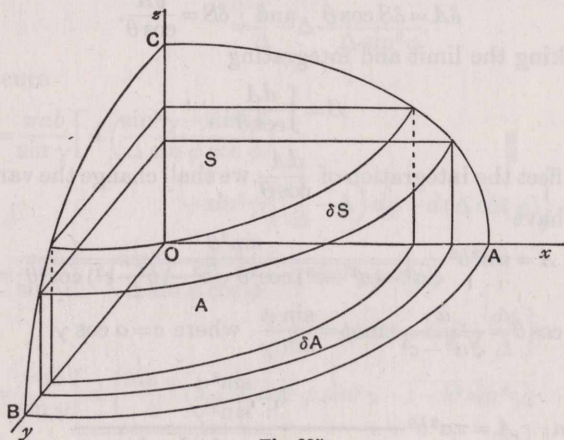


Fig. 285.

Let a cone be drawn whose vertex is at the origin O , and cutting the ellipsoid at all those points at which the normal makes a constant angle θ with the z -axis. Its equation is

$$\frac{pz}{c^2} = \cos \theta \quad \text{or} \quad \frac{z^2}{c^4 \cos^2 \theta} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}.$$

Let S be the area of the ellipsoidal cap cut off by this cone.

If we eliminate z between the equation of the cone and the equation of the ellipsoid, we obtain the projection of this curve of intersection upon the plane of xy , viz.

$$\frac{\sec^2 \theta}{c^2} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{1}{c^2} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right)$$

$$\text{or } \frac{x^2}{a^4 \sin^2 \theta} (a^2 \sin^2 \theta + c^2 \cos^2 \theta) + \frac{y^2}{b^4 \sin^2 \theta} (b^2 \sin^2 \theta + c^2 \cos^2 \theta) = 1,$$

viz. an ellipse of area

$$A = \pi a^2 b^2 \frac{\sin^2 \theta}{\sqrt{a^2 \sin^2 \theta + c^2 \cos^2 \theta} \sqrt{b^2 \sin^2 \theta + c^2 \cos^2 \theta}}.$$

If we increase θ to $\theta + \delta\theta$, we increase S and A respectively to $S + \delta S$ and $A + \delta A$. Now δA , the difference between the areas of two ellipses, is the projection of δS upon the x - y plane. And when $\delta\theta$ is indefinitely small, all elements of δS cut off by contiguous meridian planes make the same angle θ with their projections, which are the corresponding elements of δA .

Hence

$$\delta A = \delta S \cos \theta \quad \text{and} \quad \delta S = \frac{\delta A}{\cos \theta},$$

and taking the limit and integrating

$$S = \int \frac{dA}{\cos \theta}.$$

To effect the integration of $\frac{dA}{\cos \theta}$, we shall change the variable.

We have

$$A = \pi a^2 b^2 \frac{\sin^2 \theta}{\sqrt{a^2 - (a^2 - c^2) \cos^2 \theta} \sqrt{b^2 - (b^2 - c^2) \cos^2 \theta}}.$$

$$\text{Put } \cos \theta = \frac{a}{\sqrt{a^2 - c^2}} \sin \phi = \frac{\sin \phi}{\sin \gamma}, \quad \text{where } c = a \cos \gamma.$$

$$\text{Then } A = \pi a^2 b^2 \frac{1 - \frac{\sin^2 \phi}{\sin^2 \gamma}}{a \cos \phi \cdot b \sqrt{1 - \frac{a^2}{b^2} \frac{b^2 - c^2}{a^2 - c^2} \sin^2 \phi}}$$

$$= \frac{\pi a b}{\sin^2 \gamma} \frac{\sin^2 \gamma - \sin^2 \phi}{\cos \phi \sqrt{1 - k^2 \sin^2 \phi}}$$

$$= \frac{\pi a b}{\sin^2 \gamma} \frac{\sin^2 \gamma - \sin^2 \phi}{\cos \phi \Delta}, \quad \text{where } k^2 = \frac{a^2 (b^2 - c^2)}{b^2 (a^2 - c^2)}$$

which is < 1 , and $\Delta^2 = 1 - k^2 \sin^2 \phi$.

$$\begin{aligned}
\text{And } dS &= \sin \gamma \frac{dA}{\sin \phi} \\
&= \sin \gamma \left[d \frac{A}{\sin \phi} + \frac{A \cos \phi}{\sin^2 \phi} d\phi \right] \\
&= \sin \gamma \left[d \frac{A}{\sin \phi} + \frac{\pi ab}{\sin^2 \gamma} \frac{\sin^2 \gamma - \sin^2 \phi}{\Delta \cos \phi} \cdot \frac{\cos \phi}{\sin^2 \phi} d\phi \right] \\
&= \frac{\pi ab}{\sin \gamma} \left[d \left(\frac{\sin^2 \gamma - \sin^2 \phi}{\Delta \sin \phi \cos \phi} \right) + \frac{\sin^2 \gamma - \sin^2 \phi}{\Delta \sin^2 \phi} d\phi \right] \\
&= \frac{\pi ab}{\sin \gamma} \left[d \left(\frac{\sin^2 \gamma - \sin^2 \phi}{\Delta \sin \phi \cos \phi} \right) + \frac{\sin^2 \gamma}{\Delta \sin^2 \phi} d\phi - \frac{d\phi}{\Delta} \right] \dots (1)
\end{aligned}$$

$$\begin{aligned}
\text{Now } \frac{d}{d\phi} (\Delta \cot \phi) &= -\frac{k^2 \cos^2 \phi}{\Delta} \frac{\Delta}{\sin^2 \phi} \\
&= \frac{-k^2 + 1 - \Delta^2}{\Delta} \frac{1 - k^2 \sin^2 \phi}{\Delta \sin^2 \phi} \\
&= \frac{1}{\Delta} - \Delta - \frac{1}{\Delta \sin^2 \phi}.
\end{aligned}$$

Hence

$$\begin{aligned}
dS &= \frac{\pi ab}{\sin \gamma} \left[d \left(\frac{\sin^2 \gamma - \sin^2 \phi}{\Delta \sin \phi \cos \phi} \right) \right. \\
&\quad \left. + \sin^2 \gamma \left\{ \left(\frac{1}{\Delta} - \Delta \right) d\phi - d(\Delta \cot \phi) \right\} - \frac{d\phi}{\Delta} \right] \\
&= \frac{\pi ab}{\sin \gamma} \left[d \left(\frac{\sin^2 \gamma - \sin^2 \phi}{\Delta \sin \phi \cos \phi} - \Delta \sin^2 \gamma \cot \phi \right) \right. \\
&\quad \left. - \sin^2 \gamma \Delta d\phi - \cos^2 \gamma \frac{d\phi}{\Delta} \right] \\
&= \frac{\pi ab}{\sin \gamma} \left[d \left\{ \frac{\tan \phi}{\Delta} (1 - k^2 \sin^2 \phi \sin^2 \gamma - 1 - k^2 \sin^2 \gamma) \right. \right. \\
&\quad \left. \left. - \sin^2 \gamma \Delta d\phi - \cos^2 \gamma \frac{d\phi}{\Delta} \right\} \right],
\end{aligned}$$

$$\text{where } 1 - k^2 \sin^2 \gamma = 1 - \frac{a^2(b^2 - c^2)}{b^2(a^2 - c^2)} \cdot \frac{a^2 - c^2}{a^2} = \frac{c^2}{b^2},$$

and the limits for θ are 0 to $\frac{\pi}{2}$ for the upper half of the ellipsoid, and the consequent limits for ϕ are γ to 0, and double to take in the lower half of the surface.

Thus for the whole surface

$$S = \frac{2\pi ab}{\sin \gamma} \left[\frac{\tan \phi}{\Delta} \left(\overline{1 - k^2 \sin^2 \phi} \cdot \sin^2 \gamma - \frac{c^2}{b^2} \right) \right]_0^\gamma \\ + \frac{2\pi ab}{\sin \gamma} \left[\sin^2 \gamma \int_0^\gamma \Delta d\phi + \cos^2 \gamma \int_0^\gamma \frac{d\phi}{\Delta} \right] \\ = 2\pi c^2 + \frac{2\pi ab}{\sin \gamma} [\sin^2 \gamma E(\gamma, k) + \cos^2 \gamma F(\gamma, k)],$$

where $\cos \gamma = \frac{c}{a}$, a form due to Legendre.*

808. Cases.

In the case of the oblate spheroid, $a = b$, $k = 1$, and the elliptic functions degenerate,

$$E \text{ becoming } \int_0^\gamma \sqrt{1 - \sin^2 \phi} d\phi = \sin \gamma$$

and $F \text{ becoming } \int_0^\gamma \frac{d\phi}{\cos \phi} = \log \tan \left(\frac{\gamma}{2} + \frac{\pi}{4} \right),$

giving $S = 2\pi c^2 + \frac{2\pi a^2}{\sin \gamma} \left[\sin^3 \gamma + \cos^2 \gamma \log \tan \left(\frac{\gamma}{2} + \frac{\pi}{4} \right) \right] \\ = 2\pi a^2 + \frac{2\pi c^2}{\sin \gamma} \log \tan \left(\frac{\gamma}{2} + \frac{\pi}{4} \right),$

and for the prolate spheroid $b = c$, $k = 0$, $E = \gamma$ and $F = \gamma$, giving

$$S = 2\pi c^2 + \frac{2\pi ab}{\sin \gamma} \cdot \gamma \\ = 2\pi c^2 \left(1 + \frac{\gamma}{\sin \gamma \cos \gamma} \right)$$

or $\frac{2\pi ac}{\sin \gamma} (\gamma + \sin \gamma \cos \gamma).$

809. Another Method for the Surface of an Ellipsoid.

From the formula $S = \int \frac{1}{\cos \theta} dA$

we may deduce another form of expression for the area of an ellipsoid. Substituting the value of dA , we have

$$dS = \pi a^2 b^2 \frac{1}{\cos \theta} d \frac{1}{\sqrt{(a^2 + c^2 \cot^2 \theta)(b^2 + c^2 \cot^2 \theta)}}.$$

Put $\cot \theta = \frac{\sqrt{\lambda}}{c}.$

* See Serret, *Calcul Intégral*, pages 338-342; Legendre, *Exercices du Calcul Intégral*, p. 193.

Then

$$\begin{aligned} \frac{dS}{\pi a^2 b^2} &= \frac{\sqrt{\lambda+c^2}}{\sqrt{\lambda}} d \frac{1}{\sqrt{(a^2+\lambda)(b^2+\lambda)}} \\ &= -\frac{\sqrt{\lambda+c^2}}{\sqrt{\lambda}} \left[\frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} \right] \frac{d\lambda}{2\sqrt{(a^2+\lambda)(b^2+\lambda)}} \\ &= -(\lambda+c^2) \left(\frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} \right) \frac{d\lambda}{2\sqrt{\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \\ &= -\frac{c^2}{2} \left(\frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} + \frac{1}{c^2+\lambda} \right) \frac{d\lambda}{\sqrt{\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \\ &\quad - \frac{1}{2} \left(\frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} + \frac{1}{c^2+\lambda} - \frac{1}{\lambda} \right) \frac{\lambda d\lambda}{\sqrt{\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}, \end{aligned}$$

and the limits of integration for the upper half of the ellipsoid are $\theta=0$ to $\theta=\frac{\pi}{2}$, i.e. $\lambda=\infty$ to $\lambda=0$. The result must be doubled to include the lower half of the surface.

$$\begin{aligned} \text{Now } \int_0^\infty \left(\frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} + \frac{1}{c^2+\lambda} - \frac{1}{\lambda} \right) \frac{\sqrt{\lambda} d\lambda}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \\ = -2 \int_0^\infty \frac{d}{d\lambda} \frac{\sqrt{\lambda}}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} d\lambda \\ = -2 \left[\frac{\sqrt{\lambda}}{\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}} \right]_0^\infty = 0. \end{aligned}$$

Hence

(See Art. 363, Ex. 5.)

$$S = \pi a^2 b^2 c^2 \int_0^\infty \left(\frac{1}{a^2+\lambda} + \frac{1}{b^2+\lambda} + \frac{1}{c^2+\lambda} \right) \frac{d\lambda}{\sqrt{\lambda(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}}, *$$

for the whole area of the surface of the ellipsoid.

810. We now revert to the consideration of the generalised system of orthogonal coordinates discussed in Art. 789.

It will be remembered that we there obtained expressions for the direction cosines of the elements $\frac{\delta\lambda}{h_1}, \frac{\delta\mu}{h_2}, \frac{\delta\nu}{h_3}$ in terms of partial differential coefficients of x, y, z with regard to λ, μ, ν .

We may also readily express the same direction cosines in terms of partial differential coefficients of λ, μ, ν with regard to x, y, z .

* Mathematical Tripos, 1896.

Regard $\frac{\delta\lambda}{h_1}$, $\frac{\delta\mu}{h_2}$, $\frac{\delta\nu}{h_3}$ as the directions of a new set of three coordinate axes OA , OB , OC .

Referred to such axes the direction cosines of the original axes are :

for Ox ; l_1, l_2, l_3 ,

for Oy ; m_1, m_2, m_3 ,

for Oz ; n_1, n_2, n_3 ;

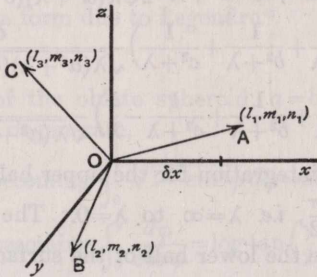


Fig. 286.

then $l_1\delta x$ is the projection of δx upon OA

= a small element on OA due to an increase of x to $x + \delta x$, y and z remaining unaltered,

$$= \frac{1}{h_1} \frac{\partial \lambda}{\partial x} \delta x.$$

Similarly $l_2\delta x = \frac{1}{h_2} \frac{\partial \mu}{\partial x} \delta x$ and $l_3\delta x = \frac{1}{h_3} \frac{\partial \nu}{\partial x} \delta x$,

$$m_1\delta y = \frac{1}{h_1} \frac{\partial \lambda}{\partial y} \delta y, \text{ etc. ;}$$

and we have the system of equations

$$\begin{aligned} l_1 &= \frac{1}{h_1} \frac{\partial \lambda}{\partial x}, & m_1 &= \frac{1}{h_1} \frac{\partial \lambda}{\partial y}, & n_1 &= \frac{1}{h_1} \frac{\partial \lambda}{\partial z}, \\ l_2 &= \frac{1}{h_2} \frac{\partial \mu}{\partial x}, & m_2 &= \frac{1}{h_2} \frac{\partial \mu}{\partial y}, & n_2 &= \frac{1}{h_2} \frac{\partial \mu}{\partial z}, \\ l_3 &= \frac{1}{h_3} \frac{\partial \nu}{\partial x}, & m_3 &= \frac{1}{h_3} \frac{\partial \nu}{\partial y}, & n_3 &= \frac{1}{h_3} \frac{\partial \nu}{\partial z}; \end{aligned}$$

whence it follows that J' , i.e. $\frac{\partial(\lambda, \mu, \nu)}{\partial(x, y, z)}$,

$$= h_1 h_2 h_3 \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm h_1 h_2 h_3,$$

which might have been anticipated from the theorem $JJ'=1$ (*Diff. Calc.*, Art. 540).

We thus have the following relations between the several partial differential coefficients, by comparing with Art. 789, viz.

$$h_1^2 \frac{\partial x}{\partial \lambda} = \frac{\partial \lambda}{\partial x}, \quad h_1^2 \frac{\partial y}{\partial \lambda} = \frac{\partial \lambda}{\partial y}, \quad h_1^2 \frac{\partial z}{\partial \lambda} = \frac{\partial \lambda}{\partial z},$$

$$h_2^2 \frac{\partial x}{\partial \mu} = \frac{\partial \mu}{\partial x}, \quad h_2^2 \frac{\partial y}{\partial \mu} = \frac{\partial \mu}{\partial y}, \quad h_2^2 \frac{\partial z}{\partial \mu} = \frac{\partial \mu}{\partial z},$$

$$h_3^2 \frac{\partial x}{\partial \nu} = \frac{\partial \nu}{\partial x}, \quad h_3^2 \frac{\partial y}{\partial \nu} = \frac{\partial \nu}{\partial y}, \quad h_3^2 \frac{\partial z}{\partial \nu} = \frac{\partial \nu}{\partial z},$$

$$h_1^2 = \lambda x^2 + \lambda y^2 + \lambda z^2 = h_1^4 (x\lambda^2 + y\lambda^2 + z\lambda^2),$$

and

$$\frac{1}{h_1^2} = x\lambda^2 + y\lambda^2 + z\lambda^2.$$

Similarly

$$\frac{1}{h_2^2} = x\mu^2 + y\mu^2 + z\mu^2,$$

$$\frac{1}{h_3^2} = x\nu^2 + y\nu^2 + z\nu^2.$$

811. It is plain that the areas of the three faces of the elementary cuboid which lie on the surfaces $\lambda = \text{const.}$, $\mu = \text{const.}$, $\nu = \text{const.}$, are respectively

$$\frac{\delta\mu \delta\nu}{h_2 h_3}, \quad \frac{\delta\nu \delta\lambda}{h_3 h_1}, \quad \frac{\delta\lambda \delta\mu}{h_1 h_2},$$

and that the infinitesimal distance between x, y, z and $x + \delta x, y + \delta y, z + \delta z$, viz. the diagonal through P of the elementary cuboid, is

$$\delta s^2 = \delta x^2 + \delta y^2 + \delta z^2 = \frac{\delta\lambda^2}{h_1^2} + \frac{\delta\mu^2}{h_2^2} + \frac{\delta\nu^2}{h_3^2}.$$

[See Todhunter, *Functions of Laplace, Lamé and Bessel*, pages 210-233; also E. J. Routh, *Anal. Statics*, vol. ii., Arts. 109, 110.]

812. **Elliptic Coordinates.**

The most remarkable case of these orthogonal surfaces is that of the three confocal conicoids, ($a > b > c$),

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1, \quad \frac{x^2}{a^2 + \mu} + \frac{y^2}{b^2 + \mu} = 1, \quad \frac{x^2}{a^2 + \nu} + \frac{y^2}{b^2 + \nu} = 1,$$

viz. an ellipsoid, a hyperboloid of one sheet and a hyperboloid of two sheets respectively, so that λ is $\leq -c^2$, μ between $-c^2$ and $-b^2$, and ν between $-b^2$ and $-a^2$.

To express x, y and z in terms of the parameters λ, μ, ν , we resort to a well-known algebraical device, viz.

Consider the equality

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1 + \frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(a^2+\theta)(b^2+\theta)(c^2+\theta)}, \dots (A)$$

where x, y, z have the values obtained from the above equations. This is either an equation to find θ , or it is an identity true for all values of θ .

If an equation, it is of quadratic nature; for θ^3 disappears upon multiplying up by $(a^2+\theta)(b^2+\theta)(c^2+\theta)$. Hence it could not be satisfied by more values of θ than two. This equality, however, is obviously satisfied by $\theta = \lambda, \theta = \mu$ and $\theta = \nu$, i.e. more than two values. Hence it is not an equation, but an identity and true for all values of θ .

Multiply then by $\theta + a^2$.

$$x^2 \equiv \left(1 - \frac{y^2}{b^2+\theta} - \frac{z^2}{c^2+\theta}\right)(a^2+\theta) + \frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(b^2+\theta)(c^2+\theta)}.$$

In this identity put $\theta = -a^2$; hence

$$x^2 = \frac{(\lambda+a^2)(\mu+a^2)(\nu+a^2)}{(a^2-b^2)(a^2-c^2)}.$$

Similarly

$$y^2 = \frac{(\lambda+b^2)(\mu+b^2)(\nu+b^2)}{(b^2-c^2)(b^2-a^2)}$$

and

$$z^2 = \frac{(\lambda+c^2)(\mu+c^2)(\nu+c^2)}{(c^2-a^2)(c^2-b^2)}.$$

Hence

$$2x \frac{\partial x}{\partial \lambda} = \frac{(\mu+a^2)(\nu+a^2)}{(a^2-b^2)(a^2-c^2)} = \frac{x^2}{a^2+\lambda},$$

that is

$$2 \frac{\partial x}{\partial \lambda} = \frac{x}{a^2+\lambda},$$

and similarly

$$2 \frac{\partial y}{\partial \lambda} = \frac{y}{b^2+\lambda}, \quad 2 \frac{\partial z}{\partial \lambda} = \frac{z}{c^2+\lambda}.$$

Again, if we differentiate the identity (A) with regard to θ , we obtain another identity, viz.

$$\begin{aligned} \frac{x^2}{(a^2+\theta)^2} + \frac{y^2}{(b^2+\theta)^2} + \frac{z^2}{(c^2+\theta)^2} &= \frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(a^2+\theta)(b^2+\theta)(c^2+\theta)} \\ &\times \left[\frac{1}{\lambda-\theta} + \frac{1}{\mu-\theta} + \frac{1}{\nu-\theta} + \frac{1}{a^2+\theta} + \frac{1}{b^2+\theta} + \frac{1}{c^2+\theta} \right], \end{aligned}$$

and putting $\theta = \lambda$ in this result,

$$4 \left[\left(\frac{\partial x}{\partial \lambda} \right)^2 + \left(\frac{\partial y}{\partial \lambda} \right)^2 + \left(\frac{\partial z}{\partial \lambda} \right)^2 \right] = \frac{(\lambda - \mu)(\lambda - \nu)}{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)},$$

i.e.
$$h_1^2 = 4 \frac{\Delta_\lambda}{(\lambda - \mu)(\lambda - \nu)},$$

where $\Delta_\lambda \equiv (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)$, $\Delta_\mu \equiv$ etc., $\Delta_\nu \equiv$ etc.

Hence

$$h_1 = \frac{2}{\sqrt{(\nu - \lambda)(\lambda - \mu)}} \sqrt{-\Delta_\lambda}, \quad h_2 = \frac{2}{\sqrt{(\lambda - \mu)(\mu - \nu)}} \sqrt{-\Delta_\mu},$$

$$h_3 = \frac{2}{\sqrt{(\mu - \nu)(\nu - \lambda)}} \sqrt{-\Delta_\nu}.$$

We thus have for an expression for a volume divided up into elementary cuboids defined by the faces of the three confocals λ, μ, ν , and the three contiguous confocals

$$\lambda + \delta\lambda, \quad \mu + \delta\mu, \quad \nu + \delta\nu,$$

$$V = \iiint \frac{d\lambda \, d\mu \, d\nu}{h_1 h_2 h_3} = \frac{1}{8} \iiint \frac{(\mu - \nu)(\nu - \lambda)(\lambda - \mu)}{\sqrt{-\Delta_\lambda \Delta_\mu \Delta_\nu}} d\lambda \, d\mu \, d\nu$$

813. In case of integration throughout the volume contained by the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the limits are: for λ , from $\lambda = 0$ to $\lambda = -c^2$;
 for μ , from $\mu = -c^2$ to $\mu = -b^2$;
 for ν , from $\nu = -b^2$ to $\nu = -a^2$.

814. If any function $F(x, y, z)$ is to be integrated through any specific region bounded, say, by confocals $\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2$, we must convert F into a function of λ, μ, ν by substituting for x, y, z their values, obtaining, say, $F_1(\lambda, \mu, \nu)$, and then the required summation will be

$$\frac{1}{8} \int_{\lambda_1}^{\lambda_2} \int_{\mu_1}^{\mu_2} \int_{\nu_1}^{\nu_2} F_1(\lambda, \mu, \nu) \frac{(\mu - \nu)(\nu - \lambda)(\lambda - \mu)}{\sqrt{-\Delta_\lambda \Delta_\mu \Delta_\nu}} d\lambda \, d\mu \, d\nu.$$

815. For instance, if the function to be integrated be

$$F_1(\lambda, \mu, \nu) \equiv \frac{\sqrt{-\Delta_\lambda \Delta_\mu \Delta_\nu}}{(\mu - \nu)(\nu - \lambda)(\lambda - \mu)} \lambda \mu \nu,$$

we have

$$I = \frac{1}{8} \int_{\lambda_1}^{\lambda_2} \int_{\mu_1}^{\mu_2} \int_{\nu_1}^{\nu_2} \lambda \mu \nu \, d\lambda \, d\mu \, d\nu$$

$$= \frac{1}{84} (\lambda_2^2 - \lambda_1^2) (\mu_2^2 - \mu_1^2) (\nu_2^2 - \nu_1^2).$$

816. In particular we may gather from the known volume of an ellipsoid, viz. $\frac{4}{3}\pi abc$, that the value of the definite integral

$$\int_0^{-c^2} \int_{-c^2}^{-b^2} \int_{-b^2}^{-a^2} \frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{\sqrt{-\Delta_\lambda \Delta_\mu \Delta_\nu}} d\lambda d\mu d\nu \text{ is } \frac{32}{3}\pi abc.$$

817. The elements of surface of the three confocals at a point of intersection are respectively

$$dS_1 = \frac{\delta\mu \delta\nu}{h_2 h_3} = \frac{1}{4}(\mu-\nu) \frac{\sqrt{(\nu-\lambda)(\lambda-\mu)}}{\sqrt{\Delta_\mu \Delta_\nu}},$$

$$dS_2 = \frac{\delta\nu \delta\lambda}{h_3 h_1} = \frac{1}{4}(\nu-\lambda) \frac{\sqrt{(\lambda-\mu)(\mu-\nu)}}{\sqrt{\Delta_\nu \Delta_\lambda}},$$

$$dS_3 = \frac{\delta\lambda \delta\mu}{h_1 h_2} = \frac{1}{4}(\lambda-\mu) \frac{\sqrt{(\mu-\nu)(\nu-\lambda)}}{\sqrt{\Delta_\lambda \Delta_\mu}}.$$

818. We may thus, for instance, express the area of any portion of the ellipsoid $\lambda=0$, bounded by confocals $\mu_1, \mu_2, \nu_1, \nu_2$, as

$$S = \frac{1}{4} \int_{\mu_1}^{\mu_2} \int_{\nu_1}^{\nu_2} (\mu-\nu) \sqrt{-\frac{\mu\nu}{\Delta_\mu \Delta_\nu}} d\mu d\nu.$$

819. The distance δs from λ, μ, ν to $\lambda+\delta\lambda, \mu+\delta\mu, \nu+\delta\nu$ is given by

$$\begin{aligned} \delta s^2 &= \delta x^2 + \delta y^2 + \delta z^2 \\ &= \frac{\delta\lambda^2}{h_1^2} + \frac{\delta\mu^2}{h_2^2} + \frac{\delta\nu^2}{h_3^2} \\ &= \frac{1}{4} \left[\frac{(\nu-\lambda)(\lambda-\mu)}{-\Delta_\lambda} \delta\lambda^2 + + \right]. \end{aligned}$$

And

$$s = \frac{1}{2} \left[\int \left[\frac{(\lambda-\mu)(\lambda-\nu)}{\Delta_\lambda} d\lambda^2 + \frac{(\mu-\nu)(\mu-\lambda)}{\Delta_\mu} d\mu^2 + \frac{(\nu-\lambda)(\nu-\mu)}{\Delta_\nu} d\nu^2 \right]^{\frac{1}{2}} \right]$$

In the case where the line lies on the ellipsoid $\lambda=0$,

$$s = \frac{1}{2} \int \left\{ \frac{\mu(\mu-\nu)}{\Delta_\mu} d\mu^2 + \frac{\nu(\nu-\mu)}{\Delta_\nu} d\nu^2 \right\}^{\frac{1}{2}}.$$

And when the curve on the ellipsoid is further defined by a relation between μ and ν , further reduction may be effected. For instance, along the line of curvature which is the intersection of the intersection of $\lambda=0$ with $\mu=\text{const.}=\mu_0$, say,

$$s = \frac{1}{2} \int_{\nu_1}^{\nu_2} \sqrt{\frac{\nu(\nu-\mu_0)}{\Delta_\nu}} d\nu,$$

or writing

$$\nu + a^2 = \omega^2, \quad \mu_0 + a^2 = d^2, \quad a^2 - b^2 = b_1^2, \quad a^2 - c^2 = c_1^2,$$

we have
$$s = \int_{\omega_1}^{\omega_2} \sqrt{\frac{(\omega^2 - a^2)(\omega^2 - d^2)}{(\omega^2 - b_1^2)(\omega^2 - c_1^2)}} d\omega,$$

for the length of a specified arc of a specified line of curvature upon the ellipsoid.

820. If we write

$$\begin{aligned} \lambda + a^2 &= \lambda_1^2, & \mu + a^2 &= \mu_1^2, & \nu + a^2 &= \nu_1^2, \\ \lambda + b^2 &= \lambda_1^2 - b_1^2, & \mu + b^2 &= \mu_1^2 - b_1^2, & \nu + b^2 &= \nu_1^2 - b_1^2, \\ \lambda + c^2 &= \lambda_1^2 - c_1^2, & \mu + c^2 &= \mu_1^2 - c_1^2, & \nu + c^2 &= \nu_1^2 - c_1^2, \end{aligned}$$

the conicoids become

$$\begin{aligned} \frac{x^2}{\lambda_1^2} + \frac{y^2}{\lambda_1^2 - b_1^2} + \frac{z^2}{\lambda_1^2 - c_1^2} &= 1, & \frac{x^2}{\mu_1^2} + \frac{y^2}{\mu_1^2 - b_1^2} + \frac{z^2}{\mu_1^2 - c_1^2} &= 1, \\ \frac{x^2}{\nu_1^2} + \frac{y^2}{\nu_1^2 - b_1^2} + \frac{z^2}{\nu_1^2 - c_1^2} &= 1, \end{aligned}$$

and we have a certain amount of simplification of the formulae, but with a loss of symmetry.*

Thus we obtain

$$\begin{aligned} x^2 &= \frac{\lambda_1^2 \mu_1^2 \nu_1^2}{b_1^2 c_1^2}, & y^2 &= \frac{(\lambda_1^2 - b_1^2)(\mu_1^2 - b_1^2)(\nu_1^2 - c_1^2)}{(b_1^2 - c_1^2) b_1^2}, \\ z^2 &= \frac{(\lambda_1^2 - c_1^2)(\mu_1^2 - c_1^2)(\nu_1^2 - c_1^2)}{(c_1^2 - b_1^2) c_1^2}, \end{aligned}$$

$$V = \iiint \frac{(\mu_1^2 - \nu_1^2)(\nu_1^2 - \lambda_1^2)(\lambda_1^2 - \mu_1^2) d\lambda_1 d\mu_1 d\nu_1}{\sqrt{(\lambda_1^2 - b_1^2)(\lambda_1^2 - c_1^2)(\mu_1^2 - b_1^2)(\mu_1^2 - c_1^2)(\nu_1^2 - b_1^2)(\nu_1^2 - c_1^2)}};$$

and for the volume of the ellipsoid

$$\frac{x^2}{\lambda_1^2} + \frac{y^2}{\lambda_1^2 - b_1^2} + \frac{z^2}{\lambda_1^2 - c_1^2} = 1,$$

the limits are: for λ_1 , from c_1 to λ_1 ;
 for μ_1 , from b_1 to c_1 ;
 for ν_1 , from 0 to b_1 .

Hence it follows that the value of the definite integral

$$\int_{c_1}^{\lambda_1} \int_{b_1}^{c_1} \int_0^{b_1} \frac{(\mu_1^2 - \nu_1^2)(\nu_1^2 - \lambda_1^2)(\lambda_1^2 - \mu_1^2) d\lambda_1 d\mu_1 d\nu_1}{\sqrt{(\lambda_1^2 - b_1^2)(\lambda_1^2 - c_1^2)(\mu_1^2 - b_1^2)(\mu_1^2 - c_1^2)(\nu_1^2 - b_1^2)(\nu_1^2 - c_1^2)}}$$

is
$$\frac{1}{8} \cdot \frac{4}{3} \pi \lambda_1 \sqrt{\lambda_1^2 - b_1^2} \sqrt{\lambda_1^2 - c_1^2},$$

being an octant of the ellipsoid.

*This is the notation adopted by Todhunter, *Functions of Laplace, Lamé and Bessel*; Bertrand, *Calc. Int.*

The suffix has been retained to prevent misconception as to the meanings of the several letters, but may now be dropped. For this and the values of other definite integrals of similar nature, see Todhunter, *Functions of Laplace, Lamé and Bessel*, Chapter XXI.

821. Solid Angle.

Let C be any closed curve, plane or twisted, bounding any region upon a surface, O a fixed point, and S a sphere of unit radius, with centre O . Let a cone with vertex O and generators passing through the perimeter of C , isolate on the unit sphere an area ω . Then ω is called the "solid angle" subtended at O by the portion of surface bounded by C .

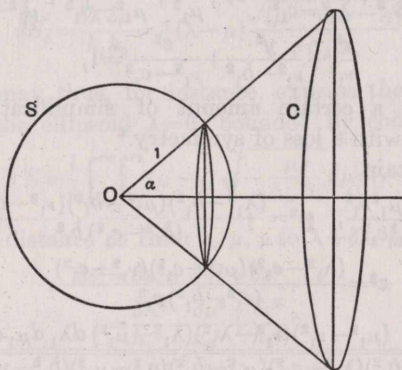


Fig. 287.

The area of a sphere being $4\pi \times (\text{radius})^2$, it follows that the solid angle subtended by any closed surface at a point within it is 4π ; at a point upon it which is not a singularity, 2π ; at a point outside, 0. The solid angle subtended at a corner of a cube by the rest of the cube is $\frac{4\pi}{8} = \frac{\pi}{2}$. At a point on the line of intersection of two planes cutting at right angles, each of the regions into which space is divided by the two planes subtends a solid angle $\frac{4\pi}{4} = \pi$. At the vertex of a right circular cone of semivertical angle a , the solid angle is the area of the portion of unit sphere, centre at the vertex, cut off by the cone, *i.e.* $2\pi \cdot 1 \cdot (1 - \cos a)$, *i.e.* $2\pi \text{ vers } a$.

A circular disc of radius a subtends at a point O on the axis whose distance from the plane of the disc is h , a solid angle

$$2\pi \left[1 - \cos \left(\tan^{-1} \frac{a}{h} \right) \right] = 2\pi \left(1 - \frac{h}{\sqrt{a^2 + h^2}} \right).$$

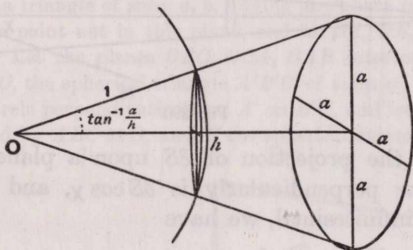


Fig. 288.

822. In the spherical polar system of coordinates, the face of the elementary cuboid $r^2 \sin \theta \delta \theta \delta \phi \delta r$, which is at right

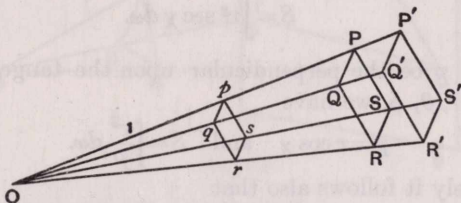


Fig. 289.

angles to the radius vector, is $r^2 \sin \theta \delta \theta \delta \phi$, and if $\delta \omega$ be the solid angle subtended at the origin O , we have

$$\frac{r^2 \sin \theta \delta \theta \delta \phi}{\delta \omega} = \frac{r^2}{1^2},$$

i.e. the area $pqrs$, viz. $\delta \omega$, intercepted upon unit sphere by radii vectores to the boundary of the element whose face is $PQRS$, viz. $r^2 \sin \theta \delta \theta \delta \phi$, is given by

$$\delta \omega = \sin \theta \delta \theta \delta \phi.$$

The element of volume $r^2 \sin \theta \delta \theta \delta \phi \delta r$ may therefore be written as $r^2 \delta \omega \delta r$, and

$$V = \iiint r^2 d\omega dr = \frac{1}{3} \int r^3 d\omega.$$

In the case of the sphere r is constant, and

$$= \frac{1}{3} r^3 \cdot 4\pi = \frac{4}{3} \pi r^3.$$

823. Let the inward drawn normal at any point of a closed surface make an angle χ with the radius vector r to the point, and let δS be an element of the surface about the

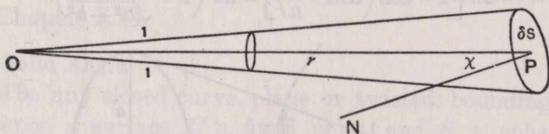


Fig. 290.

point; then the projection of δS upon a plane cutting the radius vector perpendicularly is $\delta S \cos \chi$, and in the limit when δS is infinitesimal, we have

$$\frac{\delta S \cos \chi}{\delta \omega} = \frac{r^2}{1^2} \quad \text{or} \quad \delta S = r^2 \sec \chi \delta \omega,$$

to the second order; whence

$$S = \int r^2 \sec \chi d\omega.$$

Also, if p be the perpendicular upon the tangent plane at the point r , θ , ϕ , we have

$$p = r \cos \chi \quad \text{and} \quad S = \int \frac{r^3}{p} d\omega.$$

Obviously it follows also that

$$\int \frac{\cos \chi}{r^2} dS = \int d\omega = \omega,$$

and if the closed surface surrounds the pole O , this gives

$$\int \frac{\cos \chi}{r^2} dS = 4\pi.$$

If O lies at a point on the surface where there is no singularity,

$$\int \frac{\cos \chi}{r^2} dS = 2\pi.$$

If O lies outside the closed surface,

$$\int \frac{\cos \chi}{r^2} dS = 0.$$

If O lies at a conical point of solid angle ω ,

$$\int \frac{\cos \chi}{r^2} dS = \omega.$$

These theorems are of great importance in the theory of attractions, and are due to Gauss. (See E. J. Routh, *Anal. Statics*, vol. ii., Art. 106.)

324. Solid angle subtended by a triangle at a point not in its plane.

Let ABC be a triangle of sides a, b, c lying anywhere in a given plane XY , let O be a point not in this plane, and let OA, OB, OC be respectively p, q, r . Let the planes OBC, OCA, OAB intercept on the unit sphere, centre O , the spherical triangle $A'B'C'$ of sides a', b', c' , and let p' be the great circle perpendicular from A' on $B'C'$, and let ω be the solid angle subtended by ABC at O , and E' the spherical excess of the triangle $A'B'C'$.

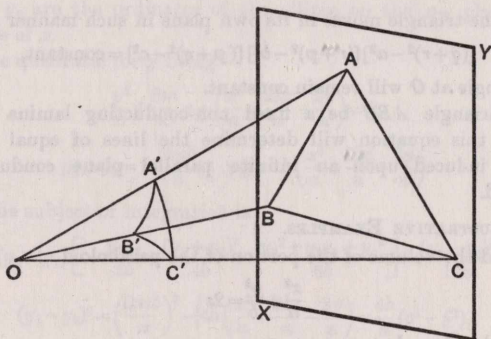


Fig. 291.

Then ω is measured by the area of $A'B'C'$, i.e.

$$\omega = E' = A' + B' + C' - \pi.$$

Hence it appears that triangles bounded by planes such that the sum of the angles between them is constant subtend the same solid angle at O .

Cagnoli's theorem gives

$$\sin \frac{E'}{2} = \frac{\sqrt{\sin s' \sin (s' - a') \sin (s' - b') \sin (s' - c')}}{2 \cos \frac{a'}{2} \cos \frac{b'}{2} \cos \frac{c'}{2}},$$

or, which is the same thing,

$$= \frac{\sin a' \sin b' \sin c'}{4 \cos \frac{a'}{2} \cos \frac{b'}{2} \cos \frac{c'}{2}}.$$

[Todhunter and Leathem, *Spherical Trigonometry*, Art. 132.]

Now let the volume of the tetrahedron $OABC$ be called V ; then

$$\frac{1}{3} \cdot \frac{1}{2} qr \sin a' \cdot p \sin p' = V,$$

i.e.

$$pqr \sin a' \sin b' \sin c' = \text{constant} = 6V.$$

Again,

$$q^2 + r^2 - a^2 = 2qr \cos \alpha',$$

i.e.

$$(q+r)^2 - a^2 = 4qr \cos^2 \frac{\alpha'}{2},$$

and if Π^2 represent $[(q+r)^2 - a^2][(r+p)^2 - b^2][(p+q)^2 - c^2]$, we have

$$\Pi^2 = 64p^2q^2r^2 \cos^2 \frac{\alpha'}{2} \cos^2 \frac{b'}{2} \cos^2 \frac{c'}{2} \quad \text{and} \quad \Pi = 8pqr \cos \frac{\alpha'}{2} \cos \frac{b'}{2} \cos \frac{c'}{2}.$$

Hence
$$\sin \frac{\omega}{2} = 12 \frac{V}{\Pi}.$$

Also, if h be the distance of O from the plane of ABC and Δ the area of the triangle,

$$V = \frac{1}{3}h\Delta \quad \text{and} \quad \sin \frac{\omega}{2} = 4h \frac{\Delta}{\Pi}.$$

If then the triangle moves in its own plane in such manner as to make

$$[(q+r)^2 - a^2][(r+p)^2 - b^2][(p+q)^2 - c^2] = \text{constant},$$

the solid angle at O will remain constant.

If the triangle ABC be a fixed non-conducting lamina uniformly electrified, this equation will determine the lines of equal density of electricity induced upon an infinite parallel plane conducting and uninsulated.

825. ILLUSTRATIVE EXAMPLES.

1. To find the volume of the portion of the paraboloid

$$\frac{x^2}{a} + \frac{y^2}{b} = 2z$$

cut off by the plane $lx + my + nz = p$.

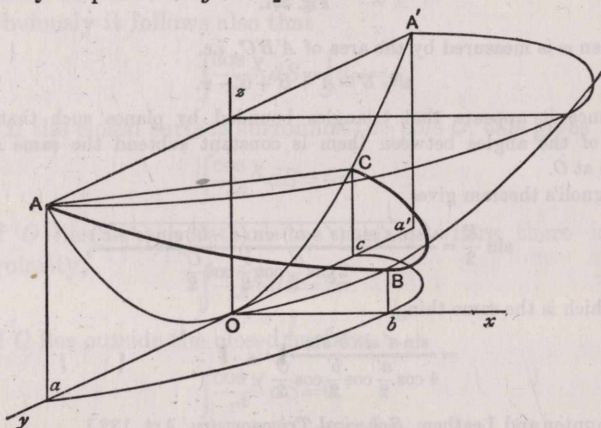


Fig. 292.

The difference of the z -ordinates of the plane and the paraboloid is

$$\zeta = \frac{p - lx - my}{n} - \frac{1}{2} \left(\frac{x^2}{a} + \frac{y^2}{b} \right).$$

3. To calculate the value of $\iiint \phi\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dx dy dz$, the integrations being conducted through the volume of the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

Take

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \delta.$$

The volume of the ellipsoidal shell bounded by the similar ellipsoids δ and $\delta + d\delta$ is

$$d\left(\frac{4}{3}\pi abc\delta^{\frac{3}{2}}\right) = 2\pi abc\delta^{\frac{1}{2}} d\delta,$$

and $\phi(\delta)$ is constant throughout this shell.

$$\text{Hence } \iiint \phi\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dx dy dz = 2\pi abc \int_0^1 \phi(\delta) \sqrt{\delta} d\delta$$

4. Find the mass of a thick focaloid,* i.e. a shell bounded by confocal ellipsoids, the layers of equal density being confocal surfaces, and the density at each point inversely proportional to the volume contained by the confocal through the point.

Let $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1$ be the confocal through the point, and let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1$$

be the outer and inner surfaces of the shell.

The volume contained by the ellipsoid λ is

$$V = \frac{4}{3}\pi \sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}.$$

The volume of the layer between the surfaces λ and $\lambda + d\lambda$ is

$$dV = \frac{4}{3}\pi \sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)} \left(\frac{1}{a^2 + \lambda} + \frac{1}{b^2 + \lambda} + \frac{1}{c^2 + \lambda} \right) d\lambda.$$

The law of density is

$$\rho = k / \frac{4}{3}\pi \sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}, \quad k \text{ being a constant.}$$

Hence the mass of the layer is

$$\rho dV = \frac{k}{2} \left(\frac{1}{a^2 + \lambda} + \frac{1}{b^2 + \lambda} + \frac{1}{c^2 + \lambda} \right) d\lambda,$$

and the mass of the thick shell is

$$\begin{aligned} M &= \int_{a'^2 - a^2}^0 \rho dV = \left[\frac{k}{2} \log(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda) \right]_{a'^2 - a^2}^0 \\ &= \frac{k}{2} \log a^2 b^2 c^2 - \frac{k}{2} \log a'^2 (b^2 + a'^2 - a^2)(c^2 + a'^2 - a^2) \\ &= k \log \frac{abc}{a'b'c'}, \quad \text{for } a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2; \end{aligned}$$

and if D be the density of the outer layer,

$$D = \frac{k}{\frac{4}{3}\pi abc}.$$

$$\text{Hence } M = \frac{4}{3}\pi abc D \log \frac{abc}{a'b'c'}.$$

*For this term see remarks by E. J. Routh, *Anal. Statics*, vol. ii., p. 97, and Thomson and Tait's *Natural Philosophy*.

The area of such a section is

$$\frac{\pi\sqrt{ab}}{n^3}(al^2 + bm^2 + 2pn)$$

[C. Smith, *Solid Geometry*, p. 99.]

The thickness of a slice is δp .

The slice of zero area is such that

$$al^2 + bm^2 + 2p_1n = 0,$$

p_1 being the corresponding value of p .

The limits of integration with respect to p are from p_1 to p .

$$\begin{aligned} \text{Hence } V &= \frac{\pi\sqrt{ab}}{n^3} \int_{p_1}^p (al^2 + bn^2 + 2pn) dp \\ &= \frac{\pi\sqrt{ab}}{n^3} \left[(al^2 + bn^2)(p - p_1) + n(p^2 - p_1^2) \right] \\ &= \frac{\pi\sqrt{ab}}{n^3} \left[-2p_1n(p - p_1) + n(p^2 - p_1^2) \right] \\ &= \frac{\pi\sqrt{ab}}{n^3} [p - p_1]^2 n = \frac{\pi\sqrt{ab}}{n^2} (p - p_1)^2 \\ &= \frac{\pi\sqrt{ab}}{4n^4} (2pn + al^2 + bn^2)^2, \end{aligned}$$

as before.

We may note that frusta of finite thickness whose bases are parallel to a given plane are such that their volumes vary as the squares of their thicknesses; also that frusta of given thickness are such that their volumes vary as the squares of the secants of the angles which the normals to their bases make with the axis of the paraboloid.

2. To calculate the value of $\iiint \phi(lx + my + nz) dx dy dz$, the integrations being conducted through the volume of the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1,$$

l, m, n being such that $l^2 + m^2 + n^2 = 1$.

Let $lx + my + nz = \delta$.

The area of this section of the ellipsoid is

$$A = \frac{\pi abc}{p} \left(1 - \frac{\delta^2}{p^2} \right),$$

where $p^2 = a^2l^2 + b^2m^2 + c^2n^2$.

Consider the ellipsoid divided into thin slices parallel to this plane. The volume of such a slice is $A d\delta$ to the first order, $d\delta$ being the thickness of the slice, and $\phi(\delta)$ is, to the first order, constant through the slice.

Hence

$$\iiint \phi(lx + my + nz) dx dy dz = \frac{\pi abc}{p} \int_{-p}^p \phi(\delta) \left(1 - \frac{\delta^2}{p^2} \right) d\delta.$$

3. To calculate the value of $\iiint \phi\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dx dy dz$, the integrations being conducted through the volume of the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$$

Take
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \delta.$$

The volume of the ellipsoidal shell bounded by the similar ellipsoids δ and $\delta + d\delta$ is

$$d\left(\frac{4}{3}\pi abc\delta^{\frac{3}{2}}\right) = 2\pi abc\delta^{\frac{1}{2}} d\delta,$$

and $\phi(\delta)$ is constant throughout this shell.

Hence
$$\iiint \phi\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) dx dy dz = 2\pi abc \int_0^1 \phi(\delta) \sqrt{\delta} d\delta$$

4. Find the mass of a thick focaloid,* i.e. a shell bounded by confocal ellipsoids, the layers of equal density being confocal surfaces, and the density at each point inversely proportional to the volume contained by the confocal through the point.

Let $\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1$ be the confocal through the point, and let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1$$

be the outer and inner surfaces of the shell.

The volume contained by the ellipsoid λ is

$$V = \frac{4}{3}\pi \sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}.$$

The volume of the layer between the surfaces λ and $\lambda + d\lambda$ is

$$dV = \frac{4}{3}\pi \sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)} \left(\frac{1}{a^2 + \lambda} + \frac{1}{b^2 + \lambda} + \frac{1}{c^2 + \lambda} \right) d\lambda.$$

The law of density is

$$\rho = k/\frac{4}{3}\pi \sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}, \quad k \text{ being a constant.}$$

Hence the mass of the layer is

$$\rho dV = \frac{k}{2} \left(\frac{1}{a^2 + \lambda} + \frac{1}{b^2 + \lambda} + \frac{1}{c^2 + \lambda} \right) d\lambda,$$

and the mass of the thick shell is

$$\begin{aligned} M &= \int_{a'^2 - a^2}^0 \rho dV = \left[\frac{k}{2} \log (a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda) \right]_{a'^2 - a^2}^0 \\ &= \frac{k}{2} \log a^2 b^2 c^2 - \frac{k}{2} \log a'^2 (b^2 + a'^2 - a^2)(c^2 + a'^2 - a^2) \\ &= k \log \frac{abc}{a'b'c'}, \quad \text{for } a^2 - a'^2 = b^2 - b'^2 = c^2 - c'^2; \end{aligned}$$

and if D be the density of the outer layer,

$$D = \frac{k}{\frac{4}{3}\pi abc}.$$

Hence
$$M = \frac{4}{3}\pi abc D \log \frac{abc}{a'b'c'}.$$

*For this term see remarks by E. J. Routh, *Anal. Statics*, vol. ii., p. 97, and Thomson and Tait's *Natural Philosophy*.

5. Consider the region bounded by

- (1) a sphere $x^2 + y^2 + z^2 = a^2$;
- (2) a right circular cylinder $x^2 + y^2 = bx$ ($a < b$);
- (3) the two planes $y = \pm x \tan \alpha$.

We shall first find the volume enclosed by these surfaces in the positive octant of space.

Take cylindrical coordinates r, θ, z .

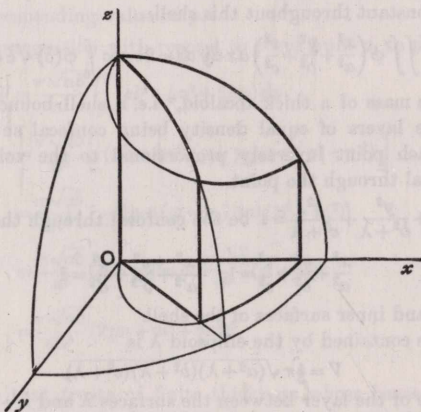


Fig. 293.

The elementary prism on base $r \delta\theta \delta r$ has volume $rz \delta\theta \delta r$ to the second order, and

$$\begin{aligned} V &= \iint r z \, d\theta \, dr \\ &= \iint r \sqrt{a^2 - r^2} \, d\theta \, dr \\ &= -\frac{1}{3} \int [(a^2 - r^2)^{\frac{3}{2}}] \, d\theta, \end{aligned}$$

and the equation of the trace of the cylinder upon the xy plane being $r = b \cos \theta$, the limits for r are 0 to $b \cos \theta$, whilst the limits for θ are from $\theta = 0$ to $\theta = \alpha$.

Hence

$$\begin{aligned} V &= \frac{1}{3} \int_0^\alpha \{a^3 - (a^2 - b^2 \cos^2 \theta)^{\frac{3}{2}}\} \, d\theta \\ &= \frac{1}{3} a^3 \alpha - \frac{a^3}{3} \int_0^\alpha \left(1 - \frac{b^2}{a^2} \cos^2 \theta\right)^{\frac{3}{2}} \, d\theta. \end{aligned}$$

Writing $\theta = \frac{\pi}{2} - \phi$ and $\alpha = \frac{\pi}{2} - \beta$ in the integral,

$$V = \frac{1}{3} a^3 \left(\frac{\pi}{2} - \beta\right) - \frac{a^3}{3} \int_\beta^{\frac{\pi}{2}} \Delta^{\frac{3}{2}} \, d\phi;$$

where

$$\Delta^2 = \left(1 - \frac{b^2}{a^2} \sin^2 \phi\right),$$

$$V = \frac{a^3}{3} \left(\frac{\pi}{2} - \beta\right) - \frac{a^3}{3} \left\{ \left(\int_0^{\frac{\pi}{2}} - \int_0^{\beta} \right) \Delta^3 \phi \, d\phi \right\},$$

and by Legendre's formula (No. 10, p. 399),

$$\int_0^{\phi} \Delta^3 d\phi = \frac{k^2}{3} \Delta \sin \phi \cos \phi + \frac{4-2k^2}{3} E - \frac{k^2}{3} F;$$

and if E_1, F_1 be the real quarter periods, we have

$$V = \frac{a^3}{3} \left(\frac{\pi}{2} - \beta\right) - \frac{a^3}{3} \left\{ -\frac{1}{3} \frac{b^2}{a^2} \sin \beta \cos \beta \sqrt{1 - \frac{b^2}{a^2} \sin^2 \beta} \right. \\ \left. + \frac{4a^2 - 2b^2}{3a^2} (E_1 - E) - \frac{a^2 - b^2}{3a^2} (F_1 - F) \right\},$$

where $E = \int_0^{\beta} \sqrt{1 - \frac{b^2}{a^2} \sin^2 \phi} \, d\phi$ and $F = \int_0^{\beta} \frac{d\phi}{\sqrt{1 - \frac{b^2}{a^2} \sin^2 \phi}}$.

And for the *whole* volume of the sphere included between the specified boundaries, we have four times this quantity.

When the cylinder just touches the sphere, *i.e.* $b=a$, the elliptic functions degenerate.

We then have for the volume in the positive octant

$$V = \frac{a^3}{3} \int_0^{\alpha} (1 - \sin^3 \theta) \, d\theta \\ = \frac{a^3}{3} \int_0^{\alpha} \left(1 - \frac{3 \sin \theta - \sin 3\theta}{4}\right) d\theta \\ = \frac{a^3}{12} [4\alpha - 3(1 - \cos \alpha) + \frac{1}{3}(1 - \cos 3\alpha)] \\ = \frac{a^3}{36} (12\alpha - 9 \operatorname{vers} \alpha + \operatorname{vers} 3\alpha);$$

and in the case where the planes $y = \pm x \tan \alpha$ coincide with the $y-z$ plane, *i.e.* $\alpha = \frac{\pi}{2}$, the whole volume cut out of the sphere by the cylinder $r = a \cos \theta$ is

$$4V = \frac{a^3}{9} (6\pi - 8) = \frac{2a^3}{9} (3\pi - 4).$$

To find the surface of the sphere thus bounded in the positive octant, we have

$$S = \iint \sec \gamma \cdot r \, d\theta \, dr,$$

γ being as usual the angle the normal to the sphere at r, θ, z makes with the z -axis; that is $\cos \gamma = \frac{z}{a} = \frac{\sqrt{a^2 - r^2}}{a}$.

Hence

$$\begin{aligned} S &= \iint \frac{ar}{\sqrt{a^2 - r^2}} d\theta dr \\ &= -a \int_0^a \left[\sqrt{a^2 - r^2} \right]_0^{b \cos \theta} d\theta \\ &= a \int_0^a \{a - \sqrt{a^2 - b^2 \cos^2 \theta}\} d\theta ; \end{aligned}$$

and putting as before $\theta = \frac{\pi}{2} - \phi$ and $a = \frac{\pi}{2} - \beta$,

$$\begin{aligned} S &= a^2 \left(\frac{\pi}{2} - \beta \right) + a^2 \int_{\frac{\pi}{2}}^{\beta} \sqrt{1 - \frac{b^2}{a^2} \sin^2 \phi} d\phi \\ &= a^2 \left(\frac{\pi}{2} - \beta \right) + a^2 \left(\int_0^{\beta} - \int_0^{\frac{\pi}{2}} \right) \sqrt{1 - \frac{b^2}{a^2} \sin^2 \phi} d\phi \\ &= a^2 \left(\frac{\pi}{2} - \beta \right) + a^2 \left\{ E \left(\beta, \frac{b}{a} \right) - E_1 \left(\frac{\pi}{2}, \frac{b}{a} \right) \right\} ; \end{aligned}$$

and when $b = a$, we have

$$\begin{aligned} S &= a^2 \int_0^a (1 - \sin \theta) d\theta \\ &= a^2 (a - \text{vers } a) ; \end{aligned}$$

and for the further particular case when $a = \frac{\pi}{2}$,

$$S = a^2 \left(\frac{\pi}{2} - 1 \right).$$

And in each case the *whole* of the surface of the sphere intercepted in this manner is four times the portion which has been found.

6. At every point of an elliptic lamina a straight line is drawn perpendicular to the plane of the lamina and of such length that the volume (μ , say) of the rectangular parallelepiped formed by this length and the distances of the point from the foci of the elliptic boundary is constant. Given that a and b are the semiaxes of the elliptic boundary, show that the volume of the solid thus formed is

$$\frac{\pi \mu}{4} \log \frac{a+b}{a-b}.$$

[COLLEGES, 1891.]

Taking $x + iy = c \cos(\theta + i\phi)$, we have

$$x = c \cos \theta \cosh \phi, \quad y = -c \sin \theta \sinh \phi,$$

and the loci $\phi = \text{constant}$, $\theta = \text{constant}$ are the confocal conics

$$\frac{x^2}{c^2 \cosh^2 \phi} + \frac{y^2}{c^2 \sinh^2 \phi} = 1 \quad \text{and} \quad \frac{x^2}{c^2 \cos^2 \theta} - \frac{y^2}{c^2 \sin^2 \theta} = 1,$$

and the focal radii r_1, r_2 are such that $r_1 + r_2 = 2c \cosh \phi$, $r_1 - r_2 = 2c \cos \theta$.

Let the elliptic area be divided up into elements by confocals in this way, taking the element bounded by $\theta, \theta + \delta\theta, \phi, \phi + \delta\phi$ as a type.

$$\text{Now} \quad \iint F(x, y) dx dy = \iint F_1(\theta, \phi) J d\theta d\phi,$$

where F_1 is the equivalent of F in terms of θ, ϕ .

Also
$$J = \frac{\partial(x, y)}{\partial(\theta, \phi)} = \begin{vmatrix} -c \sin \theta \cosh \phi, & c \cos \theta \sinh \phi \\ -c \cos \theta \sinh \phi, & -c \sin \theta \cosh \phi \end{vmatrix}$$

$$= c^2(\sin^2 \theta \cosh^2 \phi + \cos^2 \theta \sinh^2 \phi)$$

$$= c^2(\cosh^2 \phi - \cos^2 \theta)$$

$$= \frac{1}{4}\{(r_1 + r_2)^2 - (r_1 - r_2)^2\} = r_1 r_2$$

and by the condition of the question $\mu = \pi r_1 r_2$.

Thus

$$\text{Volume} = V = \iint z \, dx \, dy = \iint \frac{\mu}{r_1 r_2} \cdot r_1 r_2 \, d\theta \, d\phi = \mu [\theta] [\phi],$$

and the limits for θ are $\theta = 0$ to $\theta = \frac{\pi}{2}$, and for ϕ from $\phi = 0$ to the value for which $c \cosh \phi = a$ and $c \sinh \phi = b$, that is $\phi = \sinh^{-1} \frac{b}{\sqrt{a^2 - b^2}}$.

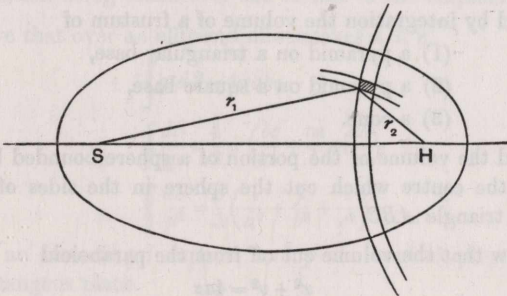


Fig. 294.

Thus

$$V = \mu \frac{\pi}{2} \sinh^{-1} \frac{b}{\sqrt{a^2 - b^2}} = \mu \frac{\pi}{2} \log \frac{b+a}{\sqrt{a^2 - b^2}} = \mu \frac{\pi}{4} \log \frac{a+b}{a-b}.$$

7. In the evaluation of such integrals as $I_n = \int \frac{dS}{p^n}$ taken over the surface of an ellipsoid of semi-axes a, b, c , where the surface is S and the volume V , p being the central perpendicular upon any tangent plane, consider three points P, Q, R on the surface, which are the extremities of three semi-conjugate diameters. Let $\delta S_1, \delta S_2, \delta S_3$ be any elements of the surface about the three points and p_1, p_2, p_3 the corresponding perpendiculars.

Then
$$I_n = \int \frac{dS_1}{p_1^n}, \text{ or } \int \frac{dS_2}{p_2^n}, \text{ or } \int \frac{dS_3}{p_3^n}$$

$$= \frac{1}{3} \int \left(\frac{dS_1}{p_1^n} + \frac{dS_2}{p_2^n} + \frac{dS_3}{p_3^n} \right).$$

Now suppose these elements of area $\delta S_1, \delta S_2, \delta S_3$ to have been so chosen that

$$\frac{\delta S_1}{p_1^{n-2}} = \frac{\delta S_2}{p_2^{n-2}} = \frac{\delta S_3}{p_3^{n-2}} = \frac{\delta S}{p^{n-2}}, \text{ say.}$$

Then, since $\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$,

we have $I_n = \frac{1}{3} \int \left(\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} \right) \frac{dS}{p^{n-2}}$,

i.e. $I_n = \frac{1}{3} \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) I_{n-2}$;

we also have $I_{-1} = \int p \, dS = 3V$, and $I_0 = S$;

whence we can readily infer the values of I_1, I_2, I_3 , etc., viz.

$$\frac{I_{2n}}{S} = \frac{I_{2n-1}}{3V} = \frac{1}{3^n} \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)^n.$$

PROBLEMS.

1. Find by integration the volume of a frustum of

- (1) a pyramid on a triangular base,
- (2) a pyramid on a square base,
- (3) a cone.

2. Find the volume of the portion of a sphere bounded by planes through the centre which cut the sphere in the sides of a given spherical triangle ABC .

3. Show that the volume cut off from the paraboloid

$$x^2 + y^2 = 4az$$

by the plane

$$x + y + z = a$$

is

$$18\pi a^3.$$

4. Show that the volume of the solid bounded by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^{2n}}{c^{2n}} = 1$$

is

$$\frac{4n}{2n+1} \pi abc.$$

5. Show that the volume bounded by the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \left(\frac{x}{a} + \frac{y}{b} \right) \frac{z^n}{c^n}$$

and the planes

$$z = 0, \quad z = h$$

is

$$\frac{2\pi abh}{2n+1} \left(\frac{h}{c} \right)^{2n}$$

6. Show that the volume of a slice of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

bounded by the parallel planes

$$lx + my + nz = \delta_1,$$

$$lx + my + nz = \delta_2,$$

is
$$\frac{\pi abc}{3p^3} (\delta_1 - \delta_2)(3p^2 - \delta_1^2 - \delta_1\delta_2 - \delta_2^2),$$

where p is the central perpendicular upon a tangent plane parallel to the faces of the slice.

7. If A be the area of a central section of an ellipsoid parallel to the tangent plane at the elementary area δS , show that

$$\int \frac{dS}{A} = 4,$$

the integration being taken over the surface of the ellipsoid.

8. Prove that over an ellipsoid of semiaxes a, b, c ,

$$\int p dS = 4\pi abc,$$

$$\int \frac{dS}{p} = \frac{4}{3} \pi \left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \right),$$

$$\int \frac{dS}{p^2} = \frac{S}{3} \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right),$$

dS being an element of surface, and p the central perpendicular upon the tangent plane.

Investigate also the value of $\int \frac{dS}{p^3}$.

9. Apply the formula $V = \frac{1}{3} \iiint (lx + my + nz) dS$ to find the volume of an ellipsoid, x, y, z being the coordinates of any point on the surface, and l, m, n the direction cosines of the normal there.

[COLLEGES *a*, 1881.]

10. If the ellipsoid of semiaxes a, b, c be very nearly spherical, then its area is, to the first order (inclusive) of the small quantities, represented by the difference of the axes

$$4\pi a^{\frac{2}{3}} b^{\frac{2}{3}} c^{\frac{2}{3}}.$$

[TRINITY, 1891.]

11. Show that a portion of a spherical surface (radius unity) may be bent into the surface of revolution defined by the equations

$$x = k \cos p \cos \frac{q}{k}, \quad y = k \cos p \sin \frac{q}{k}, \quad z = E(p, k) \left(= \int_0^p \sqrt{1 - k^2 \sin^2 p} dp \right);$$

and explain the geometrical theory, distinguishing the two cases $k < 1, k > 1$.

[MATH. TRIPOS, 1887.]

12. The curve $z=f(x)$, $y=0$ revolves about the axis of x , and the surface thus formed is intersected by the right cylinder $y=\phi(x)$, which is symmetrical with respect to the axis of x : prove that the cylinder cuts off from the first surface a portion the area of which can be determined by the evaluation of the integral

$$\int z \sqrt{1 + \left(\frac{dz}{dx}\right)^2} \sin^{-1} \frac{y}{z} dx$$

between proper limits.

[OXFORD II. P., 1888.]

13. Show that the cylinder $(x-c)^2 + y^2 = (a-c)^2$ cuts off from the sphere $x^2 + y^2 + z^2 = a^2$ a portion of which the area is

$$8a \left\{ a \cos^{-1} \left(c^{\frac{1}{2}} a^{-\frac{1}{2}} \right) - c^{\frac{1}{2}} (a-c)^{\frac{1}{2}} \right\},$$

a being supposed greater than c .

[OXFORD II. P., 1888.]

14. Prove that the volume cut off from the paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$$

by the plane

$$z = px + qy + r$$

is

$$\frac{\pi abc}{4} \left(\frac{a^2 p^2}{c^2} + \frac{b^2 q^2}{c^2} + \frac{2r}{c} \right)^2.$$

[OXFORD II. P., 1902.]

15. Show that the volume enclosed between the surface

$$z^2 \{ (x^2 + y^2 + c^2)^2 - 4c^2 x^2 \} = c^4 y^2$$

and the cylinder

$$x^2 + y^2 = c^2$$

is

$$(\pi - 2)c^3.$$

[OXFORD II. P., 1886.]

16. By application of the formulae $V = \frac{1}{3} \int p dS$, $V = \int z \cos \gamma dS$ to the evaluation of the volume of an ellipsoid, establish the results

$$(1) \int_b^c \int_0^b \frac{(\mu^2 - \nu^2) d\mu d\nu}{\sqrt{(b^2 - \nu^2)(c^2 - \nu^2)(c^2 - \mu^2)(\mu^2 - b^2)}} = \frac{\pi}{2},$$

$$(2) \int_b^c \int_0^b (\mu^2 - \nu^2) \frac{\sqrt{(c^2 - \mu^2)(c^2 - \nu^2)}}{\sqrt{(b^2 - \nu^2)(\mu^2 - b^2)}} d\mu d\nu = \frac{\pi}{6} c^2 (c^2 - b^2).$$

(See Art. 820 for the notation.)

[LAME.]

[TODHUNTER, *Functions of Laplace, Lamé and Bessel*, pages 216, 217 ;

BERTRAND, *Calc. Int.*, pages 424, 426.]

17. Show that the volume bounded by the surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \left(\frac{x}{a} + \frac{y}{b} \right) \phi(z)$$

and the planes

$$z = 0, \quad z = z_1$$

is

$$2\pi ab \int_0^{z_1} [\phi(z)]^2 dz.$$

18. A cavity is just large enough to allow of the complete revolution of a circular disc of radius c , whose centre describes a circle of the same radius c , while the plane of the disc is constantly parallel to a fixed plane, and perpendicular to that in which the centre moves. Show that the volume of the cavity is

$$\frac{2c^3}{3} (3\pi + 8).$$

19. If O be a point without a sphere of radius a and centre C , and r the distance of any point of the sphere from O , show that, integrating $\frac{1}{r^n}$ over the surface, we have

$$\int \frac{dS}{r^n} = \frac{2\pi}{n-2} \cdot \frac{a}{c} [(c-a)^{2-n} - (c+a)^{2-n}] \quad \text{if } n \neq 2,$$

and
$$2\pi \frac{a}{c} \log \frac{c-a}{c+a} \quad \text{if } n = 2.$$

What will be the results if O lies within the sphere?

20. A surface is obtained by making the diameter $2a$ of a semi-circle move parallel to itself, the path of the centre being perpendicular to the initial plane of the semicircle, whilst the plane of the semicircle rotates round the diameter; and when the plane has moved through an angle θ the distance which the diameter has moved is $c \sin \theta$. Prove that the volume of the whole surface so generated is

$$\frac{4}{3}\pi a^3 + \frac{1}{2}\pi^2 ca^2. \quad [\text{TRINITY, 1890.}]$$

21. Use the theorem

$$V = \iiint dx dy dz = \iiint J du dv dw$$

to find the volume of the parallelepiped enclosed by the planes

$$\begin{aligned} ax + by + cz = 0, & \quad a_1x + b_1y + c_1z = 0, & \quad a_2x + b_2y + c_2z = 0, \\ ax + by + cz = d, & \quad a_1x + b_1y + c_1z = d_1, & \quad a_2x + b_2y + c_2z = d_2. \end{aligned}$$

22. Prove that the area of that portion of the surface

$$(m^2 - 1)(x^2 + y^2) = z^2,$$

which is cut out by the surface

$$z = a^{-1}x^2 + b^{-1}y^2$$

where a and b are positive, is

$$\frac{\pi}{2} m (m^2 - 1) a^{\frac{1}{2}} b^{\frac{1}{2}} (a + b). \quad [\text{OXFORD II. P., 1890.}]$$

23. Show that when $f(x)$ is a slowly changing function,

$$\int_a^b f(x) dx$$

is approximately equal to

$$\frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right].$$

Prove that this formula may be used to calculate exactly the volume cut from a hyperboloid of one sheet by parallel planes meeting it in elliptic sections. [COLLEGES α , 1881.]

24. Prove that the volume included in the positive octant between the surface

$$z^2(x^2 + y^2 + a^2)^{2n+2}(x^2 + 2a^2)^{2n} = a^{8n+4}y^2$$

and the planes $x=0$, $x=\infty$, $y=a$, $y=\infty$

is

$$\frac{\pi a^3}{2^{n+1}n} \frac{1 \cdot 3 \cdot 5 \dots (4n-3)}{2 \cdot 4 \cdot 6 \dots (4n-2)},$$

n being a positive integer.

25. Show that the area of that part of the sphere $r=1$, enclosed by the cone $\tan \frac{\theta}{2} = \sqrt{3} \cos \phi$, is π . [COLLEGES α , 1881.]

26. Show that the volume of the solid, the equation to the surface of which is

$$z^2 + ax^2 + 2\delta xy + \beta y^2 = 2\mu z,$$

is

$$\frac{4\pi}{3} \frac{\mu^3}{\sqrt{a\beta - \delta^2}}. \quad [\text{COLLEGES, 1882.}]$$

27. If in the tangent plane at the vertex of a paraboloid two ellipses be described whose axes are in the principal sections and proportional to their parameters, the cylinders whose bases are these ellipses, and whose generators are parallel to the axis of the paraboloid, will intercept on the surface a portion whose area is proportional to the difference between the radii of curvature of either of the principal sections at the points where it intersects the bounding curve. [COLLEGES, 1892.]

28. If the density of a tetrahedron at any point vary as the n^{th} power of the sum of the distances of the point from the faces of the tetrahedron, show that the mass of the tetrahedron

$$= kV \frac{1 \cdot 2 \cdot 3}{(r+1)(r+2)(r+3)} \sum \frac{p_1^{r+3}}{(p_1-p_2)(p_1-p_3)(p_1-p_4)},$$

where V is the volume; p_1, p_2, p_3, p_4 are the perpendiculars from the corners upon the opposite faces, and k the density at the centroid of the volume.

Examine what happens in the case of a regular tetrahedron.

29. Find the volume contained between any two planes perpendicular to the axis of x and the surface whose equation is

$$(y^2 + z^2)^2 = (a^2 + \beta x)y^2 + (a'^2 + \beta' x)z^2. \quad [\text{ST. JOHN'S, 1884.}]$$

30. Show that the mass contained between a paraboloid of revolution and a sphere, with centre at the vertex and diameter $2a$, equal to the latus rectum of the paraboloid, where the density at any point varies as the square of the latus rectum of the paraboloid containing it and having the same vertex and axis as the bounding paraboloid, is

$$\frac{\pi}{15} (7 - 4\sqrt{2}) a^3 \rho,$$

where ρ is the density at the external surface of the paraboloid.

[COLLEGES δ , 1883.]

31. Find the volume between the surfaces

$$\begin{aligned} x(y^2 + z^2)^{\gamma} &= a_1, & y^2 + z^2 &= 4\beta_1 x, & y &= b_1 z, \\ x(y^2 + z^2)^{\gamma} &= a_2, & y^2 + z^2 &= 4\beta_2 x, & y &= b_2 z. \end{aligned}$$

[COLLEGES δ , 1881.]

32. Prove that if a, b, c be any positive quantities in descending order of magnitude, the solid angle of that part of the cone

$$ax^2z^2 + (by^2 - cz^2)(x^2 + y^2) = 0$$

which lies on the positive side of the plane xy is equal to

$$4 \sin^{-1} \left(\frac{c}{a} \right)^{\frac{1}{2}} - 4 \left(\frac{b}{a+b} \right)^{\frac{1}{2}} \sin^{-1} \left\{ \frac{c(a+b)}{a(c+b)} \right\}^{\frac{1}{2}}.$$

[COLLEGES β , 1891.]

33. Prove that the volume common to a sphere and a circular cylinder which touches it, and also passes through the centre, is $\frac{1}{2} - \frac{2}{3\pi}$ of the volume of the sphere.

[ST. JOHN'S, 1891.]

Also show that the sum of the two spherical caps cut off by the cylinder forms $\frac{1}{2} - \frac{1}{\pi}$ of the area of the sphere.

34. A sphere of radius a is cut by two diametral planes so as to form a lune of angle α , which is itself cut in two by a plane inclined at an angle β to its edge and passing through one end of it, and equally inclined to the two faces of the lune; show that the volume of the pointed part is

$$\frac{2}{3} a^3 \sin \beta \left\{ \left(2 + \cos^2 \beta \right) \tan^{-1} \left(\sin \beta \tan \frac{\alpha}{2} \right) + \frac{\sin \beta \cos^2 \beta \tan \frac{\alpha}{2}}{1 + \sin^2 \beta \tan^2 \frac{\alpha}{2}} \right\}.$$

[ST. JOHN'S, 1881.]

35. Prove that the moment of inertia about the axis of z of the part of the paraboloid $2z = ax^2 + by^2$, cut off by the plane

$$lx + my + nz = p, \text{ is}$$

$$\frac{\pi}{24n^6(ab)^{\frac{3}{2}}} (bl^2 + am^2 + 2pnab)^2 \{bl^2(a+7b) + am^2(7a+b) + 2pnab(a+b)\},$$

the density being taken as unity.

[MATH. TRIPOS, 1890.]

36. If $A + B + C = 0$ and the coordinate axes be rectangular, prove that

$$\iint \{(A, B, C, D, E, F \text{ \textcircled{ } } x, y, z)^2 \times (A', B', C', D', E', F' \text{ \textcircled{ } } x, y, z)^2\} d\omega \\ = \frac{8\pi}{15} (AA' + BB' + CC' + 2DD' + 2EE' + 2FF'),$$

where the integration extends over the whole surface of a sphere of unit radius whose centre is the origin of coordinates.

[COLLEGES, 1892.]

Also show that the unconditional result is

$$\frac{4\pi}{15} [A'(3A + B + C) + B'(3B + C + A) + C'(3C + A + B) \\ + 4DD' + 4EE' + 4FF'].$$

37. A flexible envelope is in the form of an oblate spheroid, such that e is the eccentricity of a meridian section: the part between two meridians, the planes of which are inclined to each other at the angle $2\pi(1 - e)$, is cut away, and the edges are then sewn together. Prove that the meridian curve of the new surface is the "curve of sines," and that the volume enclosed is changed in the ratio

$$3\pi e^2 : 8. \quad [\text{ST. JOHN'S, 1889.}]$$

38. A surface is such that $ABCD$ being any rectangle in the plane of x, y , with its sides parallel to Ox, Oy , and AP, BQ, CR, DS being drawn parallel to Oz to meet the surface in P, Q, R, S , the volume of the solid $ABCDPQRS$ is equal to the base $ABCD$, multiplied by the arithmetic mean of AP, BQ, CR, DS . Prove that the surface is a hyperbolic paraboloid.

[MATH. TRIPOS, 1876.]

39. Show that the integral

$$\iiint e^{\frac{x+y+z}{\sqrt{a^2+b^2+c^2}}} dx dy dz$$

taken over the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is

$$\frac{\pi abc}{4} (e^2 + 3e^{-2}).$$

[COLLEGES, 1885.]

Prove more generally that

$$\iint e^{k \frac{lx+my+nz}{\sqrt{a^2+b^2+c^2}}} dx dy dz$$

over the volume of the ellipsoid

$$= \frac{4\pi abc}{k^3} (k \cosh k - \sinh k),$$

and find the values of

$$\iiint e^x dx dy dz; \quad \iiint e^{x+y} dx dy dz; \quad \iiint e^{x+y+z} dx dy dz$$

through the same space.

40. On a closed oval surface of volume V and surface S , whose curvature is everywhere finite, rolls a sphere of radius a ; the surface of the envelope of the sphere is S' . Prove that the volume of the envelope is

$$V + a(S' + S) - \frac{1}{3}\pi a^3. \quad [\text{MATH. TRIPOS, 1886.}]$$

41. Show that the volume of the pedal of an ellipsoid taken with the centre as origin is less than that taken with regard to any other origin; and that the sum of the volumes of the pedals, taken with regard to the extremities of three semi-conjugate diameters, is six times that taken with regard to the centre. [MATH. TRIPOS, 1887.]

42. Show that the moment of inertia of the ellipsoid

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$$

about the axis of x is

$$\frac{1}{2}M(ca - g^2 + ab - h^2)(abc + 2fgh - af^2 - bg^2 - ch^2)^{-1},$$

where M is the mass of the ellipsoid.

[TRINITY, 1890.]

43. Find the envelope of the conics $x^2 \sec^3 \theta - y^2 \tan^3 \theta = a^2$, where θ is the variable parameter. Show that in addition to certain lines it consists of a curve whose asymptotes are $x = \pm a$. Also, if the area between the axis of x , an asymptote, and the corresponding branch of the curve be A , and the volume generated by the revolution of this branch about the axis of x be V , prove that

$$V = \pi a A = \frac{1}{2} \pi a^3 \int_0^{\frac{\pi}{2}} (\sin \phi)^{\frac{1}{2}} d\phi. \quad [\text{COLLEGES } \beta, 1890.]$$

44. Show that the value of

$$\iiint \frac{xyz dx dy dz}{\sqrt{x^2 + y^2 + z^2}},$$

taken throughout the positive octant of the ellipsoid

$$a^{-2}x^2 + b^{-2}y^2 + c^{-2}z^2 = 1$$

is

$$\frac{a^2b^2c^2}{15} \frac{bc + ca + ab}{(b+c)(c+a)(a+b)}. \quad [\text{OXFORD II. P., 1888.}]$$

45. Prove that the mass of a sphere of radius a , whose density at any point P is $\frac{k}{AP}$, where k is a constant and A is a fixed point distant f ($> a$) from the centre of the sphere, is equal to

$$\frac{4}{3} \frac{\pi k a^3}{f}. \quad [\text{OXF. I. P., 1914.}]$$

46. Prove that the volume which lies within the sphere

$$x^2 + y^2 + z^2 = a^2$$

and the ellipsoid

$$x^2 \sin^2 a \operatorname{cosec}^2 \beta + y^2 \cos^2 a \sec^2 \beta + z^2 = a^2,$$

where $0 < a < \beta < \frac{1}{2}\pi$, is

$$\frac{4}{3} a^3 (\pi - 2\beta + 2a \sin 2\beta \operatorname{cosec} 2a). \quad [\text{OXF. I. P., 1916.}]$$

47. P is a point of abscissa x (> 0) on the parabola

$$x^2 = 2ay, \quad z = 0,$$

and Sa^2 is the area of the segment bounded by the arc OP and the radius vector OP ; the straight line PQ of length $2Sa$ is drawn parallel to Oz . The locus of Q being a curve which passes through the origin, prove that

(1) the length of the arc OQ is $x + x^3/6a^2$;

(2) the cylindrical area bounded by the arcs OP , OQ and the straight line PQ is

$$a^2/45 + (3x^2 - 2a^2)(x^2 + a^2)^{3/2}/90a^3. \quad [\text{OXF. I. P., 1916.}]$$

48. Show that the two cylinders $x^2/a^2 + z^2/c^2 = 1$ and $y^2 = 2b(c - z)$ intercept on the plane $z = k$ (where $k^2 < c^2$), a rectangle of area

$$4a(1 - k/c)\sqrt{2b(c + k)}.$$

Show that the volume cut off from the cylinder $x^2/a^2 + z^2/c^2 = 1$ by the cylinder $y^2 = 2b(c - z)$ is

$$\frac{1}{15} \frac{28}{5} ac\sqrt{bc}. \quad [\text{OXF. I. P., 1917.}]$$

49. The sphere $x^2 + y^2 + z^2 = a^2$ is intersected by the cylinder

$$x^2 + y^2 = az.$$

Prove that the ratio of the spherical area cut off by the cylinder to the cylindrical area cut off by the sphere is

$$\pi - 2 : 2. \quad [\text{OXF. I. P., 1915.}]$$

50. Integrate $\int_0^1 \left[\int_0^{x^2} \frac{dy}{\sqrt{x^2 - y^2}} \right] dx.$ [Oxf. I. P., 1915.]

51. Find the value of $\iint \frac{dx dy}{(a^2 + x^2 + y^2)^p}$ taken all over the plane $x, y; p$ being greater than unity. [Oxf. I. P., 1915.]

52. Find the four points where any line parallel to the axis of z intersects the surface $(x^2 + y^2 + z^2)^2 = 4(a^2 z^2 + x^2 y^2).$

Prove that the volume enclosed by that part of the surface which lies above the plane $z = 0$ is $\frac{1}{3} a^3.$ [Oxf. II. P., 1915.]

53. If the coordinates of a point on a certain surface be expressed as

$$x = a \sin u, \quad y = a \sin v, \quad z = a \cos u + a \cos v,$$

prove that the area of the portion of the surface bounded by

$$u = 0, \quad u = \frac{1}{2}\pi, \quad v = 0, \quad v = \frac{1}{2}\pi,$$

is

$$\frac{\pi a^2}{4} \left(1 - \frac{c_2^3}{1} - \frac{c_4^3}{3} - \frac{c_6^3}{5} - \frac{c_8^3}{7} - \dots \right),$$

where

$$c_{2r} = \frac{(2r-1)(2r-3) \dots 1}{2r(2r-2) \dots 2}. \quad [\text{Oxf. II. P., 1915.}]$$