CHAPTER XIX.

MOVING CURVES.

Quadrature and Rectification of Loci of Carried Points and Envelopes of Carried Lines.

649. **"Instantaneous Centre."**

It is a very well-known geometrical theorem that if two triangles *ABC*, *abc* are equal in all respects and lie in the same plane, the one can be superposed upon the other by a rotation about some point in the plane.

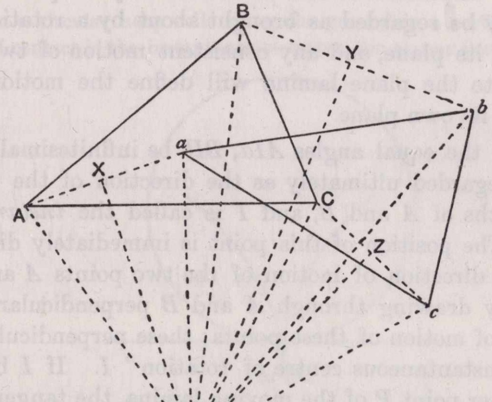

Fig. 180.

Let *XI*, *YI*, the perpendicular bisectors of *Aa*, *Bb*, meet at *I*. Join *IA*, *Ia*; *IB*, *Ib*; *IC*, *Ic*; and join *I* to the mid-point *Z* of *Cc*.

Then IA, AB, BI being respectively equal to Ia, ab, bI, the triangles IAB, Iab are congruent, and angle $I\hat{B}A = I\hat{b}a$. Hence $I\hat{B}C = I\hat{b}c$, and having also IB, BC respectively equal to Ib, bc, the triangles IBC, Ibc are congruent, and $IC = Ic$; whence IZ bisects Cc perpendicularly, so that the perpendicular bisectors of Aa, Bb, Cc are concurrent. Moreover angle $A\hat{I}B$ being equal to $a\hat{I}b$, and $B\hat{I}C$ being equal to $b\hat{I}c$, it is clear that

$$A\hat{I}a = B\hat{I}b = C\hat{I}c,$$

and therefore a rotation through the angle $A\hat{I}a$ about the point I in the proper direction will accomplish the superposition of the one triangle upon the other.

If Aa, Bb are parallel, I is at ∞ in the plane, and the motion is one of translation without rotation.

Two of the three points A, B, C may be regarded as fixing the position of the lamina upon which the triangle is drawn, and the third point may be regarded as *any* point carried by the lamina.

Thus a displacement of a lamina of any shape in its own plane may be regarded as brought about by a rotation about a point in its plane, and any consistent motion of two points attached to the plane lamina will define the motion of the lamina in its own plane.

650. If the equal angles AIa, BIb be infinitesimal, Aa, Bb may be regarded ultimately as the direction of the tangents to the paths of A and B, and I is called the *instantaneous centre.* The position of this point is immediately discovered when the direction of motion of the two points A and B are known, by drawing through A and B perpendiculars to the direction of motion of these points; these perpendiculars meet in the "instantaneous centre of rotation" I. If I be joined to any other point P of the moving lamina, the tangent to the path of P is at right angles to PI, and PI is the normal to the path.

651. For instance, if a hoop of any shape be in motion in a plane, and the direction of motion of two points of the hoop be known, say, PT, QT, then I is at the intersection of perpendiculars through P and Q to PT, QT respectively, and the motion of any other point of the hoop, R, is at

right angles to IR. Hence at any instant the directions of instantaneous motion of all particles on the hoop envelop the first negative pedal of the hoop with regard to the instantaneous centre. When the hoop is

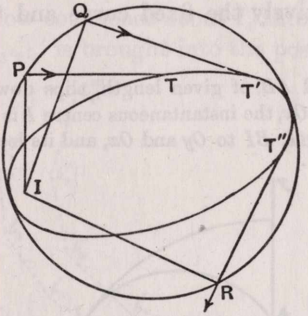

Fig. 181.

circular, this will be an ellipse if I falls within the hoop, a hyperbola if I falls without the hoop, and a point if I falls upon the hoop.

652. The instantaneous centre itself is not in general a fixed point. If it has a path upon the fixed plane, it has another path relatively to the moving lamina.

When a circular hoop rolls along the ground in a vertical plane, the point of contact is the instantaneous centre, for at any instant the point

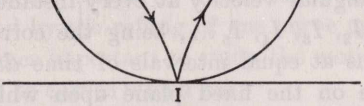

Fig. 182.

of the hoop in contact with the ground is not moving *along* the ground, for by supposition there is no slipping, and it has just ceased to approach the ground, and is on the point of beginning to leave the ground, and therefore for the instant it has no motion *at right angles* to the ground. The path of the instantaneous centre on the fixed plane is evidently the line on which the hoop rolls. The path on the plane of the hoop is the hoop itself.

653. Exactly the same is true when any curve traced upon a lamina is made to roll without sliding upon a fixed curve. The point of contact is the instantaneous centre. The two I-loci are respectively the fixed curve and the moving curve themselves.

654. When a rod AB, of given length, slips down between two perpendicular axes Oy, Ox, the instantaneous centre I is at the intersection of the perpendiculars AI, BI to Oy and Ox, and its locus on the fixed plane

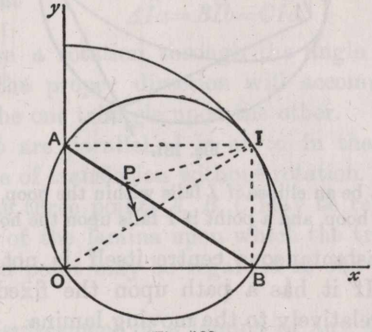

Fig. 183.

is a circle with centre at O and radius equal to the rod. The path relative to the rod is a circle of radius half the rod, described on the rod for diameter.

Any point P attached to the rod describes an ellipse, of which IP is the normal and a perpendicular through P to IP is the tangent.

655. **General Motion of a Lamina reduced to a Case of Rolling.**

Let us define the manner of motion of the lamina to be such that its angular velocity at every instant is some given quantity; I_1, I_2, I_3, I_4, I_5 ..., being the corresponding successive positions at equal intervals of time dt of the instantaneous centre on the fixed plane upon which the lamina moves.

Let $d\psi_1$, $d\psi_2$, $d\psi_3$, ... be the infinitesimal angles turned through in successive rotations about I_1, I_2, I_3, ... Then

(a) Let there be a rotation $d\psi_2$ about I_2.

Then a line on the moving lamina, which was originally coincident with I_2I_1, will be brought by rotation about I_2 into the position $I_2\iota_1$.

(*b*) Now let rotation commence about I_3 through $d\psi_3$.

Then the line $I_3 I_2 \iota_1$ on the moving lamina is brought into the position $I_3 \iota_2 \iota_1'$.

(*c*) Let rotation now commence about I_4 through $d\psi_4$.

Then the line $I_4 I_3 \iota_2 \iota_1'$ is brought into the position $I_4 \iota_3 \iota_2' \iota_1''$.

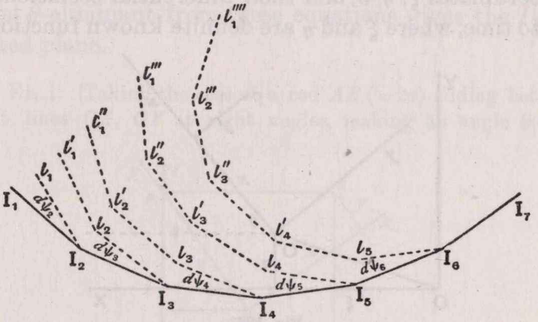

Fig. 184.

(*d*) Let rotation now commence about I_5 through $d\psi_5$.

Then the line $I_5 I_4 \iota_3 \iota_2' \iota_1''$ is brought to the position $I_5 \iota_4 \iota_3' \iota_2'' \iota_1'''$, and so on.

Hence it is clear that when the intervals of time are infinitesimally small, and the chords $I_1 I_2$, $I_2 I_3$, etc., indefinitely diminished, the motion of the lamina may be constructed by the rolling of the curve locus of the instantaneous centres relative to the lamina, viz. $I_5 \iota_4 \iota_3' \iota_2'' \iota_1'''$ upon the curve locus of the instantaneous centres upon the fixed plane, viz. $I_5 I_4 I_3 I_2 I_1$.

Hence the general motion of a lamina in its own plane may be constructed by the rolling of one curve upon another. It therefore becomes important to study the motion of points and lines attached to curves which roll.

656. **The Two Loci of the Instantaneous Centre.**

The locus of I both on the lamina itself and on the fixed plane upon which the lamina moves becomes important. Each may be readily found.

Let OX, OY be fixed rectangular axes upon the fixed plane.

Let $O'x$, $O'y$ be rectangular axes attached to the moving lamina.

Let ξ, η be the coordinates of O' relatively to OX, OY; x, y the coordinates of any point P on the lamina relatively to $O'x$, $O'y$.

Let θ be the inclination of $O'x$ to OX.

The motion of the lamina will then be fully defined by the three coordinates ξ, η, θ, and their differential coefficients with regard to time, where ξ and η are definite known functions of θ

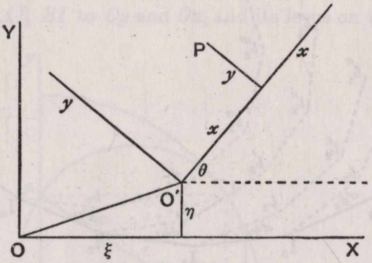

Fig. 185.

The coordinates of P relatively to OX, OY will be

$$\left.\begin{aligned} X&=\xi+x\cos\theta-y\sin\theta,\\ Y&=\eta+x\sin\theta+y\cos\theta.\end{aligned}\right\}\quad\text{.....................(1)}$$

Differentiating,

$$dX=d\xi+(dx-y\,d\theta)\cos\theta-(dy+x\,d\theta)\sin\theta,$$
$$dY=d\eta+(dx-y\,d\theta)\sin\theta+(dy+x\,d\theta)\cos\theta.$$

To find the position of I about which the lamina is turning at any instant, we must remember that

(a) it is for the moment stationary in space,

(b) it is for the moment stationary in the lamina.

Hence for this point

$$dX=dY=0 \quad\text{and}\quad dx=dy=0.$$

Therefore $\left.\begin{aligned} d\xi-y\,d\theta\cos\theta-x\,d\theta\sin\theta=0,\\ d\eta-y\,d\theta\sin\theta+x\,d\theta\cos\theta=0,\end{aligned}\right\}$ at such a point,

and ξ, η being known functions of θ, x and y are found from

$$\left.\begin{aligned} x&=\frac{d\xi}{d\theta}\sin\theta-\frac{d\eta}{d\theta}\cos\theta,\\ y&=\frac{d\xi}{d\theta}\cos\theta+\frac{d\eta}{d\theta}\sin\theta,\end{aligned}\right\}\quad\text{.....................(2)}$$

and the θ-eliminant from these equations gives the locus of I on the lamina.

Next, substituting in equation (1),

$$\left.\begin{aligned} X &= \xi - \frac{d\eta}{d\theta}, \\ Y &= \eta + \frac{d\xi}{d\theta}, \end{aligned}\right\} \quad \text{..............................(3)}$$

and the θ-eliminant from these equations gives the I-locus on the fixed plane.

657. Ex. 1. Taking the case of a rod AB $(=2a)$ sliding between two straight lines OX, OY at right angles, making an angle θ with the

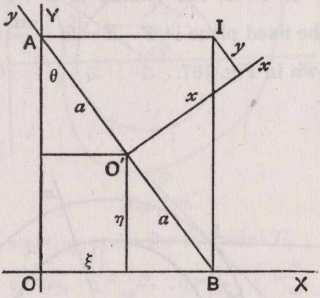

Fig. 186.

latter, and taking the centre of the rod O' as origin for the moving axes and the rod itself as the y-axis,

$$\xi = a\sin\theta, \quad \eta = a\cos\theta\,;$$

$$\therefore \left.\begin{aligned} x &= a\cos\theta\sin\theta + a\sin\theta\cos\theta = a\sin 2\theta, \\ y &= a\cos\theta\cos\theta - a\sin\theta\sin\theta = a\cos 2\theta, \end{aligned}\right\} \text{ from equations (2)};$$

$$\left.\begin{aligned} X &= a\sin\theta + a\sin\theta = 2a\sin\theta, \\ Y &= a\cos\theta + a\cos\theta = 2a\cos\theta, \end{aligned}\right\} \text{ from equations (3)},$$

and the locus of I on the lamina is

$$x^2 + y^2 = a^2,$$

and on the fixed plane $\quad x^2 + y^2 = 4a^2\,;$

as is geometrically obvious (see Art. 654); as indeed are also all the equations established, the point I being at the intersection of the perpendiculars at B and A to OX, OY respectively.

All carried points which lie on the circle with AB for diameter describe two cusped hypo-cycloids, *i.e.* straight lines, and all points attached to the line itself describe ellipses (see Besant, *Conic Sections*, Art. 245).

Ex. 2. Taking the case of an involute of a circle of radius a, sliding between two perpendicular lines OX, OY, let the radius of the circle through the cusp make an angle θ with the line OX. Then

$$\xi = a\left(\frac{\pi}{2}+\theta\right), \quad \eta = a\theta;$$

$$\therefore \left.\begin{aligned} x &= a(\sin\theta - \cos\theta), \\ y &= a(\cos\theta + \sin\theta), \end{aligned}\right\} \text{ from equations (2);}$$

$$\left.\begin{aligned} X &= a\left(\frac{\pi}{2}+\theta\right) - a, \\ Y &= a\theta + a, \end{aligned}\right\} \text{ from equations (3).}$$

Hence the locus of I on the lamina is $x^2+y^2=2a^2$, *i.e.* a circle; the locus of I on the fixed plane is $Y-X=2a-\frac{\pi a}{2}$, *i.e.* a straight line.

These loci are shown in Fig. 187.

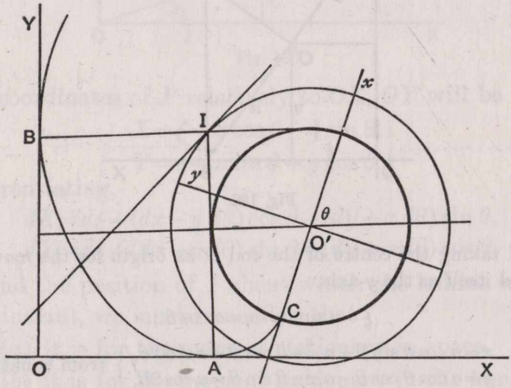

Fig. 187.

The first of the loci is geometrically obvious, as the tangents from I to the generating circle of the involute are at right angles.

The motion is that of the rolling of a circle of radius $a\sqrt{2}$ upon a straight line which makes an angle $\frac{\pi}{4}$ with the axes OX, OY and an intercept $\left(2-\frac{\pi}{2}\right)a$ on the Y-axis. The locus of the starting-point C of the involute is plainly a trochoid, and the locus of the centre of the generating circle a straight line. Points on the circular I-locus describe cycloids, all other attached points describe trochoids.

The student will find this example done (in a different way) in Besant's *Roulettes and Glisettes*, p. 37. The object here is to illustrate the use of the general formulae of the preceding article.

Ex. 3. Consider a case of motion of apparently different nature.

Let a lamina PQR rotating at a constant angular velocity ω be moving so that an attached point C describes a straight line with uniform velocity v.

Take the path of C as the axis of X, and ξ, η the coordinates of the centre, and θ the angle turned through in time t, and suppose that initially ξ and θ both vanish. Let accents denote differentiations with regard to θ

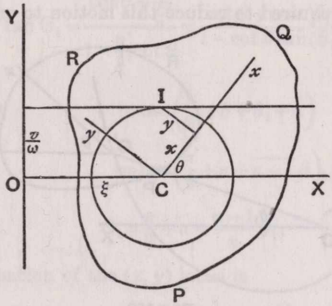

Fig. 188.

Then, O being the starting point for the point C,

$$\xi=vt,\quad \theta=\omega t,\quad \xi=\frac{v}{\omega}\theta,\quad \eta=0,\quad \xi'=\frac{v}{\omega},\quad \eta'=0.$$

The equations of Art. 656 give

$$X=\frac{v}{\omega}\theta,\quad Y=\frac{v}{\omega};\quad x=\frac{v}{\omega}\sin\theta,\quad y=\frac{v}{\omega}\cos\theta\,;$$

$\therefore$ the I-loci are a straight line, $Y=\dfrac{v}{\omega}$, on the fixed plane, and

$$x^2+y^2=\frac{v^2}{\omega^2},$$

i.e. a circle whose centre is C on the lamina.

The motion is therefore that of a circle rolling on a fixed straight line, All carried points describe cycloids or trochoids.

658. In the same way, if the point C be made to describe a circle of radius a with angular velocity ω, whilst the lamina rotates with an angular velocity ω', we have, taking rectangular axes through the centre of the fixed circle, and rectangular moving axes through the point C attached to the lamina, and supposing η and θ to vanish together,

$$\xi=a\cos\omega t,\quad \eta=a\sin\omega t,\quad \theta=\omega' t\,;$$

$$X=\xi-\frac{d\eta}{d\theta}=a\,\frac{\omega'-\omega}{\omega'}\cos\frac{\omega\theta}{\omega'},\qquad Y=\eta+\frac{d\xi}{d\theta}=a\,\frac{\omega'-\omega}{\omega'}\sin\frac{\omega\theta}{\omega'}\,;$$

$$x=-\frac{a\omega}{\omega'}\cos\frac{\omega'-\omega}{\omega'}\theta,\qquad y=\frac{a\omega}{\omega'}\sin\frac{\omega'-\omega}{\omega'}\theta,$$

and the motion is that of a circle of radius $\frac{a\omega}{\omega'}$ rolling upon a fixed circle of radius $a\frac{\omega'-\omega}{\omega'}$, and therefore all carried points on the lamina trace epi- or hypo-cycloids or epi- or hypo-trochoids.

659. Ex. Suppose that a point O' of a lamina PQR travel upon an equiangular spiral, with pole O, fixed upon a plane over which the lamina slides. Suppose that the lamina rotates at $\frac{1}{n}$th of the rate of the radius vector OO'. It is required to reduce this motion to one of rolling.

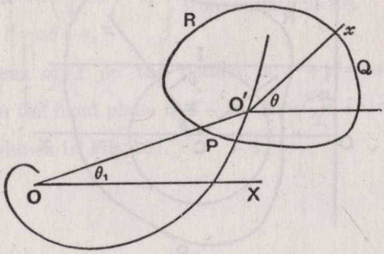

Fig. 189.

Let OO' make an angle θ_1 with the initial line, and let a line $O'x$ fixed on the rotating lamina make an angle θ with the axis OX fixed in space. Suppose OX be taken such that θ_1, θ vanish together. Then $\theta_1 = n\theta$.

If ξ. η be the coordinates of O', we have

$$\xi = ae^{\theta_1 \cot\alpha}\cos\theta_1, \quad \eta = ae^{\theta_1\cot\alpha}\sin\theta_1, \quad \frac{d\theta_1}{d\theta} = n,$$

with the usual notation as to the spiral.

Then $$\xi' \equiv \frac{d\xi}{d\theta} = nae^{\theta_1\cot\alpha}(\cot\alpha\cos\theta_1 - \sin\theta_1),$$

$$\eta' \equiv \frac{d\eta}{d\theta} = nae^{\theta_1\cot\alpha}(\cot\alpha\sin\theta_1 + \cos\theta_1);$$

$\therefore$ by Art. 656,

$$X = \xi - \eta' = ae^{\theta_1\cot\alpha}[(1-n)\cos\theta_1 - n\cot\alpha\sin\theta_1],$$

$$Y = \eta + \xi' = ae^{\theta_1\cot\alpha}[(1-n)\sin\theta_1 + n\cot\alpha\cos\theta_1].$$

Putting $1 - n = k\cos\beta$, $n\cot\alpha = k\sin\beta$,

$$\left.\begin{aligned} X &= ae^{\theta_1\cot\alpha}k\cos(\theta_1+\beta), \\ Y &= ae^{\theta_1\cot\alpha}k\sin(\theta_1+\beta), \end{aligned}\right\}$$

i.e. the locus of X, Y is $R = kae^{(\Theta-\beta)\cot\alpha}$, where R, Θ are current coordinates, and

$$k = \sqrt{(1-n)^2 + n^2\cot^2\alpha}, \quad \tan\beta = \frac{n}{1-n}\cot\alpha,$$

i.e. the fixed I-locus is an equal equiangular spiral.

Again, $x=\xi'\sin\theta-\eta'\cos\theta$, $y=\xi'\cos\theta+\eta'\sin\theta$,

and $$x^2+y^2=\xi'^2+\eta'^2=\frac{n^2a^2e^{2\theta_1\cot\alpha}}{\sin^2\alpha},$$

and if R_1, Θ_1 be the polar coordinates of a point on the I-locus upon the lamina,

$$R_1\cos\Theta_1=\xi'\sin\theta-\eta'\cos\theta,$$
$$R_1\sin\Theta_1=\xi'\cos\theta+\eta'\sin\theta,$$

$$\tan\Theta_1=\frac{\cot\theta+\dfrac{\eta'}{\xi'}}{1-\dfrac{\eta'}{\xi'}\cot\theta}=\frac{\cot\theta+\tan(\theta_1+\alpha)}{1-\cot\theta\tan(\theta_1+\alpha)}$$

$$=\tan\left(\frac{\pi}{2}-\theta+\theta_1+\alpha\right)$$

$$=\tan\left(\frac{\pi}{2}+\alpha+\overline{n-1}\,\theta\right),$$

$$\Theta_1=\frac{\pi}{2}+\alpha+\frac{n-1}{n}\theta_1;$$

$\therefore$ the polar equation of the (x, y) locus is

$$R_1=\frac{na}{\sin\alpha}e^{\frac{n\cot\alpha}{n-1}\left(\Theta_1-\alpha-\frac{\pi}{2}\right)},$$

i.e. another equiangular spiral, but of different angle, which is replaced by the straight line

$$\Theta_1=\frac{\pi}{2}+\alpha,\quad\text{when } n=1.$$

The motion is therefore that of one equiangular spiral with angle α, rolling upon another of different angle, or when $n=1$, upon a straight line. The case when $n=1$ is that in which the lamina rotates at the same rate as the radius vector of the original spiral.

660. The Curvatures of the two Loci. Analytical Consideration.

It will be found in later articles that we frequently have to find the difference of the curvatures of these two I-loci. And for convenience of drawing it is customary, as in Arts. 665, 667, and in *Diff. Calc.*, Ch. XX., to consider the concavities of the fixed and rolling curves as being in opposite directions. That is, the expressions $\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}$ which occur in theorems on Roulettes and Glisettes are the algebraic differences of curvatures as measured in the same direction.

For the present we consider the concavities in the *same* direction. Both the I-loci have been found in the form

$x=F(\theta)$, $y=f(\theta)$, and therefore the curvatures can readily be obtained from the formula

$$\frac{1}{\rho}=\frac{f'(\theta)F''(\theta)-f''(\theta)F'(\theta)}{[\{F'(\theta)\}^2+\{f'(\theta)\}^2]^{\frac{3}{2}}}.$$

Representing by accents differentiations with regard to θ, we have

(a) For the I-locus on the fixed plane,

$$X=\xi-\eta', \quad Y=\eta+\xi',$$
$$X'=\xi'-\eta'', \quad Y'=\eta'+\xi'',$$
$$X''=\xi''-\eta''', \quad Y''=\eta''+\xi''';$$
$$\therefore\ X'^2+Y'^2=(\xi''+\eta')^2+(\eta''-\xi')^2,$$

and $X'Y''-X''Y'=(\xi'-\eta'')(\eta''+\xi''')-(\eta'+\xi'')(\xi''-\eta''')$;

and if ρ_1 be the radius of curvature of this fixed I-locus,

$$\frac{1}{\rho_1}=\frac{(\xi'-\eta'')(\eta''+\xi''')-(\eta'+\xi'')(\xi''-\eta''')}{[(\xi''+\eta')^2+(\eta''-\xi')^2]^{\frac{3}{2}}}.$$

(b) For the locus of I on the moving lamina,

$$x=\xi'\sin\theta-\eta'\cos\theta, \quad y=\xi'\cos\theta+\eta'\sin\theta,$$
$$\left.\begin{aligned}x'&=(\xi''+\eta')\sin\theta-(\eta''-\xi')\cos\theta,\\ y'&=(\xi''+\eta')\cos\theta+(\eta''-\xi')\sin\theta,\end{aligned}\right\}$$
$$\left.\begin{aligned}x''&=(\xi'''+2\eta''-\xi')\sin\theta-(\eta'''-2\xi''-\eta')\cos\theta,\\ y''&=(\xi'''+2\eta''-\xi')\cos\theta+(\eta'''-2\xi''-\eta')\sin\theta,\end{aligned}\right\}$$

and $$x'^2+y'^2=(\xi''+\eta')^2+(\eta''-\xi')^2,$$

$$x'y''-y'x''=(\xi''+\eta')(\eta'''-2\xi''-\eta')-(\eta''-\xi')(\xi'''+2\eta''-\xi').$$

And if ρ_2 be the radius of curvature of the I-locus on the moving lamina estimated in the same direction as I,

$$\frac{1}{\rho_2}=\frac{(\xi''+\eta')(\eta'''-2\xi''-\eta')-(\eta''-\xi')(\xi'''+2\eta''-\xi')}{[(\xi''+\eta')^2+(\eta''-\xi')^2]^{\frac{3}{2}}}.$$

Hence $$\frac{1}{\rho_2}\sim\frac{1}{\rho_1}=\frac{\xi''^2+\eta''^2+\xi'^2+\eta'^2-2\xi'\eta''+2\xi''\eta'}{[(\xi''+\eta')^2+(\eta''-\xi')^2]^{\frac{3}{2}}}$$

$$=\frac{1}{[(\xi''+\eta')^2+(\eta''-\xi')^2]^{\frac{1}{2}}}$$

$$=\frac{1}{(X'^2+Y'^2)^{\frac{1}{2}}},$$

which gives the difference of the curvatures sought.

Finally, $x'^2+y'^2=(\xi''+\eta')^2+(\eta''-\xi')^2=X'^2+Y'^2$,

and therefore if ds be the elementary arc of either curve,

$$\frac{ds}{d\theta}=\sqrt{(\xi''+\eta')^2+(\eta''-\xi')^2} \quad\text{and}\quad s=\int\sqrt{(\xi''+\eta')^2+(\eta''-\xi')^2}\,d\theta,$$

whence $$\frac{1}{\rho_2}\sim\frac{1}{\rho_1}=\frac{1}{\frac{ds}{d\theta}}.$$

661. Geometrical Consideration.

This last result may be seen at once geometrically; for $\dfrac{ds}{\rho_2}$ and $\dfrac{ds}{\rho_1}$ are the angles turned through by ρ_2 and ρ_1, and their difference is the angle turned through by the moving lamina, *i.e.*

$$\frac{ds}{\rho_2}\sim\frac{ds}{\rho_1}=d\theta. \quad \text{(See Fig. 190.)}$$

662. (1) Thus, in the case of the sliding rod of Art. 657, Ex. 1, we have

$$\xi=\quad a\sin\theta,\quad \eta=\quad a\cos\theta,$$
$$\xi'=\quad a\cos\theta,\quad \eta'=-a\sin\theta,$$
$$\xi''=-a\sin\theta,\quad \eta''=-a\cos\theta,$$

and $$\frac{1}{\rho_2}-\frac{1}{\rho_1}=\frac{1}{\sqrt{4a^2\sin^2\theta+4a^2\cos^2\theta}}=\frac{1}{2a}$$

which agrees with the previous result for which $\rho_1=2a$, $\rho_2=a$.

(2) In the case of the sliding involute (Art. 657, Ex. 2),

$$\xi=a\left(\frac{\pi}{2}+\theta\right),\quad \eta=a\theta,$$

$$\xi'=a,\quad \eta'=a,\quad \xi''=\eta''=0,$$

and $$\frac{1}{\rho_2}-\frac{1}{\rho_1}=\frac{1}{\sqrt{a^2+a^2}}=\frac{1}{a\sqrt{2}},$$

which agrees with the previous result, for which $\rho_1=\infty$, $\rho_2=a\sqrt{2}$; and

$$s=\int\sqrt{(\xi''+\eta')^2+(\eta''-\xi')^2}\,d\theta$$

gives $2a\theta$ in case (1) above, and $a\theta\sqrt{2}$ in case (2).

663. Besant's Equations for the Fixed *I*-locus for sliding curves.

When the motion of the lamina is defined by two curves attached to the lamina making sliding contact with fixed perpendicular axes OX, OY, the equations

$$X=\xi-\eta',\quad Y=\eta+\xi'$$

give and
$$\left.\begin{aligned} X' &= \xi' - \eta'' = Y - \eta - \eta'' \\ Y' &= \eta' + \xi'' = \xi - X + \xi'' \end{aligned}\right\} \text{respectively,}$$
and show that
$$\left.\begin{aligned} X' - Y &= -(\eta + \eta'') = -\rho_1, \\ Y' + X &= \quad \xi + \xi'' = \quad \rho_2, \end{aligned}\right\} \text{by Legendre's formula,}$$

where ρ_1 and ρ_2 are the radii of curvature of the sliding curves at the points of contact with the straight lines OX, OY.

These equations are obtained by geometrical considerations by Mr. Besant (*Roulettes and Glisettes*, Art. 51), and are the equations he uses for the determination of the I-locus on the fixed plane in such cases of sliding contact. They require the integration of two simultaneous differential equations for the determination of the locus.

When the intrinsic equations of the two curves are known, viz. $s=f_1(\psi)$, $s=f_2(\psi)$, Mr. Besant's equations are very convenient, and the fixed I-locus can be deduced by solving the simultaneous equations

$$\frac{dX}{d\psi} - Y = -f_1'(\psi), \quad \frac{dY}{d\psi} + X = f_2'\left(\psi + \frac{\pi}{2}\right),$$

the constant being determined by the starting conditions.

664. "**Roulettes and Glisettes.**"

The path of a point carried by a curve which rolls upon another curve is called the Roulette of the point. (See *Diff. Calc.*, Art. 561.)

The path of a point carried by a lamina which moves so that a curve drawn upon it slides in such a manner as to touch two given fixed curves is called a Glisette.

The latter name is due to Mr. W. H. Besant.

The terms Roulette and Glisette have been extended to include the case of the envelope of a carried curve.

A very full and interesting account of the principal properties of Roulettes and Glisettes was given by Mr. Besant in his *Tract on Roulettes and Glisettes* (1870).

665. **The Curvature of the Roulettes described by a Carried Point, and as the Envelope of a Carried Curve** are worked out in Articles 564 and 565 respectively of the *Diff. Calc.* The student who has not access to Mr. Besant's tract, should revise

these articles before reading the articles which follow, which are mainly concerned with quadrature and rectification.

The formula established for the radius of curvature of the envelope of a curve carried by another curve which rolls without sliding upon a fixed curve is shown to be

$$\frac{\cos\phi}{R-r}+\frac{\cos\phi}{r+\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

Here ρ_1, ρ_2 are the radii of curvature of the fixed and rolling curves respectively, ρ that of the carried curve, R that of its envelope, whilst r is the normal distance of the point of contact of the carried curve with its envelope from the point of contact of the rolling and fixed curves, and ϕ is the angle r makes with the common normal of the latter.

If all these several quantities can be expressed in terms of ψ', the angle which r makes with any fixed line, then $\int R\,d\psi'$ gives the length of an arc of the envelope, *i.e.*

$$\text{Arc}=\int\frac{r\left(\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}\right)+\dfrac{\rho}{r+\rho}\cos\phi}{\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}-\dfrac{\cos\phi}{r+\rho}}\,d\psi'.$$

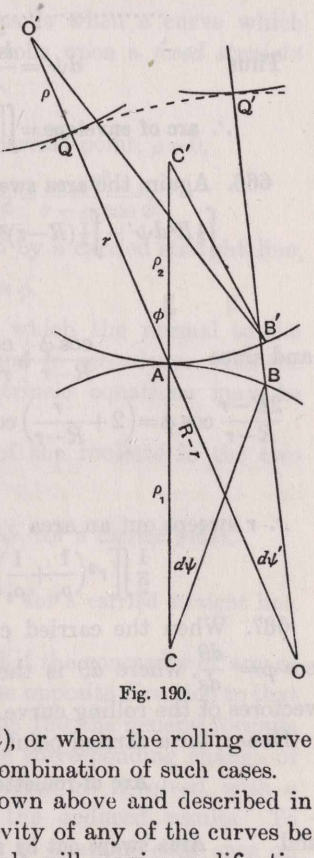

Fig. 190.

This is the **general result.** It includes the roulette of a carried point, viz. when $\rho=0$, or of a carried straight line (when $\rho=\infty$), or the case when the fixed curve is a straight line ($\rho_1=\infty$), or when the rolling curve is a circle ($\rho_2=a$), or when the rolling curve is a straight line ($\rho_2=\infty$), or any combination of such cases.

The standard figure is that shown above and described in *Diff. Calc.*, Art. 565. If the concavity of any of the curves be in the opposite direction, the formula will require modification by the change of sign in the particular radius of curvature or particular radii of curvature involved.

It must be remembered from *Diff. Calc.*, Art. 565, that the angle between

$$\text{two consecutive positions of } \rho_1 \text{ is } \frac{ds}{\rho_1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \rho_2 \text{ is } \frac{ds}{\rho_2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots r \text{ is } \frac{ds\cos\phi}{R-r},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \rho \text{ is } \frac{ds\cos\phi}{r+\rho}.$$

Thus $$d\psi'=\frac{ds\cos\phi}{R-r};$$

$$\therefore \textbf{ arc of envelope}=\int\left[r\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{\rho}{r+\rho}\cos\phi\right]ds.$$

666. Again, the **area swept out by** r is plainly

$$\int\tfrac{1}{2}R^2d\psi'-\int\tfrac{1}{2}(R-r)^2d\psi'=\frac{1}{2}\int(2Rr-r^2)\cdot\frac{ds\cos\phi}{R-r}$$

$$=\frac{1}{2}\int r\frac{2R-r}{R-r}\cos\phi\, ds,$$

and since $$\frac{\cos\phi}{R-r}+\frac{\cos\phi}{r+\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2},$$

$$\frac{2R-r}{R-r}\cos\phi=\left(2+\frac{r}{R-r}\right)\cos\phi=2\cos\phi+r\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}-\frac{\cos\phi}{r+\rho}\right)$$

$$=r\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{2\rho+r}{\rho+r}\cos\phi;$$

$\therefore$ r sweeps out an area

$$\frac{1}{2}\int\left[r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+r\frac{2\rho+r}{\rho+r}\cos\phi\right]ds.$$

667. When the carried curve *reduces to a point, i.e.* $\rho=0$, $\cos\phi=\dfrac{d\theta}{ds}$, where $d\theta$ is the angle between consecutive radii vectores of the rolling curve.

Hence, for a carried point,

$$\textbf{Arc of roulette}=\int r\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds,$$

and $$\textbf{Area swept out by } r=\frac{1}{2}\int r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds+\frac{1}{2}\int r^2d\theta.$$

Hence the area swept out by r exceeds the corresponding portion of the sectorial area of the rolling curve,

$$\text{viz. } \frac{1}{2}\int r^2 d\theta, \text{ by } \frac{1}{2}\int r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds.$$

And if the rolling curve be a straight line, $\rho_2=\infty$, and these expressions reduce further to

$$\text{Arc}=\int\frac{r}{\rho_1}ds \quad \text{and} \quad \text{Area swept}=\frac{1}{2}\int\frac{r^2}{\rho_1}ds+\frac{1}{2}\int r^2 d\theta$$

respectively.

668. **Important Cases.**

The most important case, perhaps, is when a curve which carries a point or a straight line rolls upon a *fixed straight line.*

In this case $\rho_1=\infty$.

If also the roulette be that of a carried point, $\rho=0$,

$$R=r+\rho_2\cos\phi\,\frac{r}{r-\rho_2\cos\phi}=\frac{r^2}{r-\rho_2\cos\phi}.$$

If the roulette be that enveloped by a carried straight line, $\rho=\infty$, and

$$R=r+\rho_2\cos\phi.$$

In these cases ϕ is the angle which the normal to the roulette makes with a fixed line, and in accordance with the usual custom in dealing with intrinsic equations may be written ψ.

Hence the intrinsic equations of the roulette in the two cases will be respectively

$$s=\int\left(r+\rho_2\cos\psi\,\frac{r}{r-\rho_2\cos\psi}\right)d\psi, \text{ for a carried point,}$$

$$s=\int(r+\rho_2\cos\psi)d\psi, \qquad \text{for a carried straight line.}$$

669. It is to be further noted that if the concavity of any of the curves concerned be turned in the opposite direction to that in which they are represented in Fig. 190, the general formula for R will need modification by the corresponding change of sign of the particular radii of curvature involved with a corresponding modification in all the deduced results. To avoid error it is therefore desirable to examine each case on

its own merits, rather than to deduce the formulae required from the general result

$$\frac{\cos\phi}{R-r}+\frac{\cos\phi}{r+\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

Moreover, special cases have their own special geometrical peculiarities. Hence, in succeeding articles, we adopt this course though it necessitates some repetition. This will also have the advantage of exhibiting a somewhat different treatment.

670. Ex. 1. A circular wheel rolls in a vertical plane along a straight line. To find the intrinsic equation of the envelope of a given diameter.

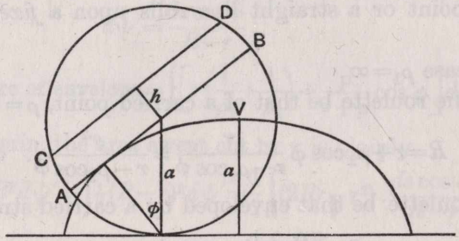

Fig. 191.

Here $\rho_2=$ the radius of the wheel $=a$, say;

$r=a\cos\phi$;

$\therefore\ R=r+\rho_2\cos\phi=2a\cos\phi$;

$\therefore\ s=2a\sin\phi$;

i.e. the envelope is a cycloid with an axis of length a, s being measured from the vertex of the cycloidal envelope.

For a parallel chord at a distance h from this diameter, we have

$$r=h+a\cos\phi$$

and $$s=h\phi+2a\sin\phi,$$

viz. a parallel to a cycloid. Moreover, the cycloid which is the envelope of the diameter of the rolling circle, is itself an involute of another cycloid. Hence the parallels to the cycloid are involutes of a cycloid. This then is the result for any carried line.

Ex. 2. Let the rolling curve be $r^n=a^n\cos n\theta$, and suppose the initial position be that in which the vertex of a foil of the curve is in contact with the line.

First, let us find the roulette of the pole.

We have $$p=\frac{r^{n+1}}{a^n},\qquad \rho_2=\frac{r\,dr}{dp}=\frac{1}{n+1}\frac{a^n}{r^{n-1}}.$$

Let P be the point of contact, O the pole, A its initial position, ψ the angle turned through by the tangent at O to the roulette, $x'Cx$ the fixed line.

Then $\quad \tan OPx' = \frac{r\,d\theta}{dr} = -\cot n\theta; \quad \therefore \quad OPx' = \frac{\pi}{2} + n\theta;$

$$\therefore \ \phi = n\theta, \ \rho_2 \cos\phi = \frac{1}{n+1} r \quad \text{and} \quad \phi = \psi.$$

$$R = \frac{r^2}{r - \rho_2 \cos\phi} = \frac{n+1}{n} r;$$

$$\therefore \ \frac{ds}{d\psi} = \frac{n+1}{n} a \cos^{\frac{1}{n}} \psi,$$

and $s = \frac{n+1}{n} a \int \cos^{\frac{1}{n}} \psi \, d\psi$ is the intrinsic equation sought.

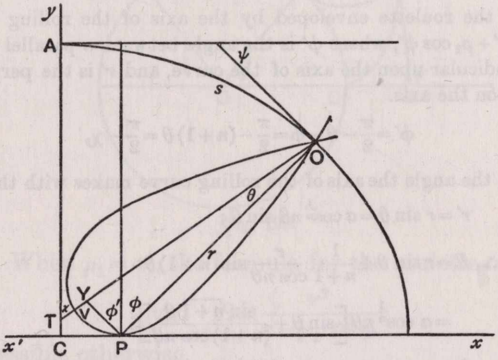

Fig. 192.

If $n=1$, we have the case of a rolling circle of diameter a, and the intrinsic equation of the cycloid traced is $s = 2a\sin\psi$.

In the general case, if we refer the curve to tangential polar coordinates, we can perform one integration. For taking A as pole.

$$\frac{d^2p}{d\psi^2} + p = \frac{ds}{d\psi} = \frac{n+1}{n} a \cos^{\frac{1}{n}} \psi.$$

Multiplying by $\sin\psi$,

$$\sin\psi \frac{d^2p}{d\psi^2} + p\sin\psi = \frac{n+1}{n} a \cos^{\frac{1}{n}} \psi \sin\psi,$$

and integrating, $\quad \sin\psi \frac{dp}{d\psi} - p\cos\psi = -a\cos^{1+\frac{1}{n}} \psi + a,$

for p and $\frac{dp}{d\psi}$ vanish if $\psi = 0$ and p be measured from the vertex A;

$$\therefore \ \frac{dp}{d\psi} - p\cot\psi = a \operatorname{cosec}\psi (1 - \cos^{1+\frac{1}{n}} \psi).$$

Again, multiplying by $\cos\psi$ and integrating

$$\cos\psi\frac{dp}{d\psi}+p\sin\psi=\frac{n+1}{n}a\int_0^\psi \cos^{1+\frac{1}{n}}\psi\,d\psi,$$

or
$$\frac{dp}{d\psi}+p\tan\psi=\frac{n+1}{n}a\sec\psi\int_0^\psi \cos^{1+\frac{1}{n}}\psi\,d\psi.$$

Eliminating $\dfrac{dp}{d\psi}$, we obtain

$$p(\tan\psi+\cot\psi)=\frac{n+1}{n}a\sec\psi\int_0^\psi \cos^{1+\frac{1}{n}}\psi-a\operatorname{cosec}\psi(1-\cos^{1+\frac{1}{n}}\psi),$$

or
$$p=\frac{n+1}{n}a\sin\psi\int_0^\psi \cos^{\frac{n+1}{n}}\psi\,d\psi-a\cos\psi\left(1-\cos^{\frac{n+1}{n}}\psi\right),$$

as the tangential-polar equation of the roulette, the origin being at the vertex of the roulette.

To find the roulette enveloped by the axis of the rolling curve, we have $R=r'+\rho_2\cos\phi'$, where ϕ' is the angle between a parallel to CA and the perpendicular upon the axis of the curve, and r' is the perpendicular from P upon the axis.

Then
$$\phi'=\frac{\pi}{2}-\theta-\phi=\frac{\pi}{2}-(n+1)\theta=\frac{\pi}{2}-\chi,$$

where χ is the angle the axis of the rolling curve makes with the line CA, and
$$r'=r\sin\theta=a\cos^{\frac{1}{n}}n\theta\sin\theta;$$

$$\therefore\ R=r\sin\theta+\frac{1}{n+1}\frac{r}{\cos n\theta}\sin(n+1)\theta$$

$$=a\cos^{\frac{1}{n}}n\theta\left[\sin\theta+\frac{\sin\overline{n+1}\,\theta}{(n+1)\cos n\theta}\right],$$

$$\frac{ds}{d\chi}=a\cos^{\frac{1}{n}}\frac{n}{n+1}\chi\left[\sin\frac{\chi}{n+1}+\frac{\sin\chi}{(n+1)\cos\dfrac{n}{n+1}\chi}\right],$$

and the intrinsic equation of the envelope of the axis is therefore

$$s=a\int_0^\chi\left\{\sin\frac{\chi}{n+1}+\frac{\sin\chi}{(n+1)\cos\dfrac{n}{n+1}\chi}\right\}\cos^{\frac{1}{n}}\frac{n\chi}{n+1}d\chi.$$

Ex. Special Case of the Epi- and Hypo-cycloids.

Here $\rho_1=a$, the radius of the fixed circle; $\rho_2=b$, the radius of the rolling circle; $\rho=0$.

$$(\pi-2\phi)b=a\theta,\quad \psi=\theta+\frac{\pi}{2}-\phi;$$

$$\therefore\ \phi=\frac{\pi}{2}-\frac{a}{a+2b}\psi\quad\text{and}\quad r=2b\cos\phi;$$

and
$$\frac{\cos\phi}{R-r}=\frac{1}{a}+\frac{1}{b}-\frac{\cos\phi}{r}=\frac{1}{a}+\frac{1}{2b}=\frac{a+2b}{2ab};$$

$$\therefore \ R = r\left(1+\frac{a}{a+2b}\right) = 2\frac{a+b}{a+2b}\cdot 2b\cos\phi,$$

$$\frac{ds}{d\psi} = -4b\frac{a+b}{a+2b}\sin\frac{a}{a+2b}\psi$$

(*s* measured from the vertex increases as ψ diminishes);

$$s = \frac{4b}{a}(a+b)\cos\frac{a}{a+2b}\psi,$$

s being measured from the vertex (Art. 412, *Diff. Calc.*).

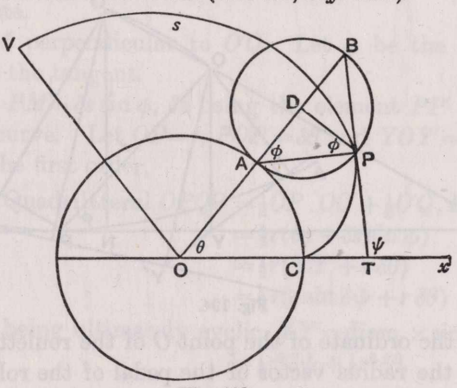

Fig. 193.

671. When $\rho_1=\infty$, the formula for the roulette of a carried point,

$$\text{viz.}\quad R = \frac{r^2}{r-\rho_2\cos\phi},$$

is expressible otherwise.

For with the usual notation, taking the carried point as the pole of the rolling curve,

$$\rho_2 = \frac{r\,dr}{dp} \quad\text{and}\quad \cos\phi = \frac{p}{r}.$$

$$\text{Hence}\qquad \frac{1}{R} = \frac{r-p\dfrac{dr}{dp}}{r^2} = \frac{d}{dp}\left(\frac{p}{r}\right),$$

which gives a convenient measure for R in this case.

672. **General Theorems with regard to Rolling on a Fixed Straight Line. Roulette of a Carried Point. Theorems of Jacob Steiner and W. H. Besant.**

Let *APB* be any curve rolling along a straight line *xz*, *P* being the point of contact, *P′* the adjacent point on the curve which will come into contact with the line at *Q*. Let *O* be a carried point and O' the point at which it arrives when the rolling of the curve has carried P' to Q.

Let OY, OY' be the perpendiculars from O upon the contiguous tangents at P and P'. Let $OO'=d\sigma$, the elementary arc traced by O as the point of contact travels from P to Q. Let $O'O$ cut xz at R. Then OY plays a double part.

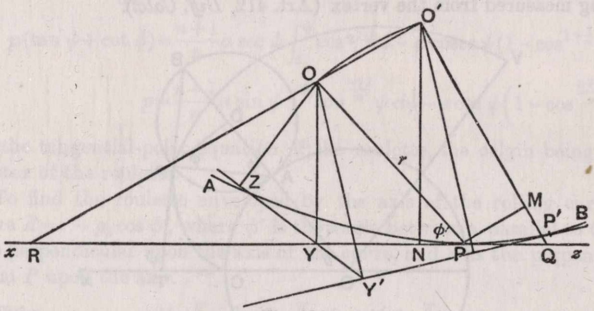

Fig. 194.

(1) It is the ordinate of the point O of the roulette of O.

(2) It is the radius vector of the pedal of the rolling curve with regard to O.

Let the elementary arc of the pedal curve, viz. YY', be called ds_p.

Then $$\frac{dy}{d\sigma}=Lt \sin z\hat{R}O=Lt\cos R\hat{P}O,$$

for OP is the normal to the roulette (Art. 562, *Diff. Calc.*)

$$=Lt\cos O\hat{Y}'Y=\frac{dOY}{ds_p}=\frac{dy}{ds_p}.$$

That is, in the limit, $d\sigma=ds_p$. ....(1)

Hence corresponding arcs of the roulette of O and of the pedal of the rolling curve with regard to O are equal.

This theorem is due to Jacob Steiner (1796-1863).*

673. Again, if OZ be the perpendicular from O on YY', we have ultimately

$$y\frac{dx}{d\sigma}=Lt\, y\cos z\hat{R}O=Lt\, y\sin R\hat{P}O=Lt\, y\sin O\hat{Y}'Y$$
$$=Lt\, y\sin O\hat{Y}Z=OZ;$$

$$\therefore\ y\,dx=OZ\,d\sigma=OZ\,ds_p,$$

i.e. the element $OYNO'$ is ultimately double the element OYY'.

* Cajori's *History of Mathematics*, p. 295.

Hence integrating, the area swept out by the ordinate of the roulette during any portion of the rolling is double the corresponding sectorial area of the pedal curve.

This theorem appears to be due to the late W. H. BESANT (Art. 26, *Roulettes and Glisettes*).

674. We consider next the **area swept out by the normal** OP **to the roulette.**

Draw PM perpendicular to $O'Q$. Let ϕ be the angle OP makes with the tangent.

We have $PM = \delta s \sin\phi$, δs being the element PP' or PQ of the rolling curve. Let $OP = r$, $POP' = \delta\theta$ and $YOY' = \delta\psi$.

Then to the first order,

$$\begin{aligned}\text{Quadrilateral } OPQO' &= \tfrac{1}{2}OP\,.\,OO' + \tfrac{1}{2}O'Q\,.\,PM\\ &= \tfrac{1}{2}r(\delta\sigma + \delta s\sin\phi)\\ &= \tfrac{1}{2}r(YY' + r\,\delta\theta)\\ &= \tfrac{1}{2}r(r\sin\delta\psi + r\,\delta\theta)\end{aligned}$$

(for $OYY'P$ being ultimately cyclic, $YY' = \text{diam.}\times\sin YOY'$)

$$= \tfrac{1}{2}r^2\delta\psi + \tfrac{1}{2}r^2\delta\theta.$$

$\therefore$ area swept out by normal in any portion of the rolling

$$= \text{corresponding sectorial area of curve} + \frac{1}{2}\int r^2 d\psi,$$

the limits for ψ being its initial and final values.

675. If the curve be a closed oval, every point of whose perimeter comes into contact with the line in one revolution, and if we suppose the rolling to start with OP at right angles to the line, so that the limits for ψ may be specified as 0 to 2π, we have for a complete revolution

$$\begin{aligned}\text{Area swept by normal} &= \text{area of rolling curve} + \frac{1}{2}\int_0^{2\pi} r^2 d\psi\\ &= \text{area of curve} + \frac{1}{2}\int_0^{2\pi}\frac{r^2}{\rho}ds.\end{aligned}$$

But by Art. 426

$$2\text{ area of pedal} = \text{area of curve} + \frac{1}{2}\int_0^{2\pi}\frac{r^2}{\rho}ds\,;$$

$$\therefore \left.\begin{array}{l}\textbf{area swept out by normal}\\ \textbf{in a complete revolution}\end{array}\right\} = 2\text{ area of pedal.}$$

This theorem is also due to **Steiner.***

* See Bertrand, *Calc. Intég.*, p 362 and Besant, *Roulettes and Glisettes*, p. 19.

676. It is worth noting also that

$$\textbf{Area of oval} = 2 \text{ area of pedal} - \frac{1}{2}\int_0^{2\pi} r^2 d\psi$$
$$= \int_0^{2\pi} p^2 d\psi - \frac{1}{2}\int_0^{2\pi} r^2 d\psi$$
$$= \frac{1}{2}\int_0^{2\pi} (2p^2 - r^2)\, d\psi.$$

(Besant, *R. and G.*, p. 19.)

677. Illustrative Examples.

1. When an ellipse rolls upon a straight line, any arc of the roulette of the focus is equal to the corresponding portion of the circumference of the circle which is the first positive pedal of the ellipse with regard to the focus, *i.e.* the auxiliary circle.

The roulette of the centre is of the same length as the corresponding arc of its central pedal, viz. $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$.

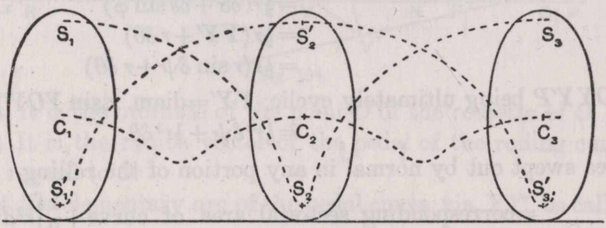

Fig. 195.

And in both cases the areas swept by the ordinate are double of the corresponding sectorial area of the pedal. In a complete revolution these areas are $2\pi a^2$ for the area swept by the ordinate of the focus in a complete revolution of the ellipse and $\pi(a^2+b^2)$ for the roulette of the centre. These paths are illustrated in the accompanying diagram.

2. The arc of the roulette of a point rigidly connected *with a circle rolling* on a straight line (*i.e. a Trochoid*) is equal to the corresponding portion of the limaçon which is the first positive pedal of the circle with regard to the point. And when the point is on the circumference of the rolling circle, we see that the arc of a cycloid is of the same length as the corresponding arc of a cardioide.

3. If *a rectangular hyperbola rolls along a straight line*, any arc of the roulette of the centre is equal to the corresponding arc of the lemniscate which is the pedal of the hyperbola with regard to the centre, and is therefore expressible as an elliptic integral (Art. 592).

4. When *a parabola rolls along a straight line*, the arc of the roulette of the vertex is equal to the arc of the cissoid which is the first positive pedal of the parabola with regard to the vertex.

Many other cases may be cited and many curves may be discovered as roulettes whose arcs can be found; this being so whenever the arc of the pedal of the rolling curve can be found.

In each of these cases we also find that the area swept out by the ordinate is double the corresponding sectorial area of the pedal.

678. General Theorems with regard to Rolling on a Curve. Rectification of Roulette of a Carried Point P.

We may prove the results for a carried point P as follows, directly and without deduction from the general formulae.

Let A be the point of contact,

B_2 an adjacent point on the fixed curve,

B_1 the point on the rolling curve which will come into contact with B_2,

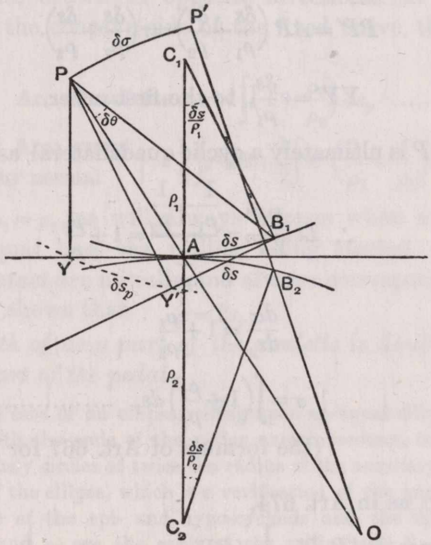

Fig. 196.

P, P' the two points on the roulette corresponding to the points of contact A and B_2, so that PA, $P'B_2$ are contiguous normals to the roulette. Let these meet in O.

Let C_1, C_2 be the centres of curvature of the rolling and fixed curves respectively at A, $P\hat{A}C_1 = \phi$,

ρ_1, ρ_2 the radii of curvature,

$r = AP$; PY, PY' perpendiculars on tangents at A and B_1,
δs, $\delta\sigma$, δs_p the elementary arcs of the fixed and rolling curves, the roulette, and the pedal of the rolling curve with regard to P; *i.e.*

$$AB_1 = AB_2 = \delta s, \quad PP' = \delta\sigma, \quad YY' = \delta s_p$$

Then when C_1B_1 comes into line with B_2C_2, PB_1 will come into line with B_2O. Let $APB_1 = \delta\theta$.

Then the angle turned through by the rolling curve is

$$A\hat{C}_1B_1 + A\hat{C}_2B_2 = \frac{\delta s}{\rho_1} + \frac{\delta s}{\rho_2}.$$

Also PB_1 turns through the same angle, and B_1B_2 is a second order small quantity. Hence, to the first order,

$$PP' = AP\left(\frac{\delta s}{\rho_1} + \frac{\delta s}{\rho_2}\right) = r\left(\frac{\delta s}{\rho_1} + \frac{\delta s}{\rho_2}\right).$$

Again, $\qquad YY' = r\dfrac{\delta s}{\rho_1}$, to the first order,

since $YY'AP$ is ultimately a cyclic quadrilateral, as in Art. 674;

$$\therefore \; Lt\,\frac{PP'}{YY'} = \frac{\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}}{\dfrac{1}{\rho_1}} = 1 + \frac{\rho_1}{\rho_2},$$

i.e. $$\frac{d\sigma}{ds_p} = 1 + \frac{\rho_1}{\rho_2}$$

and $$\sigma = \int\left(1 + \frac{\rho_1}{\rho_2}\right) ds_p \quad \text{..........................(A)}$$

(the formula of Art. 667 for $\rho_1\, ds_p = r\, ds$).

679. Also, as in Art. 674,

$$\text{Area } PAB_2P' = \frac{1}{2}r(PP' + \delta s \sin\phi), \text{ to the first order,}$$

$$= \frac{1}{2}r\left\{\left(1 + \frac{\rho_1}{\rho_2}\right)\delta s_p + \delta s \sin\phi\right\}$$

$$= \frac{1}{2}r\left\{\left(1 + \frac{\rho_1}{\rho_2}\right) r\frac{\delta s}{\rho_1} + r\,\delta\theta\right\}$$

$$= \frac{1}{2}r^2\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\delta s + \frac{1}{2}r^2\delta\theta.$$

And integrating, **the area swept out by the normal to the roulette** between the roulette and the fixed curve

$$=\frac{1}{2}\int r^2 d\theta+\frac{1}{2}\int r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds \text{ (the formula of Art. 667).} \quad \text{(B)}$$

680. When the *rolling curve is closed*, we have for the *whole* area swept by the normal in one turn of the curve, such that the original point of contact has again come into contact,

$$\textbf{Area swept} = \textbf{area of curve} + \frac{1}{2}\int r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds,$$

the limits of integration being from the initial to the final value of s.

681. It should be noted that in the investigations above, ρ_1 and ρ_2 are drawn in opposite directions. If the rolling curve be on the concave side of the fixed curve, the formulae will become

$$\textbf{Arc of roulette} = \sigma = \int\left(1-\frac{\rho_1}{\rho_2}\right)ds_p \text{ ....(A}')$$

$$\left.\begin{array}{r}\textbf{and Area swept}\\ \textbf{by normal}\end{array}\right\} = \frac{1}{2}\int r^2 d\theta+\frac{1}{2}\int r^2\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)ds. \text{ ......(B}')$$

682. If $\rho_1=\rho_2$, as will always happen when a curve rolls upon an equal one, the rolling being started so that the points of contact are initially and always corresponding points, formula (A) shows that $\sigma=2s_p$,

i.e. the *length of any part of the roulette is double the corresponding part of the pedal.*

683. In the case of an ellipse rolling upon an equal ellipse and placed at starting with the ends of the major axes in contact, the paths of the foci are obviously circles of twice the radius of the auxiliary circle, which is the pedal of the ellipse, which is a verification of the general theorem.

In the case of the epi- and hypo-cycloids and the epi- and hypotrochoids, ρ_1 and ρ_2 are the radii of the rolling and fixed circles and constant. Hence the arcs of such curves are proportional to the corresponding arcs of the first positive pedal of the rolling circle, *i.e.* to the arc of a cardioide or of a limaçon, and are therefore rectifiable in the same manner.

684. **Rolling along both sides of a Curve.**

If the rolling curve be allowed to roll first on the convex side of a fixed curve and then upon the concave side, starting with the same pair of points common and rolling in the same manner as before, so that corresponding points again come into contact, formulae (A) and (B), (A′) and (B′) show that if σ, σ' be the arcs of the roulette, and A, A' the areas described by the normal in the two cases, and A_p the corresponding area of the pedal of the rolling curve, then

$$\sigma+\sigma'=\int\left(1+\frac{\rho_1}{\rho_2}\right)ds_p+\int\left(1-\frac{\rho_1}{\rho_2}\right)ds_p=2s_p$$

and
$$A+A'=\frac{1}{2}\int r^2d\theta+\frac{1}{2}\int r^2\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)ds$$
$$+\frac{1}{2}\int r^2d\theta+\frac{1}{2}\int r^2\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)ds$$
$$=\int r^2d\theta+\int\frac{r^2}{\rho_1}ds=4A_p.$$

And both results being independent of ρ_2, are independent of the nature of the fixed curve, and therefore in each case double of the results for rolling along a straight line.

685. *In the case of a curve carried by a second curve which itself slides in contact with two other curves, or moves in its own plane in any given manner, the same formulae as those established for a roulette can be used for the curvature and rectification of the envelope of the attached curve.*

For the motion being a case of rolling of the locus of the instantaneous centre I, traced on the moving lamina, upon the locus of the instantaneous centre I traced on a fixed plane, it is a matter in general of first determining these loci and their radii of curvature; or, what is equivalent, if δs be the arc of the fixed I-locus and ϕ the angle which the normal to the I-locus makes with the normal through I to the carried curve, and if $\delta\chi$ be the angle turned through by the moving curve whilst I travels over δs on its locus,

$$d\chi=\frac{ds}{\rho_1}+\frac{ds}{\rho_2},$$

and the formula
$$\frac{\cos\phi}{R-r}+\frac{\cos\phi}{r+\rho}=\frac{1}{\rho_1}+\frac{1}{\rho_2}$$

may be written
$$\frac{\cos\phi}{R-r}+\frac{\cos\phi}{r+\rho}=\frac{d\chi}{ds}$$

the various letters having the same meanings as before, ρ_1, ρ_2 referring to the two I-loci, the values being obtainable as explained in Art. 660.

When $\frac{d\chi}{ds}$, which is $\frac{1}{\rho_1}+\frac{1}{\rho_2}$, $\cos\phi$, r and ρ have been expressed in terms of ψ, the angle which the normal to the carried curve makes with a given line, the radius of curvature of the envelope is

$$\frac{d\sigma}{d\psi}=R=r+\frac{\cos\phi}{\frac{d\chi}{ds}-\frac{\cos\phi}{r+\rho}},$$

and $\sigma=\int R\,d\psi$ gives the intrinsic equation of the envelope of the carried curve.

Also, as before, the case of a carried point is included as that for which $\rho=0$, and the case of a carried straight line is included as that for which $\rho=\infty$, which respectively give

$$\sigma=\int\left(r+\frac{\cos\phi}{\frac{d\chi}{ds}-\frac{\cos\phi}{r}}\right)d\psi \quad\text{and}\quad \sigma=\int\left(r+\cos\phi\frac{ds}{d\chi}\right)d\psi$$

as the intrinsic equations required.

686. **When a Curve slides in such a manner as always to touch a Given Straight Line at a Given Point**, the glisette of any carried point is obtainable at once.

Let the carried point be taken as a pole, and let $p=f(\psi)$ be the tangential polar equation of the curve with regard to this pole.

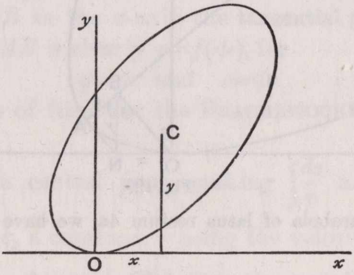

Fig. 197.

Then if the point of contact be taken as the origin and the given straight line as the x-axis, we have

$$\left.\begin{aligned}x&=\frac{dp}{d\psi}=f'(\psi),\\ y&=p=f(\psi),\end{aligned}\right\}$$

and the ψ-eliminant is the "glisette" required.

687. Illustrative Examples.

Ex. 1. If the curve be an equiangular spiral, we obviously have

$$p = r\sin\alpha \quad \text{and} \quad \frac{dp}{d\psi} = r\cos\alpha;$$

$\therefore\ y = x\tan\alpha$ is the path of the pole, as is geometrically obvious.

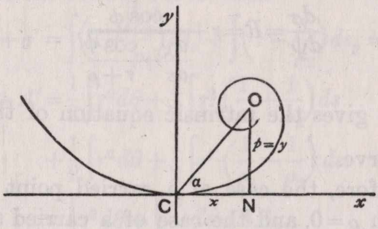

Fig. 198.

Ex. 2. If the curve be an ellipse,

$$\left.\begin{aligned} p^2 &= a^2\cos^2\psi + b^2\sin^2\psi, \\ p\frac{dp}{d\psi} &= -(a^2-b^2)\sin\psi\cos\psi, \end{aligned}\right\}$$

and the ψ-eliminant gives for the glisette of the centre the quartic

$$x^2y^2 = (a^2-y^2)(y^2-b^2).$$

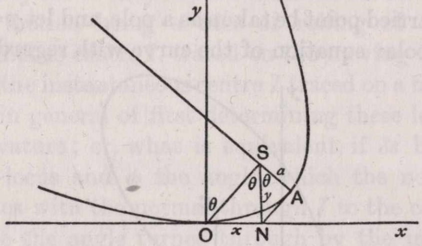

Fig. 199.

Ex. 3. In a parabola of latus rectum $4a$, we have for the glisette of the focus

$$\left.\begin{aligned} \frac{x}{y} &= \tan\theta, \\ \frac{y}{a} &= \sec\theta, \end{aligned}\right\}$$ 2θ being the angle subtended at the focus by the arc from the vertex to the point of contact (Fig. 199);

$$\therefore\ \frac{y^2}{a^2} - \frac{x^2}{y^2} = 1, \quad \text{or} \quad y^2(y^2-a^2) = a^2x^2,$$

i.e.

$$\frac{2a}{r} = (1+\cos 2\theta),$$

θ being the angle OS makes with the y-axis.

688. **(ι, y) Relations.**

In many curves the relation between the ordinate y and the angle ι between the ordinate and the tangent takes a very simple form, and is, moreover, very useful (1) in the determination of the envelope of a straight line carried by a curve which always touches a given straight line *at a given point* and also (2) in the problem of Brachistochronism for a law of force which is always in the same direction.

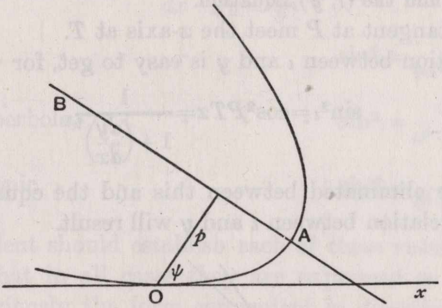

Fig. 200.

(1) Let O be the fixed point at which the curve always touches the fixed line Ox.

Let AB be the carried line.

Then if the equation of the curve has been expressed as $y=f(\iota)$, with AB as the x-axis, the **tangential polar equation of the envelope** of AB is clearly $p=f(\psi)$, for

$$y=p \quad \text{and} \quad \iota=\psi.$$

(2) The laws of force for the BRACHISTOCHRONOUS description of a curve,

(*a*) under a central force making $\int\frac{ds}{v}$ a minimum and $\frac{v}{p}=k$, a constant, v being the velocity;

(*b*) under a force parallel to a given straight line which we may take as the y-axis making $\int\frac{ds}{v}$ a minimum and $\frac{v}{\cos\psi}=u$, a constant,

are respectively

$$P=\frac{k^2}{2}\frac{dp^2}{dr} \quad \text{and} \quad P=\frac{u^2}{2}\frac{d}{dy}(\sin^2\iota).$$

These will be found in books treating of kinetics of a particle. They are placed here for the convenience of the student, and to illustrate further the use of the (ι, y) equation of a curve which is necessary for the glisette of a carried line with motion described above. The central force formula we are not now concerned with, but it will serve for practice in the use of (p, r) equations.

689. **To find the (ι, y) Equation.**

Let the tangent at P meet the x-axis at T.

The relation between ι and y is easy to get, for

$$\sin^2\iota = \cos^2 PTx = \frac{1}{1+\left(\dfrac{dy}{dx}\right)^2},$$

and if x be eliminated between this and the equation of the curve the relation between ι and y will result.

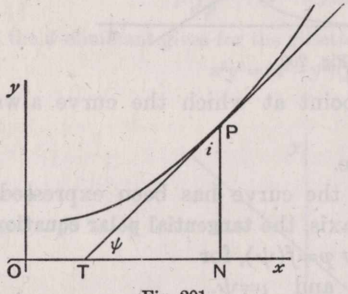

Fig. 201.

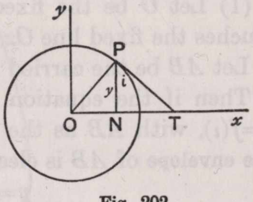

Fig. 202.

List of Common (ι, y) Equations.

Circle, - - - - - $\sin^2\iota = \dfrac{y^2}{a^2}$.

Catenary, - - - - $\sin^2\iota = \dfrac{c^2}{y^2}$.

Tractrix, - - - - $\sin^2\iota = 1 - \dfrac{y^2}{c^2}$.

Cycloid, - - - - - $\sin^2\iota = 1 - \dfrac{y}{2a}$.

Evolute of a parabola,* - - $\sin^2\iota = 1 - \dfrac{a}{y}$.

* Directrix for x-axis, Lat. Rect. $=4a/3$.

Evolute of a catenary, - - - $\sin^2\iota=1-\dfrac{4c^2}{y^2}$.

Four-cusped hypo-cycloid, - - $\sin^2\iota=1-\left(\dfrac{y}{a}\right)^{\frac{2}{3}}$.

Curves of the class $\dfrac{dy}{dx}=\dfrac{\sqrt{a^n-y^n}}{y^{\frac{n}{2}}}$, $\sin^2\iota=\dfrac{y^n}{a^n}$.

Curves of the class $\dfrac{dy}{dx}=\dfrac{y^{\frac{n}{2}}}{\sqrt{a^n-y^n}}$, $\sin^2\iota=1-\dfrac{y^n}{a^n}$.

Parabola, - - - - - $\sin^2\iota=\dfrac{y^2}{4a^2+y^2}$.

Rect. hyperbola, - - - - $\sin^2\iota=\dfrac{a^4}{a^4+y^4}$.

Biaxal conic, - - - - - $\sin^2\iota=\dfrac{a^2y^2}{b^4+(a^2-b^2)y^2}$.

The student should establish each of these results. It will be noted that in all cases they are expressed as $\sin^2\iota=f(y)$. This is obviously the form convenient in discussing Brachistochronism.

690. Ex. 1. If, for instance, a catenary slides in contact with a straight line Ox at a fixed point O, we have for the envelope of the directrix the tangential polar-equation $p=\dfrac{c}{\sin\psi}$, for $y=\dfrac{c}{\sin\iota}$ is the (ι, y) equation.

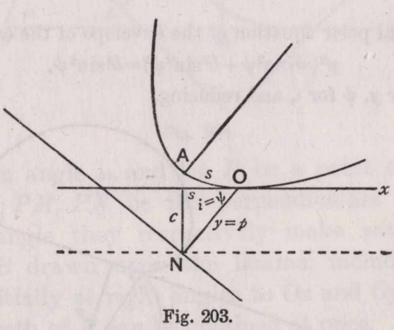

Fig. 203.

It is obvious from this equation that the directrix touches a parabola with O for focus and $4c$ for latus rectum. This is clear geometrically also, for the locus of the foot of the perpendicular upon the directrix is obviously a line at a distance c from the fixed line, and the envelope of the directrix is the first negative pedal of a fixed line, *i.e.* a parabola.

Since $\sin^2\iota=\dfrac{c^2}{y^2}$, the equation $P=\dfrac{u^2}{2}\dfrac{d}{dy}(\sin^2\iota)$ gives

$$P=-\frac{u^2}{2}\frac{2c^2}{y^3}=-\frac{c^2u^2}{y^3}.$$

Hence, the catenary is Brachistochronous for a law of force which acts perpendicularly towards the directrix and varying inversely as the cube of the distance from the directrix. The line of zero velocity in this case is at infinity.

Ex. 2. An ellipse slides, touching a straight line at a given point. What is the envelope of the axis major?

Here $$\cot^2\iota=\frac{b^4x^2}{a^4y^2}=\frac{b^2(b^2-y^2)}{a^2y^2};$$

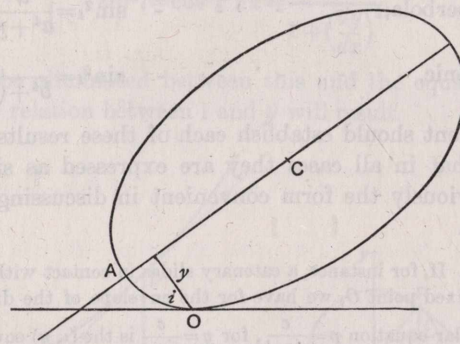

Fig. 204.

$\therefore$ the tangential polar equation of the envelope of the carried axis is
$$p^2(a^2\cos^2\psi+b^2\sin^2\psi)=b^4\sin^2\psi,$$
by writing p for y, ψ for ι, and reducing.

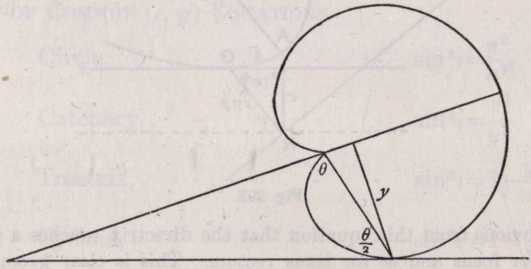

Fig. 205.

Ex. 3. A cardioide slides in contact with a fixed straight line at a fixed point. What is the envelope of the axis?

Here $$y = r\sin\theta = a(1-\cos\theta)\sin\theta$$
$$= 4a\sin^3\frac{\theta}{2}\cos\frac{\theta}{2},$$
and $$\iota = \frac{\theta}{2} + \left\{\frac{\pi}{2} - (\pi - \theta)\right\}$$
$$= \frac{3\theta}{2} - \frac{\pi}{2}.$$

Putting p for y and ψ for ι, the tangential polar equation of the envelope of the axis is
$$p = 4a\sin^3\frac{1}{3}\left(\frac{\pi}{2}+\psi\right)\cos\frac{1}{3}\left(\frac{\pi}{2}+\psi\right)$$
$$= a\sin\frac{\pi+2\psi}{3} - \frac{a}{2}\sin\frac{2\pi+4\psi}{3}.$$

691. Two Curves in the Lamina touching Fixed Straight Lines.

Let two curves be drawn upon a lamina, and let the lamina move so that the curves touch two given straight lines Ox, Oy

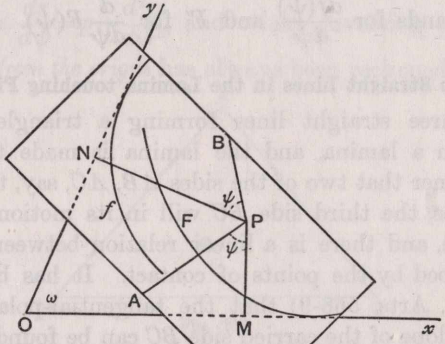

Fig. 206.

inclined at an angle ω, and let P be a point carried by the lamina. Let PM, PN be the perpendiculars upon Ox, Oy, and ψ the angle they respectively make with two initial lines PA, PB drawn upon the lamina, including an angle $\pi-\omega$, and initially at right angles to Ox and Oy respectively.

Then the path of P can be obtained at once.

Let $$p = f(\psi), \quad p = F(\psi)$$
be the tangential polar equations of the curves, with P for origin of measurement of p, and PA, PB respectively as initial lines.

Let x, y be the coordinates of P with regard to the lines Ox, Oy as coordinate axes.

Then $\qquad x\sin\omega=f(\psi)$, and $\quad y\sin\omega=F(\psi)$,

and the ψ-eliminant furnishes the path of P.

It is clear that instead of the two curves on the lamina we might have one single curve drawn, *i.e.* $f(\psi)$ and $F(\psi)$ might be identical, except as regards the initial line from which ψ is measured in the two cases.

The rectification of the path of P follows from

$$dx=\operatorname{cosec}\omega\, f'(\psi)\,d\psi,\quad dy=\operatorname{cosec}\omega\, F'(\psi)\,d\psi,$$

and $$ds^2=dx^2+2\,dx\,dy\cos\omega+dy^2,$$

whence $$s=\operatorname{cosec}\omega\int\sqrt{f'^2+2f'F'\cos\omega+F'^2}\,d\psi,$$

where f' stands for $\dfrac{df(\psi)}{d\psi}$ and F' for $\dfrac{d}{d\psi}F(\psi)$.

692. Two Straight Lines in the Lamina touching Fixed Curves.

When three straight lines forming a triangle ABC are traced upon a lamina, and the lamina is made to move in such a manner that two of the sides AB, AC, say, touch given fixed curves, the third side BC will in its motion envelop a third curve, and there is a linear relation between the three arcs described by the points of contact. It has been shown (*Diff. Calc.*, Arts. 568-9) that the tangential-polar equation of the envelope of the carried side BC can be found at once.

If α, β, γ be the trilinear coordinates of any point O, fixed in space, with regard to the triangle ABC taken as a triangle of reference, we have the relation

$$a\alpha+b\beta+c\gamma=2\Delta, \qquad\qquad(1)$$

where a, b, c are the lengths of the sides and Δ the area of the triangle, with the usual trilinear convention that α, β, γ are positive when drawn from a point within the triangle.

Hence it follows that

$$a\frac{d\alpha}{d\psi}+b\frac{d\beta}{d\psi}+c\frac{d\gamma}{d\psi}=0,\quad a\frac{d^2\alpha}{d\psi^2}+b\frac{d^2\beta}{d\psi^2}+c\frac{d^2\gamma}{d\psi^2}=0,$$

where ψ is the angle any line fixed in the lamina makes with a line fixed in space;

$$\therefore\ a\left(\alpha+\frac{d^2\alpha}{d\psi^2}\right)+b\left(\beta+\frac{d^2\beta}{d\psi^2}\right)+c\left(\gamma+\frac{d^2\gamma}{d\psi^2}\right)=2\Delta.*$$

And the increment of the angle of contingence being the same for all, we have $\pm a\rho_1 \pm b\rho_2 \pm c\rho_3 = 2\Delta$.

693. **Caution.**

Regarding O as the origin of measurement for perpendiculars for the tangential polar equations of the envelopes of BC, CA, AB, it is to be noted carefully that we are in the presence of two separate conventions with regard to the signs of the perpendiculars, which may be antagonistic.

(1) The trilinear convention is that stated above, that the perpendiculars are reckoned as positive when the point from which they are drawn lies *within* the triangle of reference.

(2) In the general treatment of curves, *i.e.* in establishing the formula $\frac{ds}{d\psi}=p+\frac{d^2p}{d\psi^2}$, and others involving p, the perpendicular *from the origin* has always been reckoned positive.

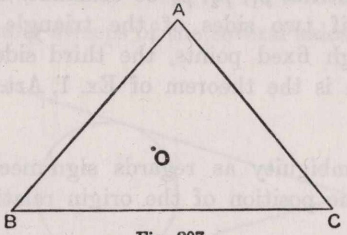

Fig. 207.

If p_1, p_2, p_3 be the positive perpendiculars from O upon the sides, we have in all cases $\frac{ds_1}{d\psi}=p_1+\frac{d^2p_1}{d\psi^2}$, etc., $\delta s_1, \delta s_2, \delta s_3$ being elements of the three arcs described by the points of contact.

Hence, so long as the origin from which the perpendiculars are measured lies within the triangle, we have $\alpha=p_1$, $\beta=p_2$, $\gamma=p_3$, and

$$\frac{ds_1}{d\psi}=\alpha+\frac{d^2\alpha}{d\psi^2},\quad \frac{ds_2}{d\psi}=\beta+\frac{d^2\beta}{d\psi^2},\quad \frac{ds_3}{d\psi}=\gamma+\frac{d^2\gamma}{d\psi^2}.$$

If, however, the origin lie between BC and AB produced and AC produced, $\alpha=-p_1$, $\beta=p_2$, $\gamma=p_3$, and

$$-\frac{ds_1}{d\psi}=\alpha+\frac{d^2\alpha}{d\psi^2},\quad \frac{ds_2}{d\psi}=\beta+\frac{d^2\beta}{d\psi^2},\quad \frac{ds_3}{d\psi}=\gamma+\frac{d^2\gamma}{d\psi^2},$$

*This method is stated by Mr. Besant to have been suggested by the late Master of Caius College, Dr. N. M. Ferrers.

with similar changes for other positions of the origin relative to the triangle of reference.

In addition to this, when we estimate the radius of curvature, it will be remembered that $\frac{ds}{d\psi}$, which is always

$$+\left(p+\frac{d^2p}{d\psi^2}\right),$$

is $\quad +\rho$ if s and ψ are increasing together,

but $\quad = -\rho$ if s and ψ are such that when one increases, the other decreases.

This point has been discussed in Art. 531.

Hence we have written

$$\pm a\rho_1 \pm a\rho_2 \pm a\rho_3 = 2\Delta,$$

the signs to be determined in each particular case. But in any case this equation is sufficient to prove that if two of the three quantities ρ_1, ρ_2, ρ_3 be constant, the third is also constant, *i.e.* if two sides of the triangle envelop circles or pass through fixed points, the third side also envelopes a circle, which is the theorem of Ex. 1, Art. 569, *Differential Calculus*.

694. The ambiguity as regards sign necessitates careful attention to the position of the origin relatively to the triangle.

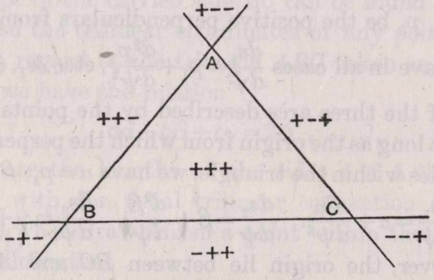

Fig. 208.

Three straight lines on a plane divide the plane into seven regions, and the signs of α, β, γ in these regions are indicated in the figure.

Accordingly we have, if we assume s_1, s_2, s_3 to be measured from points for which $\psi=0$, and to be each increasing when ψ increases,

$$as_1+bs_2+cs_3=2\Delta\psi$$

if, and so long as, the origin lies within the triangle;

$$-as_1+bs_2+cs_3=2\Delta\psi$$

if, and so long as, the origin lies in the region where the signs of α, β, γ are respectively $-++$, and so on for the other five regions.

Also, as the lamina moves the origin may pass from one region to another. Hence care must be taken in integrating between specified limits for ψ to observe the sweep of any of the three lines BC, CA, AB through the origin, and to take proper account thereof by using the appropriate case or cases of

$$\pm as_1 \pm bs_2 \pm cs_3=2\Delta\psi$$

for the intervening sweeps of the several sides.

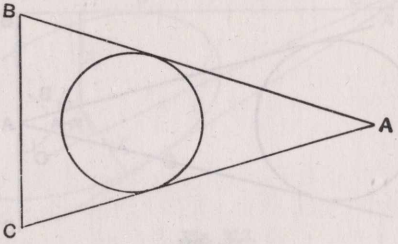

Fig. 209.

Thus, in integrating round an oval which the arms AB, AC touch, the oval lying within the triangle (Fig. 209), we have, taking the origin within the oval,

$$as_1+bs_2+cs_3=2\Delta\psi,$$

and for a complete revolution

$$aS_1+(b+c)S=4\pi\Delta,$$

where S is the perimeter of the oval

and S_1 that of the curve enveloped by BC.

695. In the same way, if the oval be always external to the triangle as in Fig. 210,

$$-aS_1'+(b+c)S=4\pi\Delta,$$

and similarly in other cases.

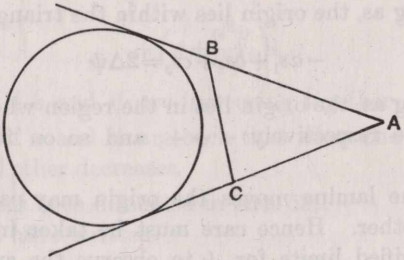

Fig. 210.

696. **A Limiting Case.**

If the triangle ABC becomes evanescent, we have the case of a line through A, viz. $B'C'$ (Fig. 211), carried by the pair of tangents AB, AQ, and making constant angles B, C with

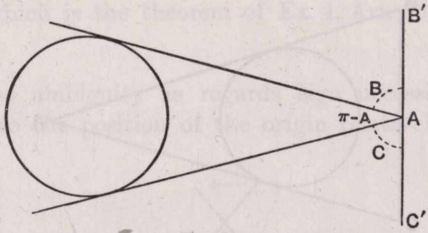

Fig. 211.

them respectively, the tangents making a constant angle A with each other. The sides a, b, c vanish in the ratio $\sin A : \sin B : \sin C$, and the theorem becomes

$$S_1' \sin A = (\sin B + \sin C)S,$$

i.e.
$$S' = \frac{\cos\left(\dfrac{B-C}{2}\right)}{\sin\dfrac{A}{2}}S.$$

697. If the carried line $B'C'$ lie within the angle PAQ, as shown in Fig. 212, it is the limit of the case in Fig. 213,

where the signs of the perpendiculars α, β, γ are respectively $-+-$, and $\alpha=-p_1$, $\beta=p_2$, $\gamma=-p_3$,

$$-as_1+bs_2-cs_3=2\Delta\psi \text{ (and ultimately } \Delta=0),$$

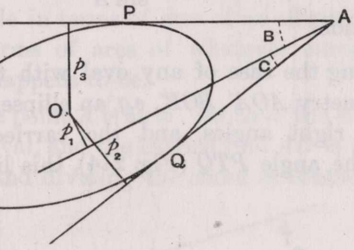

Fig. 212.

so long as BC does not sweep through the origin; and if it never does do so during the whole motion of the lamina during a complete revolution,

$$-aS''+(b-c)S=4\pi\Delta \quad \text{(and ultimately } \Delta=0),$$

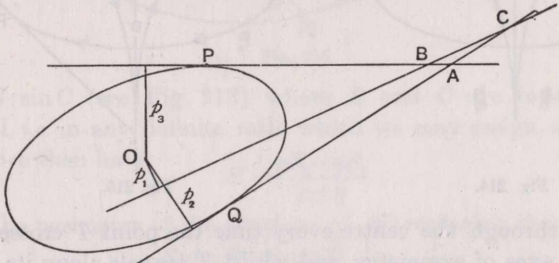

Fig. 213.

giving S'' the perimeter of the curve enveloped, or in the limit, when the triangle is evanescent,

$$S''=\frac{\sin B-\sin C}{\sin A}S=\frac{\sin\dfrac{B-C}{2}}{\cos\dfrac{A}{2}}S.$$

698. If, however, the line BC does sweep through the origin in the course of the revolution, the integration must be performed separately for the several complete portions for which the line BC moves without a sweep through the origin, and

the arcs of the envelope of BC being found thus, the positive results must be finally added together, using the formula

$$S'' = \frac{S_2 \sin B \sim S_3 \sin C}{\sin A}$$

for each portion.

699. Taking the case of any oval with two perpendicular axes of symmetry AOA', BOB', *e.g.* an ellipse, TP, TQ a pair of tangents at right angles, and the carried line being the bisector of the angle PTQ (Fig. 214), this line will obviously

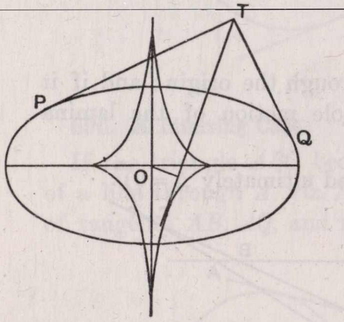

Fig. 214.

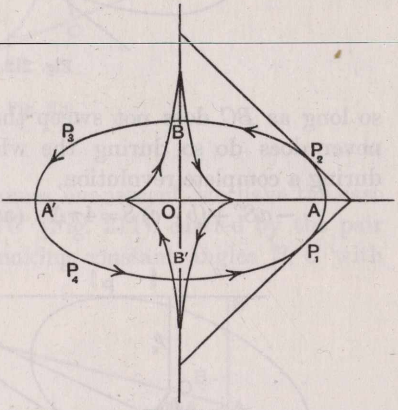

Fig. 215.

sweep through the centre every time the point T crosses one of the axes of symmetry, and whilst T travels along its locus over the first quadrant, the perimeter of the corresponding portion of the envelope of the carried line is

$$S'' = \frac{\sin\frac{\pi}{4}\operatorname{arc} P_1P_2 \sim \sin\frac{\pi}{4}\operatorname{arc} P_2P_3}{\sin\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}}(\operatorname{arc} P_1P_2 \sim \operatorname{arc} P_2P_3)$$

$$= \sqrt{2}(\operatorname{arc} BP_2 \sim \operatorname{arc} AP_2),$$

where P_1, P_2, P_3, P_4 are the points of contact of tangents which make an angle of $\frac{\pi}{4}$ with the x-axis (Fig. 215).

It is to be noted that the arc in question is described in the opposite order to that of description of the ellipse by the several points of contact.

The whole perimeter is then $4\sqrt{2}(\text{arc } BP_2 - \text{arc } AP_2)$, and the curve is rectifiable in terms of arcs of an ellipse if the oval be elliptic, or in terms of arcs of whatever curve the doubly symmetric oval happens to be.

700. When the point A is at ∞, we have the case of parallel tangents to the oval, and the carried line AD is a line parallel to the tangents and dividing the chord of contact in the ratio

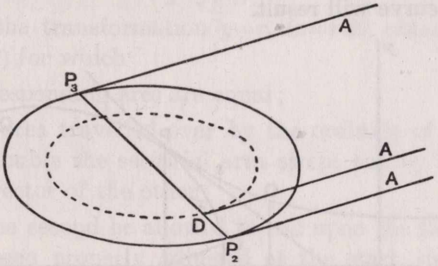

Fig. 216.

$\sin B : \sin C$ (see Fig. 213), where B and C are indefinitely small, *i.e.* in any definite ratio which we may assign, say $p:q$, and we then have

$$S'' = \frac{pS_2 - qS_3}{p+q}$$

for the perimeter of the envelope of AD replacing the result

$$S'' = \frac{S_2 \sin B - S_3 \sin C}{\sin(B+C)} \text{ of Art. 698.}$$

701. A Case of Isoperimetric Companionship of Curves.

Let us consider the form of a curve $O'PQ$ with pole N, which will be such that, when it rolls upon the fixed curve OP whose equation is known, $y=f(x)$, the pole N will travel along a straight line, say the x-axis.

Let O and O' be the points originally in contact, Ax, Ay the axes, P the point of contact, PN, OM ordinates, PT the tangent at P making an angle ψ with Ax, $O'N$ the radius vector of the rolling curve from which θ is measured and r the radius vector NP, ϕ the angle between the tangent to the rolling curve and its radius vector.

Then
$$r = y,$$
$$\frac{r\,d\theta}{dr} = \tan\phi = \cot\psi = \frac{dx}{dy}.$$

Hence and
$$\left.\begin{aligned} r\,d\theta &= dx \\ dr &= dy. \end{aligned}\right\} \text{..............................(1)}$$

We therefore have
$$\left.\begin{aligned} \theta &= \int\frac{dx}{r} = \int\frac{dx}{f(x)}, \\ r &= y \quad = f(x), \end{aligned}\right\} \text{..............................(2)}$$

and if x be eliminated from equations (2), the polar equation of the rolling curve will result.

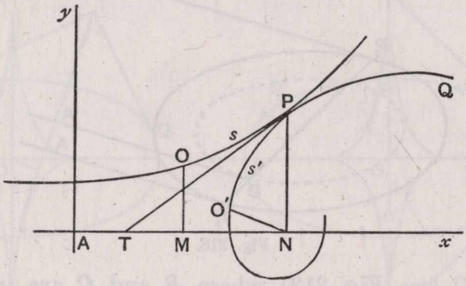

Fig. 217.

Again, if the form of the rolling curve had been given, say
$$r = F(\theta),$$

then
$$\left.\begin{aligned} x &= \int F(\theta)\,d\theta, \\ y &= F(\theta), \end{aligned}\right\} \text{..............................(3)}$$

and if θ be eliminated between these equations, the Cartesian equation of the fixed curve will result.

It follows that, since there is pure rolling without slipping, the corresponding arcs of the two curves must be equal.

This follows at once also from equation (1), for if s and s' be the respective arcs OP, $O'P$,
$$ds^2 = dx^2 + dy^2 = r^2 d\theta^2 + dr^2 = ds'^2;$$
whence $ds = ds'$ and $s = s'$ if measured from such points as have originally been in contact.

It also follows that

$$\int y\,dx=\int r\,.\,r\,d\theta=\int r^2 d\theta,$$

i.e. the area swept over by the ordinate PN, that is $MNPO$, is double of the area swept out relatively to the rolling curve by its radius vector, that is the sectorial area $O'NP$.

The polar subtangent of the rolling curve is the Cartesian subtangent of the fixed curve, and the subnormals are the same.

Hence, given $y=f(x)$, we can, by the transformation $y=r$, $dx=r\,d\theta$, obtain another curve $r=F(\theta)$ for which

(1) corresponding arcs are equal;

(2) the area travelled over by the ordinate of the one is double the sectorial area swept out by the radius vector of the other;

and (3) if the second be allowed to roll upon the first, having been properly adjusted at the start, the locus of the pole of the rolling curve is the x-axis of the other.

702. Generalisation.

More generally, if we take any polar curve

$$r=F(\theta),$$

and construct from it a Cartesian locus, such that for each

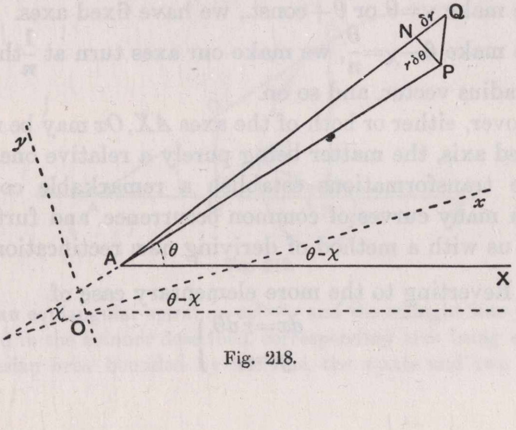

Fig. 218.

point (r, θ) on the one there is a point (x, y) on the other for which

$$dx = dr\cos\chi - r\,d\theta\sin\chi,$$
$$dy = dr\sin\chi + r\,d\theta\cos\chi,$$

where χ is any angle whatever at our choice, we have, upon elimination of r and θ, a new curve in which

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 = ds'^2,$$

where ds, ds' are corresponding elements of arcs in the two curves. It follows that

$$ds = ds' \quad\text{and}\quad s = s',$$

if the origins of measurement of arc are so chosen that s and s' vanish together.

703. The geometrical meaning of this is plain. We are projecting dr, $r\,d\theta$ upon a pair of perpendicular axes Ox, Oy with an arbitrary origin, and such that the x-axis makes an angle χ behind the radius vector of the polar curve, and therefore makes an angle $\theta - \chi$ with the initial line of the polar curve, or what is the same thing, with a fixed line through O parallel to the initial line of the polar curve; and by reserving choice of χ, we can make the new axes either fixed axes or moving in any given manner.

If we make $\chi = 0$, *i.e.* if we make the x-axis turn at the same angular rate as the radius vector of the polar curve, we have

$$dx = dr, \quad dy = r\,d\theta,$$

the transformation discussed in the last article, except that the axes of x and y are interchanged.

If we make $\chi = \theta$ or $\theta +$ const., we have fixed axes.

If we make $\theta - \chi = \dfrac{\theta}{n}$, we make our axes turn at $\dfrac{1}{n}$th the rate of the radius vector, and so on.

Moreover, either or both of the axes AX, Ox may be regarded as a fixed axis, the matter being purely a relative one.

These transformations establish a remarkable connection between many curves of common occurrence, and further will furnish us with a method of deriving new rectifications.

704. Reverting to the more elementary case of

$$\left.\begin{aligned} dx &= r\,d\theta, \\ y &= r, \end{aligned}\right\}$$

we shall find that,

A straight line $y = x \cot\alpha$ has for its analogue an equiangular spiral $r = ae^{\theta\cot\alpha}$.

A straight line $r = c \operatorname{cosec}\theta$ has a companion in a catenary.

A parabola - - - has as companion a spiral of Archimedes.

An ellipse - - - has as companion one of the Rhodoneae. (*Diff. Calc.*, Art. 385.)

A cardioide - - - has as companion a cycloid.

And when any curve is rectifiable, a companion is also rcctifiable in the same manner, and even when neither curve is rectifiable in terms of arcs of a circle or an ellipse, arcs of the one can be expressed in terms of arcs of the other.

And in addition the property as to the relative magnitude of the area swept out by the radius vector of the one and the ordinate of the other holds good.

Such pairs of curves may perhaps be termed **Isoperimetric Companions.**

As illustrative examples, we consider these examples in detail.

705. 1. Taking the straight line $y = x\cot\alpha$ as the fixed "curve,"

$$dy = dr, \quad dx = r\,d\theta\,;$$
$$\therefore\ dr = r\,d\theta\cot\alpha,$$
$$\frac{dr}{r} = d\theta\cot\alpha,$$
$$r = ae^{\theta\cot\alpha}.$$

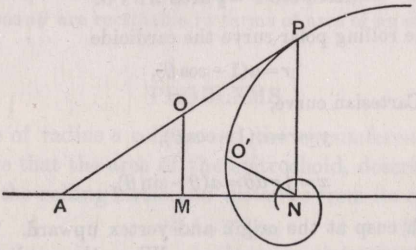

Fig. 219.

Hence an equiangular spiral $r = ae^{\theta\cot\alpha}$ and the straight line $y = x\cot\alpha$ correspond in the manner described, corresponding arcs being equal, and the Cartesian area bounded by the line, the x-axis and two ordinates being equal to double the corresponding sectorial area of the spiral. (See *Diff. Calc.*, Art. 449.)

2. Take as the rolling polar curve the straight line $r=c\sec\theta$.

Then $$y=r=c\sec\theta,\quad dx=r\,d\theta=c\sec\theta\,d\theta\,;$$

$$\therefore\ x=c\log\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right)=c\,\mathrm{gd}^{-1}\theta\,;$$

$$\therefore\ \cos\theta\cosh\frac{x}{c}=1\quad(\text{Art. }69),$$

or $$y=c\cosh\frac{x}{c},\ \text{the catenary,}$$

which is therefore the isoperimetric companion to the straight line, and rectifiable as has been seen (Art. 538). See also *Diff. Calc.*, Art. 444.

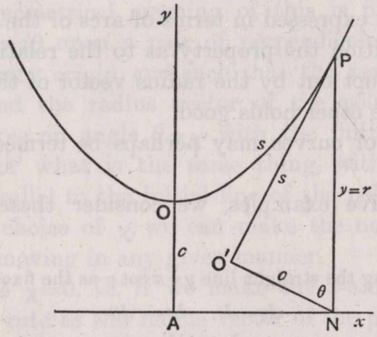

Fig. 220.

We note in addition to properties proved in *Diff. Calc.*, Art. 444, that

$$\text{Area } NO'P=\tfrac{1}{2}\text{ area } ANPO.$$

3. Take as the rolling polar curve the cardioide
$$r=a(1-\cos\theta).$$

Then, for the Cartesian curve,
$$y=r=a(1-\cos\theta),$$
$$x=\int r\,d\theta=a(\theta-\sin\theta),$$

i.e. a cycloid with cusp at the origin and vertex upward. These curves are therefore isoperimetric companions. When the cardioide is placed with its vertex in contact with the vertex of the cycloid on the concave side and allowed to roll inside the cycloid, the roulette of the pole is the line of cusps of the cycloid and the propositions of Art. 701 with regard to equality of corresponding arcs and the relative magnitudes of the areas swept by the ordinates of the cycloid and the radius vector of the cardioide both hold good.

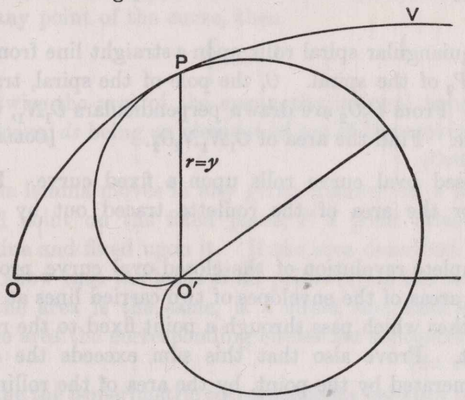

Fig. 221.

4. Take as fixed curve the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then $y=r$, $dx=r\,d\theta$ give

$$x=\frac{a}{b}\sqrt{b^2-r^2}, \quad \text{and} \quad r\,d\theta=-\frac{a}{b}\frac{r\,dr}{\sqrt{b^2-r^2}};$$

$$\therefore \frac{b\theta}{a}=\cos^{-1}\frac{r}{b},$$

i.e.

$$r=b\cos\frac{b}{a}\theta,$$

which is the isoperimetric companion of the ellipse. Hence the Rhodoneae $r=A\cos n\theta$ are rectifiable in terms of arcs of an ellipse.

PROBLEMS.

1. A circle of radius a rolls round the circumference of an equal circle. Prove that the area of the epitrochoid, described by a point carried with the rolling circle and distant c from its centre, is

$$(4a^2+2c^2)\pi. \qquad \text{[Oxf. I. P., 1918.]}$$

2. If a circle roll on the convex side of a parabola from one extremity of the latus rectum to the other, and can just pass between the vertex and the directrix, prove that four times the area traced out by that radius of this circle, which always passes through the

point of contact, exceeds the area of the circle by half the rectangle contained by the latus rectum and a line equal to the arc it cuts off. [R. P.]

3. An equiangular spiral rolls upon a straight line from a point P_1 to a point P_2 of the spiral. O, the pole of the spiral, traces out the path O_1O_2. From O_1O_2 are drawn perpendiculars O_1N_1, O_2N_2 on the straight line. Find the area of $O_1N_1N_2O_2$. [Colleges α, 1881.]

4. A closed oval curve rolls upon a fixed curve. Find an expression for the area of the roulette traced out by any carried point.

In a complete revolution of the closed oval curve, prove that the sum of the areas of the envelopes of two carried lines at right angles to one another which pass through a point fixed to the rolling curve is constant. Prove also that this sum exceeds the area of the roulette generated by the point, by the area of the rolling curve. [Colleges γ, 1887.]

5. If a closed oval curve roll with angular velocity ω on a straight line, while a point moves along its evolute with relative velocity $\omega\rho'$, prove that the area included in any portion of a revolution between the straight line, the curve generated by the moving point, and the perpendiculars to the former drawn through the extremities of the latter, is double the corresponding portion of the area between the curve and its evolute, bounded by the initial and final radii of curvature, provided the moving point is initially at the centre of curvature of the point of contact; ρ' being the radius of curvature of the evolute at the point corresponding to the point of the rolling curve in contact with the straight line. [Colleges δ, 1883.]

6. The cardioide $r=a(1-\cos\theta)$ rolls on a straight line; prove that the intrinsic equation of the roulette of the cusp is

$$2s=3a(2\psi-\sin 2\psi),$$

measuring from the point of contact of the cusp.

Prove also that its Cartesian equation is

$$\frac{4a-x}{2a}=\left\{2+\left(\frac{y}{2a}\right)^{\frac{2}{3}}\right\}\sqrt{1-\left(\frac{y}{2a}\right)^{\frac{2}{3}}},$$

that its area is $\frac{15}{4}\pi a^2$, and that the radius of curvature of the roulette of the cusp is three times its distance from the point of contact. [Trinity, 1888.]

Find the evolute of the roulette of the pole and the intrinsic equation of the envelope of the axis.

7. A closed curve is moving in any manner in its own plane. Show that if ρ be the radius of curvature of the envelope of the tangent at any point of the curve, then

$$\int \rho\, ds$$

is equal to twice the area of the curve, the integral being taken all round the curve, ds being an element of arc of the moving curve.

[COLLEGES, 1879.]

8. A plane lamina moves in any given manner on a fixed plane: O is a fixed point on the fixed plane, P a point attached to the moving lamina and fixed upon it. If the area described by P about O be given, show that the locus of all points (P) in the moving plane for which the area is the same, is a circle, and that for different values of the area the corresponding circles are concentric.

[ST. JOHN'S, 1881.]

9. Examine the isoperimetric correspondence between the parabola $y^2=4ax$ and the Archimedean spiral $r=2a\theta$, showing that the spiral can be made to roll upon the parabola in such manner that the pole of the spiral travels along the axis of the parabola.

10. Show that the reciprocal spiral $r\theta=a$ and the exponential curve $y=ae^{-\frac{x}{a}}$ are isoperimetric companions, both curves being rectifiable and corresponding arcs equal, and interpret the result by reference to the locus of the pole of the spiral when suitably started rolling.

11. Establish isoperimetric companionship between the curve

$$\left.\begin{aligned} \frac{x}{a} &= \log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)-\sin\phi, \\ \frac{y}{a} &= \frac{\sin^2\phi}{\cos\phi} \end{aligned}\right\}$$

and the cissoid $\quad r=a\dfrac{\sin^2\theta}{\cos\theta}$.

12. Establish isoperimetric companionship between the semi-cubical parabola $ay^2=x^3$ and the spiral $8au+\theta^3=0$.

13. Show that the curve

$$r=a\log\sec t,$$
$$\theta=\int\frac{dt}{\log\sec t}$$

is rectifiable and in isoperimetric companionship with the catenary of equal strength

$$y=a\log\sec\frac{x}{a}.$$

14. Show that the curves

$$\left.\begin{aligned} 4x &= a\left(\cos\phi - 9\cos\frac{\phi}{3}\right), \\ 4y &= a\left(3\sin\frac{\phi}{3} - \sin\phi\right) \end{aligned}\right\}$$

and $$r^{\frac{1}{3}} = a^{\frac{1}{3}} \sin\tfrac{1}{3}\theta$$

are rectifiable and isoperimetric companions.

15. Show that the curve

$$2\theta + 6\frac{\sqrt{a^{\frac{2}{3}} + r^{\frac{2}{3}}}}{r^{\frac{1}{3}}} = 3\log\frac{\sqrt{a^{\frac{2}{3}} + r^{\frac{2}{3}}} + r^{\frac{1}{3}}}{\sqrt{a^{\frac{2}{3}} + r^{\frac{2}{3}}} - r^{\frac{1}{3}}}$$

is rectifiable and in isoperimetric companionship with

$$x^{\frac{2}{3}} - y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

16. Show that the curve

$$\left.\begin{aligned} r &= 4a\sin\frac{t}{2}\cos^3\frac{t}{2}, \\ \theta &= \tan\frac{t}{2} - 2t \end{aligned}\right\}$$

is rectifiable and in isoperimetric companionship with the cardioide $r = a(1+\cos\theta)$, its pole travelling along the axis of the cardioide as it rolls within the cardioide, the two poles being initially coincident.

17. Show, by taking $r = a\theta$ and $\chi = n\theta$ in Art. 702, that

$$x = \frac{a}{n^2}[n\theta\cos n\theta + (n-1)\sin n\theta],$$

$$y = \frac{a}{n^2}[n\theta\sin n\theta - (n-1)(\cos n\theta - 1)]$$

is an isoperimetric companion of the Archimedean spiral $r = a\theta$.

Hence show

(1) that $x^2 = 2ay$ is isoperimetric with $r = a\theta$;

(2) that $r = a\operatorname{cosec}\left\{\frac{\sqrt{r^2-a^2}}{a} - \theta\right\}$ is rectifiable and in isoperimetric companionship with $r = 4a\theta$.

18. Show that an ellipse of semiminor axis b and eccentricity e can be made to roll upon the curve

$$\frac{b}{y} = \operatorname{dn}\frac{x}{b} \quad (\text{mod. } e),$$

so that the path of the centre of the ellipse is the x axis.

Show that if the origin be taken at the point for which the end of the major axis a is in contact with the curve, this may be reduced to the form

$$\frac{y}{a}=\operatorname{dn}\frac{x}{b}.$$

[Write $Kb-x$ for x and reduce, see Ch. XXXI., Art. 1352. See also Greenhill, *Elliptic Functions*, p. 72.]

19. Show that the perimeter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equal to twice the perimeter of one outer foil of the curve

$$r=b\cos\frac{b\theta}{a},$$

and that the area of the ellipse is equal to four times the area of one outer foil of the same curve.

Show further, that if the vertex of the foil be placed in contact with the inner side of the ellipse at the end of the minor axis, and the foil roll without sliding upon the ellipse, the pole of the rolling foil will traverse the major axis of the ellipse.

Deduce a well-known proposition as to a circle rolling in the interior of another circle of double its radius.

20. An involute of a circle is made to slide, touching the rectangular axes Ox, Oy. Show that the locus of the instantaneous centre on the plane x, y is a straight line. What is the locus of the instantaneous centre relatively to the curve.

Show that the glisettes of carried points are cycloids and trochoids, and the envelopes of carried straight lines are either cycloids or involutes of cycloids. [Besant, *Roulettes and Glisettes.*]

21. A cycloid rolls along a straight line. Show that the intrinsic equations of the envelopes of (1) the axis, (2) the line of cusps, (3) the tangent at the vertex are respectively

$$\begin{aligned}&(1)\ \ s=a\psi^2+3a\sin^2\psi,\\ &(2)\ \ s=3a(\psi+\tfrac{1}{2}\sin 2\psi),\\ &(3)\ \ s=a(\psi+\tfrac{3}{2}\sin 2\psi),\end{aligned}$$

measuring s in each case from the point on its locus for which $\psi=0$.

Trace each of these curves, supposing the cycloid to be continued both ways, and the rolling to continue with successive arches of the cycloid, and find the positions of their cusps.

Show that the whole perimeter of the last of these curves is

$$8a\sqrt{2} + 8a\sin^{-1}\sqrt{\tfrac{2}{3}} - 2\pi a,$$

and its area $= \frac{1}{2}\pi a^2$.

Show that the first evolutes of the second and third curves, and the second evolute of the first are four-cusped hypocycloids.

22. A parabola rolls on a straight line; show that

(1) the locus of the focus is a catenary (Art. 517),

(2) the envelope of the directrix is an equal catenary,

(3) the tangent at the vertex and the latus-rectum envelop parallels to a catenary,

(4) the intrinsic equation of the envelope of the axis is

$$s = a\,(2\log\sec\psi + \tan^2\psi).$$

23. If the cardioide $r = a(1-\cos\theta)$ move so as to touch a straight line always at the same point, show that the locus of the pole is

$$r = 2a\sin^2\theta,$$

and that the intrinsic equation of the envelope of the axis is

$$\frac{3s}{a} = 12\sin^2\frac{\psi}{3} - 7\sin^4\frac{\psi}{3}.$$

24. If an ellipse slide in contact with a given straight line at a given point, the glisette of the foci is

$$(x^2+y^2)(b^2+y^2)^2 = 4a^2y^4,$$

and that of the centre is $x^2y^2 = (a^2-y^2)(y^2-b^2)$.

25. A lamina moves in such manner that a certain point in it describes the path

$$\left.\begin{aligned}\xi &= c\sin\psi - c\cos\psi\log(\sec\psi+\tan\psi),\\ \eta &= c\cos\psi + c\sin\psi\log(\sec\psi+\tan\psi) - c,\end{aligned}\right\}$$

referred to fixed axes OX, OY in its plane, whilst a straight line through this point attached to the lamina makes an angle ψ with the Y-axis.

Reduce this motion to rolling. Also show that the difference of the curvatures of the loci of the instantaneous centre on the lamina and on the fixed plane is $\dfrac{\cos^2\psi}{c}$.

Show further that the intrinsic equation of the envelope of the line attached to the lamina is

$$\frac{ds}{d\psi} = c\sec\psi\tan\psi + c\log(\sec\psi+\tan\psi).$$

26. A lamina moves in its own plane, so that a point O' upon it traces out a cissoid,

$$\left.\begin{aligned} \xi &= -2a\cos^2\frac{\theta}{2}, \\ \eta &= 2a\frac{\cos^3\frac{\theta}{3}}{\sin\frac{\theta}{2}}, \end{aligned}\right\} \quad i.e.\ \eta^2(2a+\xi)+\xi^3=0,$$

upon a fixed plane with reference to a pair of fixed rectangular axes OX, OY in that plane, whilst a straight line $O'x$ attached to the moving lamina rotates, making an angle θ with OX. Show that the motion is that of rolling of one parabola upon another equal parabola, and deduce from the formula of Art. 660, for the difference of curvature of the I-loci, the radius of curvature of a parabola.

27. A catenary moves in its own plane so as always to touch a given straight line at a given point. Show that the tangential polar equation of the envelope of the axis is

$$\frac{p}{c}=\text{gd}^{-1}\psi,$$

where c is the parameter of the catenary.

28. The centre of a circular disc of radius a travels along a parabolic path $y^2=2ax$, spinning at an angular velocity ω in a clockwise direction, the centre receding from the axis with a velocity $a\omega$. Show that the motion thus produced is that of the rolling of an involute of the circle upon the axis of the parabola, and that the velocity of the point of contact is the same as the velocity with which the centre of the circle recedes from the tangent at the vertex.

29. A Bernoulli's lemniscate moves so as to touch a fixed axis at a given point. Show that the tangential polar equation of the envelope of the axis is

$$p^2=a^2\sin^2\frac{\psi}{3}\cos\frac{2\psi}{3},$$

and that the glisette of the pole is

$$r^2=a^2\sin\theta.$$

30. A circle rolls on an equal circle and carries with it a fixed tangent. Find the intrinsic equation of the envelope of the carried tangent. [Oxford II. P., 1887.]

31. A triangle of area Δ moves so that two of its sides (a, b) touch an oval of perimeter l at points where the radii of curvature are ρ, ρ'; prove that the radius of curvature and the perimeter of the envelope of the third side c are

$$\frac{1}{c}(2\Delta - a\rho - b\rho') \quad \text{and} \quad \frac{1}{c}\{4\pi\Delta - (a+b)l\}.$$

[St. John's, 1883.]

32. An ellipse rolls on a fixed horizontal straight line (the axis of x). Show that the locus of the highest point of the ellipse will be

$$x = \int \frac{y^4 - 8a^2b^2}{y^2\sqrt{(4a^2 - y^2)(y^2 - 4b^2)}}\,dy,$$

and reduce the integral to the standard form.

[St. John's Coll., 1881.]

33. Prove that the intrinsic equation of the envelope of the directrix of a catenary of parameter c, rolling on a circle of radius c, will be found by eliminating α between the equations

$$\left.\begin{aligned} \frac{s}{c} &= \tfrac{1}{2}\tan\alpha\sec\alpha + \tfrac{1}{4}\log\frac{1+\sin\alpha}{1-\sin\alpha} \\ \text{and} \qquad \psi &= \alpha + \tan\alpha. \end{aligned}\right\}$$

[St. John's, 1886.]

34. A given right-angled triangle is made to slide round the outside of a fixed oval curve with the point P on the curve, the side PR touching it and the side PQ normal to it. If s be the perimeter of the oval, prove that the length of the curve enveloped by QR is equal to $(s + 2\pi PQ)\sin PQR$. [St. John's, 1889.]

35. When a curve rolls on a straight line, show how to find the locus of the centre of curvature at the point of contact, and prove that, in the case of a cardioide, the locus is an ellipse.

[St. John's, 1889.]

36. When a curve rolls on a fixed curve, prove that the locus of the centre of curvature is inclined to the common tangent at the angle

$$\tan^{-1}\{\rho\, d\rho'/(\rho+\rho')\,ds\},$$

where ρ, ρ' are the radii of curvature of the fixed and rolling curves at the point of contact. [St. John's, 1889]

37. A cardioide $r = a(1-\cos\theta)$ rolls upon an equal cardioide, the vertices coinciding during the roll. Show that the roulette of the pole of the rolling curve is

$$r = 4a\sin^2\left(\frac{\pi}{6} + \frac{\theta}{2}\right),$$

that the tangential polar equation of the envelope of the axis is

$$p = 4a \sin\psi \sin^3 \frac{\pi + \psi}{6},$$

and that the area of the roulette of the pole is

$$\frac{a^2}{2}(3\pi + 4\sqrt{3}).$$

38. A cardioide of perimeter $8a$ rolls on the outer side of a cycloid of equal perimeter from cusp to cusp, the vertices coinciding during the roll. Show that the area of the roulette of the cusp of the cardioide between the roulette and the cycloid $= \frac{9}{2}\pi a^2$.

Show also that the arc of any portion of the roulette of the cusp measured from the vertex of the curve is double the distance of the point of contact of the two curves from the axis of the cycloid.

Show further that the tangential polar equation of the envelope of the axis of the cardioide is

$$p = 2a \sin\psi(\psi + 2\cos^3\psi),$$

where p is drawn from the vertex of the cycloid and ψ is measured from its axis.

39. A cycloid of length $8a$ rolls on the outside of a cardioide of equal length, a cusp of the cycloid starting from the cusp of the cardioide. Show that the intrinsic equation of the envelope of the line joining the cusps of the cycloid is

$$2s = 3a\psi + 6a \sin\frac{\psi}{2},$$

ψ being measured from the tangent at the vertex of the cardioide.

[OXF. II. P., 1913.]