CHAPTER XXI.

VOLUMES AND SURFACES OF SOLIDS OF REVOLUTION, AND THEIR CENTROIDS.

745. Volumes.

Supposing the z-axis to be the axis of revolution, the typical equation of such a surface is

$$x^2+y^2=f(z).$$

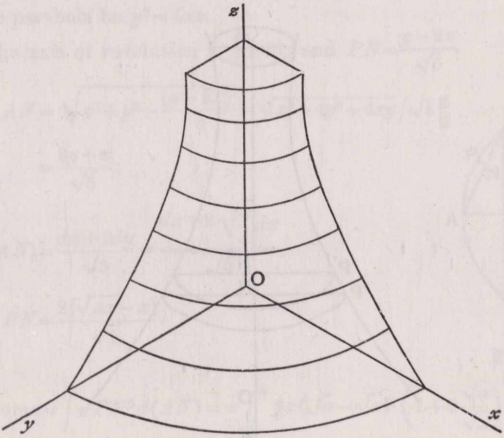

Fig. 250.

It is formed by the revolution about the z-axis of the curve $y^2=f(z)$ which lies in the y-z plane.

It was shown in Art. 24 that the solid contained by this surface and the planes $z=z_1$, $z=z_2$, is to be obtained by the formula

$$V=\int_{z_1}^{z_2}\pi y^2\,dz,$$

y being the perpendicular from any point of the revolving curve upon the axis of revolution.

It is obvious that if we regard the surface as defined by its three-dimension equation $x^2+y^2=f(z)$, we must replace the y^2 and the dx of Art. 12 by x^2+y^2 and dz respectively. The formula therefore will stand as

$$V=\pi\int_{z_1}^{z_2}(x^2+y^2)\,dz,$$

i.e.
$$\pi\int_{z_1}^{z_2}f(z)\,dz.$$

746. More generally, if the revolution be about any line AB in the plane of the curve, and if PN be any perpendicular drawn from a point P of the curve upon the line AB, and $P'N'$ be a contiguous perpendicular, the volume is expressed as

$$Lt_{NN'=0}\,\Sigma\pi PN^2\,.\,NN',$$

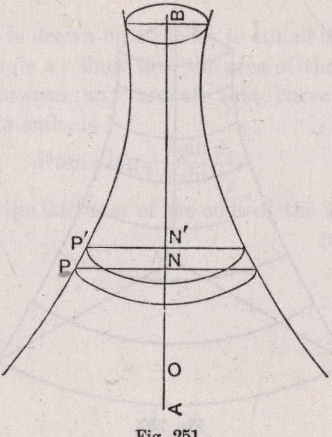

Fig. 251.

or if O be a given point on the line AB,

$$V=\int\pi PN^2\,d(ON),$$

the limits being the values of ON which mark the terminal planes of the solid formed.

747. Illustrative Examples.

1. Find the volume formed by the revolution of the loop of the curve $y^2=x^2\dfrac{a-x}{a+x}$ (Art. 403, Ex. 3) about the x-axis, *i.e.* the volume bounded by the closed portion of the surface

$$(y^2+z^2)(a+x)=x^2(a-x).$$

Here volume$=\pi\displaystyle\int_0^a x^2\frac{a-x}{a+x}\,dx.$

Putting $a+x=u$, this becomes

$$=\pi\int_a^{2a}\frac{(u-a)^2(2a-u)}{u}\,du$$

$$=\pi\int_a^{2a}\left(\frac{2a^3}{u}-5a^2+4au-u^2\right)du$$

$$=\pi\left[2a^3\log u-5a^2u+2au^2-\frac{u^3}{3}\right]_a^{2a}$$

$$=2\pi a^3[\log 2-\tfrac{2}{3}].$$

2. Find the volume of the spindle formed by the revolution of a parabolic arc about the line joining the vertex to one extremity of the latus rectum.

Let the parabola be $y^2=4ax$.

Then the axis of revolution is $y=2x$, and $PN=\dfrac{y-2x}{\sqrt{5}}.$

Also $AN=\sqrt{x^2+y^2-\dfrac{(y-2x)^2}{5}}=\sqrt{x^2+4y^2+4xy}\,/\sqrt{5}$

$$=\frac{2y+x}{\sqrt{5}};$$

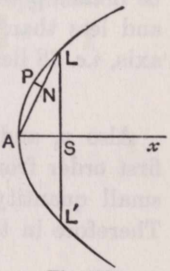

Fig. 252.

$$\therefore\ d(AN)=\frac{dx+2dy}{\sqrt{5}}=\frac{dx+2\sqrt{\dfrac{a}{x}}\,dx}{\sqrt{5}},$$

and $$PN=\frac{2(\sqrt{ax}-x)}{\sqrt{5}}.$$

Hence

$$\text{Volume}=\int\pi PN^2 d(AN)=\pi\int_0^a \tfrac{4}{5}x(\sqrt{a}-\sqrt{x})^2\left(1+2\frac{\sqrt{a}}{\sqrt{x}}\right)\frac{1}{\sqrt{5}}\,dx$$

$$=\frac{4\pi}{5\sqrt{5}}\times\frac{a^3}{6}=\frac{2\pi a^3\sqrt{5}}{75}.$$

748. Surfaces of Revolution.

Again, if S be the area of the curved surface of the solid traced out by the revolution of any arc AB about a given line XY in its plane, let PN, QM be two adjacent perpendiculars from points P, Q of the arc upon the axis of revolution, δs

the elementary arc PQ, δS the area of the elementary zone or belt traced out by the revolution of PQ about XY.

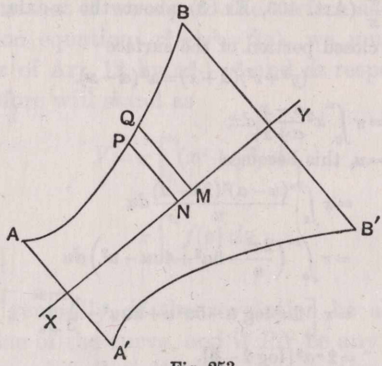

Fig. 253.

Let p_1 and p_2 be the greatest and the least of the perpendicular distances of points on the arc PQ from the axis of revolution. Then we may take it as axiomatic that the area traced out by PQ in its revolution is greater than it would be if each point of PQ were at the distance p_2 from the axis, and less than if each point were at a distance p_1 from the axis, *i.e.* δS lies between

$$2\pi p_1 \,\delta s \quad \text{and} \quad 2\pi p_2 \,\delta s.$$

Also p_1 and p_2 differ by a small quantity of at least the first order from PN. Hence $2\pi p_1 \,\delta s$ and $2\pi p_2 \,\delta s$ differ by a small quantity of at least the second order from $2\pi PN\,\delta s$. Therefore in the limit we have

$$\frac{dS}{ds} = 2\pi PN$$

or

$$S = \int 2\pi PN\, ds.$$

749. Various Forms of the Formula.

If the axis of revolution be the x-axis, this may be written as

$$S = \int 2\pi y\, ds, \quad \int 2\pi y \frac{ds}{dx} dx, \quad \int 2\pi y \frac{ds}{dy} dy,$$

$$\int 2\pi y \frac{ds}{d\theta} d\theta, \quad \int 2\pi y \frac{ds}{dr} dr, \text{ etc.},$$

as may happen to be convenient in any particular example, the values of $\frac{ds}{dx}$, $\frac{ds}{dy}$, $\frac{ds}{d\theta}$, etc., being obtained according to the rules of the Differential Calculus, viz.

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2},\quad \frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}, \text{ etc.}$$

750. Centroids.

The centroids, both of the surface and volume of a solid of revolution bounded by planes perpendicular to the axis of revolution, are plainly upon the axis of revolution, supposing the surface density and the volume density in the respective cases to be either constant or some function of the distance from a point on the axis of revolution, *i.e.* so that the distribution of density is symmetrical about the axis.

Take the x-axis as the axis of revolution, σ the surface density and ρ the volume density, both symmetrical as to the axis, and functions of x alone, so that the elementary zones in the one case and the elementary discs in the other case, into which the surface or volume is divided, have their own centroids upon the axis of x, and we have, on application of the formula $\bar{x}=\frac{\Sigma mx}{\Sigma m}$,

(1) For the Surface,

$$\bar{x}=\frac{\int(\sigma 2\pi y\, ds)x}{\int(\sigma 2\pi y\, ds)}=\frac{\int \sigma xy\, ds}{\int \sigma y\, ds};$$

(2) For the Volume,

$$\bar{x}=\frac{\int(\rho\pi y^2 dx)x}{\int(\rho\pi y^2 dx)}=\frac{\int \rho xy^2 dx}{\int \rho y^2 dx}.$$

It is to be noted that in the first case s is left as the independent variable; in the second case, x.

If it be desirable to take x or θ as the independent variable in the first case, we must replace ds by

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \text{ or } \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta,$$

as the case may be.

In cases where ρ or σ are constants, they of course disappear from the formulae.

751. Ex. 1. Find the surface of a zone of a sphere bounded by parallel planes $z=z_1$, $z=z_1+h$.

If a be the radius of the sphere, and θ be the latitude of any point P on the sphere, we have (Fig. 254)

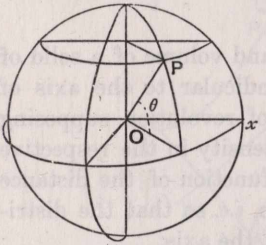

Fig. 254.

$$S=\int 2\pi a\cos\theta . ds \quad \text{and} \quad ds=a\,d\theta$$

$$=2\pi a^2\Big[\sin\theta\Big]_{\theta_1}^{\theta_2}$$

$$=2\pi a^2[\sin\theta_2-\sin\theta_1]$$

$$=2\pi a^2\left[\frac{z_1+h}{a}-\frac{z_1}{a}\right]=2\pi ah,$$

and therefore equal to the corresponding belt intercepted upon the enveloping cylinder by the same planes, the z-axis being the axis of the cylinder. This is the result usually arrived at in a Newtonian manner in books on Mensuration. It has already been used in Art. 734.

Ex. 2. Find the surface of a belt of the paraboloid formed by the revolution of the curve $y^2=4ax$ about the x-axis.

Here $$\frac{dy}{dx}=\sqrt{\frac{a}{x}},\quad \frac{ds}{dx}=\sqrt{1+\frac{a}{x}},$$

and $$S=2\pi\int_{x_1}^{x_2} y\frac{ds}{dx}\,dx=4\pi\sqrt{a}\int_{x_1}^{x_2}\sqrt{x+a}\,dx$$

$$=\tfrac{8}{3}\pi a^{\frac{1}{2}}\{(x_2+a)^{\frac{3}{2}}-(x_1+a)^{\frac{3}{2}}\};$$

and since for the parabola the radius of curvature is given by

$$\rho=\frac{2}{\sqrt{a}}(x+a)^{\frac{3}{2}},$$

we have $$S=\frac{4\pi a}{3}(\rho_2-\rho_1),$$

where ρ_1, ρ_2 are the radii of curvature of the generating curve at the points where it is cut by the planes bounding the belt.

Ex. 3. The curve $r=a(1+\cos\theta)$ revolves about the initial line. Find the volume and surface of the figure formed.

Here $$V=\int\pi y^2\,dx=\pi\int r^2\sin^2\theta\,d(r\cos\theta)$$

$$=\pi\int a^2(1+\cos\theta)^2\sin^2\theta . a\,d(\cos\theta+\cos^2\theta),$$

the limits being such that the radius vector sweeps over the upper half of the cardioide.

Hence $V=\pi a^3\int_0^\pi (1+\cos\theta)^2(1+2\cos\theta)\sin^3\theta\,d\theta$

$$=\pi a^3\int_0^\pi (1+4\cos\theta+5\cos^2\theta+2\cos^3\theta)\sin^3\theta\,d\theta$$

$$=2\pi a^3\int_0^{\frac{\pi}{2}} (1+5\cos^2\theta)\sin^3\theta\,d\theta$$

$$=2\pi a^3\left\{\frac{2}{3}+5\frac{\Gamma(\frac{3}{2})\Gamma(2)}{2\Gamma(\frac{7}{2})}\right\}=2\pi a^3\cdot\frac{4}{3}$$

$$=\frac{8\pi a^3}{3}.$$

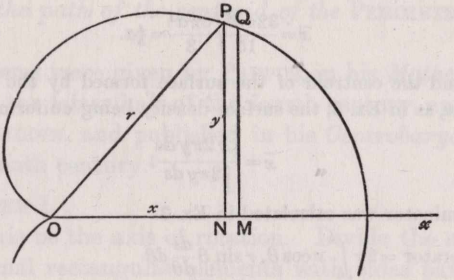

Fig. 255.

Again,

$$S=2\pi\int y\,ds=2\pi\int_0^\pi r\sin\theta\frac{ds}{d\theta}d\theta$$

$$=2\pi\int_0^\pi a(1+\cos\theta)\sin\theta\sqrt{a^2(1+\cos\theta)^2+a^2\sin^2\theta}\,d\theta$$

$$=2\pi a^2\int_0^\pi (1+\cos\theta)\sin\theta\,.\,2\cos\frac{\theta}{2}\,d\theta$$

$$=16\pi a^2\int_0^\pi \cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\,d\theta$$

$$=32\pi a^2\left[-\frac{1}{5}\cos^5\frac{\theta}{2}\right]_0^\pi=\tfrac{32}{5}\pi a^2.$$

Ex. 4. Find the centroid of the solid formed in the last example, the volume density being uniform.

The centroid obviously lies upon the axis. To find its abscissa $\bar{x}$ we have

$$\bar{x}=\frac{\int x\,.\,\pi y^2dx}{\int \pi y^2dx}.$$

The denominator has just been calculated, viz. $=\frac{8}{3}\pi a^3$.

The numerator

$$=\pi\int r\cos\theta\,.\,r^2\sin^2\theta\,d(r\cos\theta)$$

$$=\pi a^4\int (1+\cos\theta)^3\cos\theta\sin^2\theta\,d(\cos\theta+\cos^2\theta),$$ the limits being π and 0

$$=\pi a^4\int_0^\pi (1+\cos\theta)^3\cos\theta(1+2\cos\theta)\sin^3\theta\,d\theta$$

$$=\pi a^4\int_0^\pi (\cos\theta+5\cos^2\theta+9\cos^3\theta+7\cos^4\theta+2\cos^5\theta)\sin^3\theta\,d\theta$$

$$=2\pi a^4\int_0^{\frac{\pi}{2}} (5\cos^2\theta+7\cos^4\theta)\sin^3\theta\,d\theta$$

$$=2\pi a^4\left[5\frac{\Gamma(2)\Gamma(\frac{3}{2})}{2\Gamma(\frac{7}{2})}+7\cdot\frac{\Gamma(2)\Gamma(\frac{5}{2})}{2\Gamma(\frac{9}{2})}\right]$$

$$=2\pi a^4\left[\frac{5}{2}\cdot\frac{2}{5}\cdot\frac{2}{3}+\frac{7}{2}\cdot\frac{2}{7}\cdot\frac{2}{5}\right]=\frac{32\pi a^4}{15}.$$

Hence $$\bar{x}=\frac{32\pi a^4}{15}\Big/\frac{8\pi a^3}{3}=\tfrac{4}{5}a.$$

Ex. 5. Find the centroid of the surface formed by the revolution of the cardioide, as in Ex. 3, the surface density being uniform.

Here $$\bar{x}=\frac{\int x\,2\pi y\,ds}{\int 2\pi y\,ds}.$$

The denominator was calculated in Ex. 3.

The numerator $=2\pi\int_0^\pi r\cos\theta\,.\,r\sin\theta\,\dfrac{ds}{d\theta}\,d\theta$

$$=2\pi\int_0^\pi a^2(1+\cos\theta)^2.\cos\theta.\sin\theta.\,2a\cos\frac{\theta}{2}\,d\theta$$

$$=32\pi a^3\int_0^\pi \cos^6\frac{\theta}{2}\sin\frac{\theta}{2}\left(2\cos^2\frac{\theta}{2}-1\right)d\theta$$

$$=64\pi a^3\left[\frac{\cos^7\frac{\theta}{2}}{7}-2\frac{\cos^9\frac{\theta}{2}}{9}\right]_0^\pi$$

$$=\tfrac{320}{63}\pi a^3.$$

Hence $$\bar{x}=\tfrac{320}{63}\pi a^3/\tfrac{32}{5}\pi a^2=\tfrac{50}{63}a.$$

Ex. 6. As an example of the case when x and y are given in terms of a third variable, consider the case of the surface of the solid formed by the revolution of a cycloid about the line of cusps.

Here $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, $ds=2a\cos\dfrac{\theta}{2}d\theta$, and the perpendicular from x, y upon the line of cusps$=a(1+\cos\theta)$.

Hence $$S=2\int_0^\pi 2\pi a(1+\cos\theta)\,2a\cos\frac{\theta}{2}\,d\theta$$

$$=16\pi a^2\int_0^\pi \cos^3\frac{\theta}{2}\,d\theta$$

$$=32\pi a^2\int_0^{\frac{\pi}{2}}\cos^3\phi\,d\phi,\quad\text{where } \phi=\frac{\theta}{2},$$

$$=\frac{64\pi a^2}{3}.$$

752. The Theorems of Pappus or Guldin.

When any plane closed curve revolves about a straight line in its own plane which does not cut the curve we have the following theorems:

I. *The* Volume *of the ring formed is equal to that of a cylinder whose base is the revolving curve and whose height is the length of the path of the centroid of the* Area *af the curve.*

II. *The* Surface *of the ring formed is equal to that of a cylinder whose base is the revolving curve and whose height is the length of the path of the centroid of the* Perimeter *of the curve.*

These theorems were given by Pappus, in his *Mathematical Collections*, in the latter half of the fourth century, and rediscovered by Guldin, and published in his *Centrobaryca* early in the seventeenth century.[1]

753. Theorem I.

Let the x-axis be the axis of rotation. Divide the area (A) into infinitesimal rectangular elements with sides parallel to the coordinate axes, such as $P_1P_2P_3P_4$ in the accompanying figure, each of area δA.

Let the ordinate $P_1N_1=y$.

Let rotation take place through an infinitesimal angle $\delta\theta$.

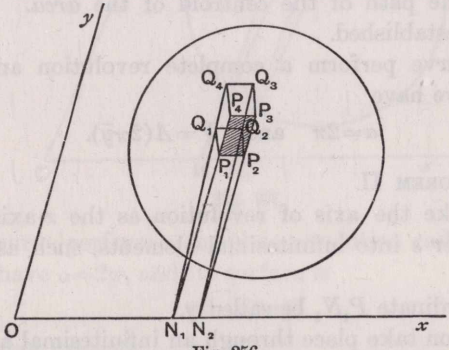

Fig. 256.

Then the elementary solid formed is on base δA, and its height to first order infinitesimals, is $y\,\delta\theta$, and therefore to infinitesimals of the third order its volume is $\delta A\,.\,y\,\delta\theta$.

[1] Cajori, *History of Mathematics*, pages 59, 167.

If the rotation be through any finite angle a, we obtain by summation or integration,

$$\delta A \,.\, y \,.\, a.$$

If this be integrated over the whole area of the curve, we have for the volume of the solid formed,

$$a\int y\, dA.$$

Now the formula for the ordinate of the centroid of a number of masses $m_1, m_2, \ldots$, with ordinates $y_1, y_2, \ldots$, is $\bar{y}=\frac{\Sigma my}{\Sigma m}$.

Hence the ordinate of the centroid of the *area* of the revolving curve is

$$\bar{y}=\frac{\int y\, dA}{\int dA}=\frac{\int y\, dA}{A},$$

and therefore $$\int y\, dA = A\bar{y}.$$

Hence the volume formed $= A(a\bar{y})$.

But A is the area of the revolving figure, and $a\bar{y}$ is the length of the path of the centroid of the *area*. Hence the theorem is established.

If the curve perform a complete revolution and form a solid ring, we have

$$a=2\pi \quad \text{and} \quad V=A(2\pi\bar{y}).$$

754. Theorem II.

Again, take the axis of revolution as the x-axis. Divide the perimeter s into infinitesimal elements, such as P_1P_2, of length δs.

Let the ordinate P_1N_1 be called y.

Let rotation take place through an infinitesimal angle $\delta\theta$.

Then the elementary area formed, $P_1P_2Q_2Q_1$, is ultimately a rectangle with sides δs and $y\,\delta\theta$, and to infinitesimals of the second order its area is $\delta s \,.\, y\,\delta\theta$.

If the rotation be through any finite angle a we obtain, by summation or integration, $\delta s \,.\, ya$.

If this be integrated over the whole perimeter of the curve, we have for the curved surface of the solid formed,

$$\alpha\int y\,ds.$$

If $\bar{y}$ be the ordinate of the centroid of the *perimeter* of the curve in the plane of x-y, we have

$$\bar{y}=\frac{\int y\,ds}{\int ds}=\frac{\int y\,ds}{s}.$$

Then $$\int y\,ds=s\bar{y}$$

and the surface formed $=s(\alpha\bar{y})$.

But s is the perimeter of the revolving figure, and $\alpha\bar{y}$ is the length of the path of the centroid of the *perimeter* of the revolving curve. Hence the theorem is established.

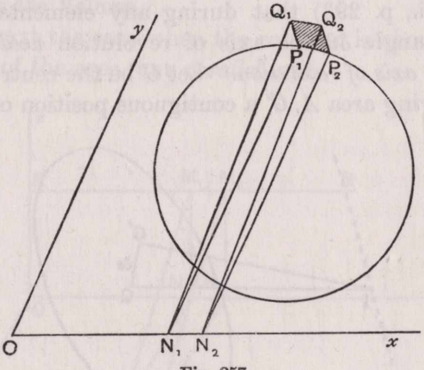

Fig. 257.

If the curve perform a complete revolution and form a solid ring, we have $\alpha=2\pi$, and its surface is

$$S=s(2\pi\bar{y}).$$

Illustrative Example.

The volume and surface of an "**Anchor-ring**" or "**Tore**" formed by the revolution of a circle of radius a about a line in the plane of the circle at distance d from the centre $(d>a)$ are respectively,

$$\text{Volume} = \pi a^2\times 2\pi d=2\pi^2a^2d,$$

$$\text{Surface} =2\pi a\times 2\pi d=4\pi^2ad.$$

In this case the centroid of the perimeter and the centroid of the area are at the same point, viz. the centre of the revolving figure. This of course *would not generally be the case.*

755. Precautions.

In these theorems it has been stated that the axis of revolution does not cut the curve. If the curve consists of more than one closed oval, it is to be further noted that the whole portion to which the rules apply must lie on one side of the axis of revolution.

When the axis of revolution cuts the curve, or when regions bounded by the curve lie on opposite sides of the axis of revolution, the theorems, both as to volume and surface, give the difference of the volumes or surfaces traced by the portions on opposite sides of the axis of revolution.

756. Note by Mr. Routh.

Again, it has been pointed out by Mr. E. J. Routh (*Anal. Statics*, vol. i., p. 293) that during any elementary rotation through an angle $\delta\theta$, the axis of revolution *need only be an instantaneous axis of revolution.* Let G be the centre of gravity of the revolving area A, G' a contiguous position of the centre

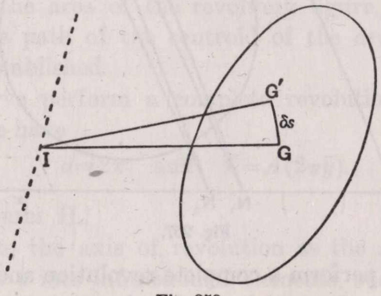

Fig. 258.

of gravity, $\delta s = GG'$, and let the plane of A be always at right angles to the tangent to the path of G. Let I be the centre of curvature of G's path. The rotation through $\delta\theta$ may be regarded as about a straight line through I perpendicular to the plane GIG', and the volume generated is

$$A \times IG\,\delta\theta \quad \text{or} \quad A\,\delta s.$$

And integrating, the volume generated is

Area $\times$ length of the path of the *centroid of the area.*

And further, for the theorem with regard to the surface; if the plane of the revolving curve be always at right angles to the tangent to the path of the centroid of the *perimeter*, the surface generated is the perimeter of the revolving curve $\times$ length of the path of the *centroid of the perimeter.*

Ex. A circle of radius c ($<b$) moves with its centre on the ellipse $x^2/a^2+y^2/b^2=1$, the plane of the circle being perpendicular to the direction of the tangent to the ellipse at the centre of the circle. The volume and surface of the ring generated are $\pi c^2 P$ and $2\pi c P$ respectively, where P is the perimeter of the ellipse, *i.e.* $4aE\left(\frac{\pi}{2}, e\right)$ when $b^2=a^2(1-e^2)$.

In this case the centroids of *area* and *perimeter* of the moving curve are the same point, viz. the centre of the circle.

757. Axis not in the Plane of the Curve. Extension for the Theorem as to the Volume.

Consider next the case when the rotation is about a line *not* in the plane of the area, but *parallel to it.*

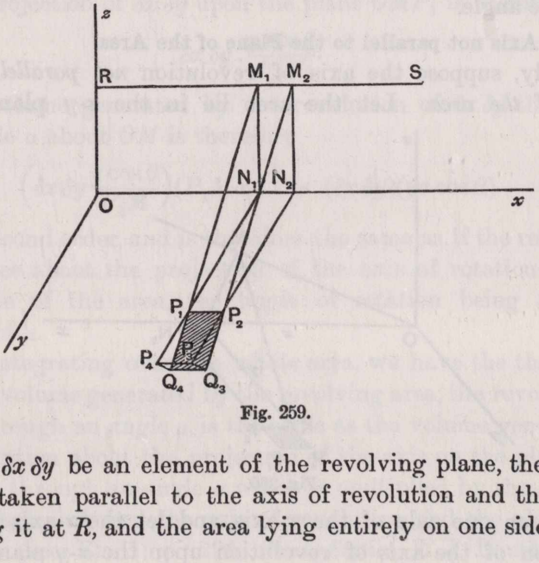

Fig. 259.

Let $\delta x\, \delta y$ be an element of the revolving plane, the x-axis being taken parallel to the axis of revolution and the z-axis cutting it at R, and the area lying entirely on one side of the x-axis.

Let $P_1P_2P_3P_4$ be the element $\delta x\,\delta y$, and let P_1N_1, P_2N_2 be perpendiculars on the x-axis and P_1M_1, P_2M_2 perpendiculars on the axis of revolution, and let θ be the angle $N_1P_1M_1$.

Let $P_1P_2Q_3Q_4$ be the projection of $\delta x\,\delta y$ on the plane $M_1M_2P_2P_1$, that is, the normal section of the elementary ring formed, and let a represent the angular extent of the revolution.

Then, to the second order the volume traced out by the revolution of $\delta x\,\delta y$ about RS is

$$P_1P_2Q_3Q_4 \times (aP_1M_1),$$

i.e. $$\delta x\,\delta y \cos\theta \times \left(a\frac{P_1N_1}{\cos\theta}\right) \quad \text{or} \quad \delta x\,\delta y \times aP_1N_1,$$

and is the same as that of $\delta x\,\delta y$ about the x-axis.

Hence, taking the limit when δx, δy are infinitesimally small, and integrating over any area which lies on one side of the x-axis in the x-y plane, we have the theorem that the volume generated by the area revolving about a line parallel to the plane of the area, but not in its own plane, is the same as would be traced out if the revolution were about the projection of the axis of revolution upon the plane of the area through the same angle.

758. **Axis not parallel to the Plane of the Area.**

Finally, suppose the axis of revolution *not parallel to the plane of the area.* Let the area lie in the x-y plane and

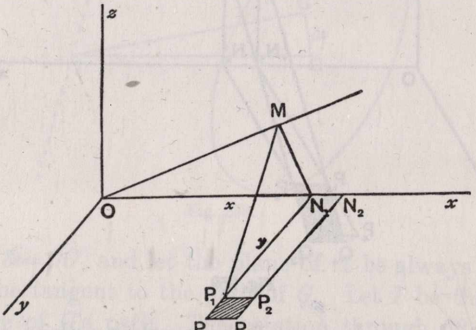

Fig. 260.

entirely on one side of the x-axis, and let the x-axis be the projection of the axis of revolution upon the x-y plane, and the origin the point where that axis cuts the x-y plane and θ its inclination to the plane. Then the equations of the axis of revolution are

$$\frac{x}{\cos\theta}=\frac{y}{0}=\frac{z}{\sin\theta}.$$

The perpendicular upon this from P_1, $(x, y, 0)$, viz. P_1M, is

$$P_1M=\sqrt{x^2+y^2-x^2\cos^2\theta}=\sqrt{x^2\sin^2\theta+y^2}.$$

The equation of the plane OMP_1 is

$$-X\sin\theta+\frac{x}{y}\sin\theta Y+Z\cos\theta=0.$$

The direction ratios of the normal are

$$-\sin\theta,\ +\frac{x}{y}\sin\theta,\ +\cos\theta.$$

The direction cosines of the normal to the element $P_1P_2P_3P_4$, *i.e.* $\delta y\,\delta z$, are $(0, 0, 1)$.

The angle between these normals is

$$\cos^{-1}\frac{\cos\theta}{\sqrt{1+\dfrac{x^2}{y^2}\sin^2\theta}}=\cos^{-1}\left(\frac{y\cos\theta}{P_1M}\right).$$

The projection of $\delta x\,\delta y$ upon the plane OMP_1 is therefore

$$\delta x\,\delta y\,.\,\frac{y\cos\theta}{P_1M}.$$

The volume generated by the revolution of $\delta x\,\delta y$ through any angle α about OM is therefore

$$\left(\delta x\,\delta y\,\frac{y\cos\theta}{P_1M}\right)(P_1M\alpha),\quad i.e.\ (\delta x\,\delta y)(y\alpha\cos\theta)$$

to the second order, and is therefore the same as if the rotation took place about the projection of the axis of rotation upon the plane of the area, the angle of rotation being $\alpha\cos\theta$ instead of α.

And integrating over the whole area, we have the theorem that the volume generated by the revolving area, the revolution being through an angle α, is the same as the volume generated by revolution about the projection of the axis on the plane of the area through an angle α which is multiplied by the cosine of the angle between the axis of revolution and the plane of the area (see Ex. 1, p. 294, *Anal. Statics*, E. J. Routh); or, which is the same thing, the volume may be found by revolution through an angle α about the projection and then multiplied by $\cos\theta$. This supposes the revolving area to be entirely on one side of the projection of the axis on its plane.

759. Ex. 1. A quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves (1) about its semiminor axis. The area is $\frac{\pi ab}{4}$. The abscissa of the centroid is $\frac{4a}{3\pi}$. The volume traced out in a complete revolution is

$$\frac{\pi ab}{4}\cdot 2\pi\frac{4a}{3\pi}=\frac{2}{3}\pi a^2 b.$$

(2) If the revolution were about a straight line outside the plane of x-y but parallel to the minor axis, and which projects upon the minor axis, the volume would still be $\frac{2}{3}\pi a^2 b$.

(3) If the revolution were about a straight line through the centre at right angles to the major axis, and making an angle θ with the minor axis, the volume would be $\frac{2}{3}\pi a^2 b\cos\theta$.

Ex. 2. An ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves about its tangent

$$x\cos\alpha+y\sin\alpha=p,$$

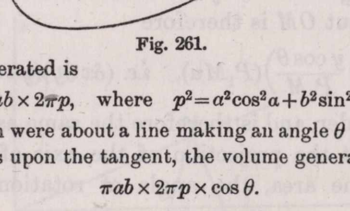

Fig. 261.

The volume generated is

$$\pi ab\times 2\pi p, \quad \text{where} \quad p^2=a^2\cos^2\alpha+b^2\sin^2\alpha.$$

If the revolution were about a line making an angle θ with this tangent, and which projects upon the tangent, the volume generated would be

$$\pi ab\times 2\pi p\times\cos\theta.$$

PROBLEMS.

1. Prove that the volume of the solid generated by the revolution of an ellipse round its minor axis is a mean proportional between those generated by the revolution of the ellipse and of the auxiliary circle round the major axis. [I. C. S., 1881.]

2. Find the volume of the solid formed by the revolution of a cycloid round a tangent at the vertex.

3. The loop of the curve $2ay^2 = x(x-a)^2$ revolves about the straight line $y = a$; find the volume of the solid generated.

[OXFORD I. P., 1890.]

4. Show that the volume of the solid formed by the revolution of the cissoid $y^2(2a-x) = x^3$ about its asymptote is equal to $2\pi^2 a^3$.

[TRINITY, 1886.]

5. Find the volume of the solid produced by the revolution of the loop of the curve $y^2 = x^2 \dfrac{a+x}{a-x}$ about the axis of x. [I. C. S., 1882.]

Prove that the areas of the oblate and prolate spheroids formed by rotating an ellipse of major axis $2a$ and eccentricity e about its principal axes are

$$2\pi a^2\left(1 + \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right)$$

and $$2\pi a^2\left(1 - e^2 + \frac{\sqrt{1-e^2}}{e}\sin^{-1}e\right).$$

[OXFORD II. P., 1914.]

Prove also that of all prolate spheroids formed by the revolution of an ellipse of given area, the sphere has the greatest surface.

[I. C. S., 1891.]

Find the surface of any zone of an ellipsoid of revolution cut off by planes perpendicular to the axis of revolution.

[COLLEGES α, 1888.]

7. If the evolute of a catenary revolve about the directrix of the catenary, show that the area of any portion of the surface generated, cut off by two planes perpendicular to the directrix, varies as the difference of the cubes of the radii of its bounding circles. [COLLEGES α, 1892.]

8. Find the volume of the solid formed by the revolution about the prime radius of the loop of the curve

$$r^3 = a^3\theta\cos\theta$$

between $\theta = 0$ and $\theta = \dfrac{\pi}{2}$.

[OXFORD II. P., 1890.]

9. If the cardioide $r = a(1-\cos\theta)$ revolve round the line $p = r\cos(\theta - \gamma)$, prove that the volume generated is

$$3p\pi^2 a^2 + \tfrac{5}{2}\pi^2 a^3 \cos\gamma,$$

assuming that the line does not cut the cardioide. [ST. JOHN'S, 1882.]

10. Prove that the area of the surface generated by the revolution of a portion of the arc of a cycloid about the normal at one extremity is equal to the area of the cycloid multiplied by

$$\tfrac{16}{3}[(\beta-\gamma)\sin\beta\cos\gamma+\tfrac{3}{4}\cos(\beta+\gamma)-\tfrac{1}{12}\cos(3\beta-\gamma)-\tfrac{2}{3}\cos 2\gamma],$$

where γ and β are the angles of inclination of the axis of revolution, and of the normal at the other extremity of the arc, to the axis of the cycloid.

Deduce the areas of the surfaces generated by the revolution of the whole cycloid about its axis and about its base.

[COLLEGES ϵ, 1884.]

11. Find the volume of the solid formed by revolving a loop of a lemniscate of Bernoulli about the straight line in its plane which passes through the pole and is perpendicular to the axis.

[OXFORD I. P., 1901.]

12. The lemniscate $r^2=a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the volume and surface of the solid generated are respectively $\pi^2a^3/4$ and $4\pi a^2$.

13. A surface is the locus of points which have their distances from a fixed plane inversely proportional to the fifth power of their distances from a fixed point O in that plane. Prove that its volume equals twice that of the sphere which, with its centre at O, touches the surface. [OXFORD II. P., 1880.]

14. Find the volume of the solid formed by the revolution of the curve $(a-x)y^2=a^2x$ about its asymptote. [I. C. S., 1883.]

15. Show that the rate of increase of the volume of an anchor ring when the radius of the generating circle is increased while its centre remains at a constant distance a from the axis of revolution is

$$2\pi^2 ad,$$

the diameter of the generating circle being d, increasing at unit rate. [TRINITY COLL., 1881.]

16. A loop of the curve $r=a\sin n\theta$ revolves about the initial line. Find the volume of the solid thus generated, and verify the result by deducing the volume of the ring formed by the revolution of a circle about a tangent. [COLLEGES α, 1889.]

17. If the curve $r=a+b\cos\theta$ revolve about the initial line, show that the volume generated is

$$\tfrac{4}{3}\pi a(a^2+b^2),$$

provided a be $<b$, [COLLEGES α, 1884.

18. The curve $r=a(1-\epsilon\cos\theta)$, when ϵ is very small, revolves about a tangent parallel to the initial line; prove that the volume of the solid thus generated is approximately

$$2\pi^2a^3(1+\epsilon^2).$$ [I. C. S., 1892.]

19. The curve $r^3=a^3\cos 3\theta$ revolves about $\theta=0$. Prove that the loop in the third quadrant generates a volume

$$\frac{3\pi a^3}{8}.$$ [Oxford I. P., 1902.]

20. A loxodrome is drawn from a point A on the earth's surface to a point B. If θ_1, ϕ_1 be the longitude and co-latitude of B, and θ_2, ϕ_2 the corresponding quantities for A, show that the area contained between the meridians of A and B and the loxodrome is

$$\frac{2(\theta_1-\theta_2)\log(\cos\frac{1}{2}\phi_1\div\cos\frac{1}{2}\phi_2)}{\log(\tan\frac{1}{2}\phi_1\div\tan\frac{1}{2}\phi_2)},$$

the radius of the earth being taken as unity. [St. John's, 1884.]

21. Prove that the whole area bounded by the curve

$$x^4+y^4=2axy^2$$

is $\frac{\pi a^2}{2\sqrt{2}}$. Also show that if the area revolves about the x-axis, either loop generates a solid whose volume is $\frac{2}{3}\pi a^3$. When the area revolves about the y-axis, the whole volume generated is $\frac{\pi^2a^3}{4}\sqrt{2}$.

22. Determine the curve which generates, by revolving about the axis of x, a volume proportional to the length cut off from the axis by the terminal bounding planes. [Trin. Hall and Magd., 1886.]

23. The axes of two cylinders of radius a intersect at an angle α; show that the whole volume common to the two is

$$\tfrac{16}{3}a^3\operatorname{cosec}\alpha.$$ [Trin. H. and Magd., 1886.]

24. Evaluate $\iint\frac{dS}{p^n}$, taken over the surface of a sphere of radius a, p being the perpendicular on the tangent plane from a fixed point within the sphere at a distance b from the centre; showing that

$$\iint\frac{dS}{p^n}=\frac{2\pi a^2}{(n-1)b}[(a-b)^{-n+1}-(a+b)^{-n+1}].$$

[Oxford II. P., 1892.]

25. Show that the volume traced out by the part of the area of the curve $r=f(\theta)$ which lies between $\theta=\beta$ and $\theta=\alpha$, when the curve revolves about the line $\theta=\gamma$, taking $\alpha>\beta>\gamma$, is

$$\frac{2\pi}{3}\int_\beta^\alpha [f(\theta)]^3 \sin(\theta-\gamma)\,d\theta.$$ [OXFORD I. P., 1902.]

26. In the case of any portion of a surface revolving about an axis, prove that the volume generated is the sum with the proper signs of the corresponding volumes generated by the projections of the surface on any two planes at right angles to one another through the axis of rotation. [γ, 1900.]

27. A point O is taken on a diameter of a sphere (centre C, radius a) so that $OC=c\ (c<a)$; the radius vector of length r drawn from O to any point P on the surface makes an angle θ with OC, and the radius CP makes an angle θ' with OC produced, dS is an element of area of the surface containing the point P; evaluate the integral

$$\int \frac{\cos\theta\cos\theta'}{r^2}\,dS$$

taken over the larger of the portions into which the surface is divided by a plane, through O, at right angles to OC.

[OXFORD I. P., 1901.]

28. Prove the formula $\frac{2}{3}\pi\int r^3 \sin\theta\,d\theta$ for the volume of the surface formed by the revolution of a closed plane curve about the initial line.

The outer loop of $r^{\frac{1}{3}}=a^{\frac{1}{3}}\cos\frac{1}{3}\theta$ revolves about the initial line. Show that the volume of the surface generated is

$$\frac{1}{15}\pi a^3\left(7+\frac{1}{2^{11}}\right).$$ [OXF. I. P., 1911.]

29. Find the area of the finite portion of the surface $2z=x^2+y^2$ cut off by the plane $z=h$. [OXF. I. P., 1913.]

30. Show how to find the volume of the solid formed by revolving the curve $r=f(\theta)$ about the line $\theta=\alpha$, it being assumed that the curve passes through the origin.

Prove that the volume of the solid formed by revolving one loop of the curve $r^2=a^2\cos 2\theta$ about one of the inflexional tangents is

$$\tfrac{1}{8}\pi^2 a^3.$$ [OXF. I. P., 1915.]

Show also that the distance of the centroid of this solid from the origin is $\frac{4a}{3\pi}$.