47.

ON A POINT IN THE THEORY OF VULGAR FRACTIONS.

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THE reciprocal of an integer I call a simple fraction; any other fraction, whether rational or irrational, may be termed complex; but it is to be understood that only proper fractional quantities of either sort, that is, fractions greater than zero and less than unity, will be considered in what follows.

Suppose Q to represent any fractional quantity; if Q lies between $\frac{1}{u_0 - 1}$ and $\frac{1}{u_0}$, we may make $Q = \frac{1}{u_0 + \delta} + Q'$, where δ is zero or a positive integer, and Q' will continue a proper fraction, which in like manner may be resolved into $\frac{1}{u_1 + \delta_1} + Q''$, and so on continually.

But if we make $\delta_0, \delta_1, \ldots$ each zero, the process of expansion becomes determinate. Any such determinate representation of a fractional quantity I shall term a *sorites*. It is obvious that in expanding a given fraction under the form of a sorites, the successive denominators, which I shall call the *elements*, may be obtained by a process of division; if the fraction to be expanded is rational, the real divisor will be an integer which continually decreases*, and consequently every complex rational fraction can be expanded (and only in one way) under the form of a finite sorites.

The elements of a sorites are analogous to the partial quotients of a regular continued fraction; but there is this difference between the two cases, that whilst the latter quantities are perfectly arbitrary, the elements in question are subject to a certain law which I shall proceed to examine.

* See examples of development of sorites, page [443].

Let $n, p, q, \ldots r, s, \ldots t, u$ be the elements of a sorites. It is clear that the last remainder being the reciprocal of $\frac{1}{t}+\frac{1}{u}$, we must have $\frac{1}{t}+\frac{1}{u}<\frac{1}{t-1}$, that is to say, u greater than t^2-t, that is, u is equal to or greater than t^2-t+1. Again, if we look to the residue which gives birth to the element r, that must be of the form $\frac{1}{s-\epsilon}$, where ϵ is some fraction, and we must now have $\frac{1}{r}+\frac{1}{s-\epsilon}<\frac{1}{r-1}$, or $s-\epsilon$ equal to or greater than r^2-r. Hence s is equal to or greater than r^2-r+1, so that the relation between any two contiguous elements is the same, whether they are or are not the final two; and if u_x, u_{x+1} be any two consecutive integers in a series, the one necessary and sufficient condition for the possibility of the existence of the sorites, of which those terms shall be elements, is that we must have for all values of x, u_{x+1} equal to or greater than $u_x^2-u_x+1$.

If u_{x+1} is throughout equal to $u_x^2-u_x+1$, we obtain a series which may be termed a limiting sorites.

It is obvious that any simple fraction $\frac{1}{u_0-1}$ may be expanded under the form of an infinite sorites, of which the elements are $u_0, u_1, u_2 \ldots$ subject to the above relation. An infinite sorites read in the limiting case is therefore expressible under the form of a finite fraction, and the same will be true for a sorites in which the right-hand branch beginning from any term u_i, namely, $\frac{1}{u_i}+\frac{1}{u_{i+1}}+\frac{1}{u_{i+2}}\ldots$, forms a limiting sorites.

But in every other case of a sorites the sum cannot be a finite fraction; for such fraction can be expanded in only one way under the form of a sorites, and such sorites is necessarily finite in the number of its terms.

Hence it is impossible that the sum of the reciprocals of an ascending series of positive integers, such that the square root of the difference between any one of them and its immediate antecedent is greater than the difference between that antecedent and unity, can represent a rational quantity; for if so, we have $u_{x+1}-u_x$ greater than $(u_{x-1}-1)^2$, that is, $u_{x+1}>u_x^2-u_x+1$, and the series will form a sorites not belonging to the limiting class.

I proceed to examine some of the properties of the series of terms defined by the condition $u_{x+1}=u_x^2-u_x+1$.

In the first place, I observe that any term u_{x+i} may be expressed under the form Pu_x+1: for suppose this to be true for one value of i; then, since $u_{x+i+1}-1=u_{x+i}(u_{x+i}-1)$, it is obviously true for the next above; here the proposition, being true when i is unity, is true universally.

It follows from this that each element of a limiting sorites is prime to all that follow it, and consequently any two terms of the sorites are prime to one another.

Again, for greater simplicity, let $v_0, v_1, v_2, \ldots$ be used to represent (u_0-1), (u_1-1), (u_2-1), ...; we have, then,

$$v_1 - v_0 = v_0^2, \quad v_2 - v_1 = v_1^2, \quad v_3 - v_2 = v_2^2, \ldots.$$

Hence $v_2 - v_0$, $v_3 - v_0$, ... $v_x - v_0$ (as is obvious from successive addition of the above equations) will each of them be of the form Pv_0^2, where P is a rational integral function of v_0, and v_x will be of the form $Pv_0^2 + v_0$. This conclusion leads to a representation of the sum of any given number of terms of a limiting sorites by a fraction in its lowest terms. For

$$\frac{1}{v_x} - \frac{1}{v_{x+1}} = \frac{v_{x+1} - v_x}{v_x v_{x+1}} = \frac{v_x^2}{v_x v_{x+1}} = \frac{v_x}{v_x + v_x^2} = \frac{1}{v_x + 1} = \frac{1}{u_x}.$$

Hence

$$\frac{1}{u_0} + \frac{1}{u_1} + \ldots + \frac{1}{u_x} = \frac{1}{v_0} - \frac{1}{v_{x+1}} = \frac{v_{x+1} - v_0}{v_0 v_{x+1}} = \frac{(v_{x+1} - v_0) \div v_0^2}{v_{x+1} \div v_0},$$

which is of the form $\dfrac{P}{Pv_0 + 1}$ and is consequently a fraction in its lowest terms.

Again, if we denote the product of the elements $u_0, u_1, u_2, \ldots, u_x$ by Πu_x and the sum of their $(x-1)$-ary combinations by $\Pi' u_x$, $\dfrac{\Pi' u_x}{\Pi u_x}$ will also be the same fraction in its lowest terms, because (as has been shown) all the elements of the sorites are prime to one another.

Hence we may deduce the equations

$$u_{x+1} = u_0 + (u_0 - 1)^2 \, \Pi' u_x,$$

$$u_{x+1} = 1 + (u_0 - 1) \, \Pi u_x.$$

The second of these serves to give an inferior limit to the rate of convergence of any sorites. For in the limiting case we have

$$u_1 > (u_0^2 - u_0),$$

$$u_2 > (u_0 - 1) \; u_0 u_1 \; > (u_0^2 - u_0)^2,$$

$$u_3 > (u_0 - 1) \, u_0 u_1 u_2 > (u_0^2 - u_0)^4,$$

....

and so in general $u_x > (u_0^2 - u_0)^{2^{x-1}}$, because the solution of the equation

$$\theta_i = \theta_{i-1} + \theta_{i-2} + \ldots + \theta_0 \text{ is } \theta_i = 2^{i-1} \theta_0.$$

In any other sorites in which the initial element remains u_0, the value of the element at x-places distant must be *à fortiori* greater than the value $(u_0^2 - u_0)^{2^{x-1}}$ last obtained for the limiting case.

The preceding matter was suggested to me by the chapter in Cantor's *Geschichte der Mathematik* which gives an account of the singular method in use among the ancient Egyptians for working with fractions. It was their curious custom to resolve every fraction into a sum of simple fractions according to a certain traditional method, not leading, I need hardly say, except in a few of the simplest cases, to the expansion under the special form to which I have, in what precedes, given the name of a fractional *sorites.*

I subjoin examples of development of a rational fraction under the form of a sorites.

Let $\frac{4699}{7320}$ be the fraction to be expanded. The work may be arranged as follows:—

(2)	(8)	(60)	(3660)
4699	2078	1984	1920
7320	14640	117120	7027200
9398	16624	119040	7027200

(2) is the number one unit greater than $E\frac{7320}{4699}$; 9398 is 2×4699; 2078 is $9398 - 7320$; 14640 is 2×7320.

One element (2) is now determined, and the fraction $\frac{2078}{14640}$ remains to be expanded.

(8) is the number one unit greater than $E\frac{14640}{2078}$; 16624 is 8×2078; 1984 is $16624 - 14640$; 117120 is 8×14640.

A second element (8) is now found, and $\frac{1984}{117120}$ remains over to be expanded. Proceeding in this manner, and with numerators 4699, 2078, 1984, 1920, necessarily diminishing at each step, we come at last to the element 3660 with a remainder zero. The required sorites is therefore

$$\frac{1}{2} + \frac{1}{8} + \frac{1}{60} + \frac{1}{3660}.$$

As a second example take the fraction $\frac{335}{336}$.

The work may be arranged in a similar manner to that of the foregoing example, and will be as follows:—

(2)	(3)	(7)	(48)
335	334	330	294
336	672	2016	14112
670	1002	2310	14112

and accordingly it will be found that

$$\frac{335}{336}=\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{48}.$$

Postscript.

Let ϕx represent x^2-x+1, $\phi^n c$ will then be the general term of the "limiting sorites" whose first term is c, for which, if we please, $1-c$ may be substituted. The properties of the numbers $\phi^n c$ seem to be worthy of some attention. I confine my observations in what follows to the lowest of such series, namely, where $c=2$ or -1.

The first five terms in such series then become $\bar{1}$ or 2, 3, 7, 43, 1807, 3263443, of which all but 1807, which $=13.139$, are prime numbers. Every term in the series must contain only factors of the form $6i+1$, and this, joined to the fact that a prime factor which has once appeared in any term can never reappear in any other, favours a tendency, so to say, of the numbers to remain primes, or at all events, to be of very limited frangibility into a product of primes.

It is easy to determine whether any proposed prime can occur as a factor of any term whatever in the series; for taking that number, say p, as a modulus, if r is a remainder of any term to that modulus, the remainder of the next term will be r^2-r+1, and as soon as any remainder reappears the series of remainders becomes periodic; so that necessarily in less than the number p of remainders, if p does divide any term of the sorites, we must arrive at a remainder zero, subsequent to which all the remainders are unity. I give the remainders and periods in the annexed table for all values of p of the form $6i+1$ up to 139, from which it will be seen that, under that limit, 13 and 73 are the only prime numbers which are contained as factors in the terms of the series.

p	Remainders of $\phi^n(2)$ to modulus p
2	0.
3	2, 0.
7	2, 3, 0.
13	2, 3, 7, 4, 0.
19	2, 3, 7, 5 ; 2, 3, 7, 5 ; ...
31	2, 3, 7, 12, 9, 11, 18, 28, 13 ; 2, 3, 7, ..., 13 ; ...
37	2, 3, 7 ; 6, 31 ; 6, 31 ; ...
43	2, 3, 7, 0.
61	2, 3, 7, 43, 38, 4, 13, 35, 32 ; 17, 29, 20, 15, 28, 25, 52, 30 ; ...
67	2, 3 ; 7, 43, 65 ; 7, 43, 65 ; ...
73	2, 3, 7, 43, 55, 51, 69, 21, 56, 15, 65, 0.
79	2, 3, 7 ; 43, 69, 32, 45, 6, 31, 61, 27, 71, 73 ; ...
97	2, 3, 7, 43, 61 ; 72, 69, 37 ; 72, 69, 37 ; ...
103	2, 3 ; 7, 43, 56, 94, 91, 54, 82, 51, 79, 86, 101 ; 7, 43, ... ; ....
109	2, 3, 7, 43, 63, 92, 89, 94, 23, 71, 66, 40, 35, 101, 73, 25, 56, 29, 50, 53 ; 32, 12, 24, 8 ; 32, 12, 24, 8 ; ...
127	2, 3, 7, 43, 29, 51, 11 ; 111, 19, 89, 86, 72, 33, 41, 117 ; ...
139	2, 3, 7, 43, 0.
151	2, 3, 7, 43, 146 ; 31, 25, 148, 13, 6 ; ...
157	2, 3 ; 7, 43, 80, 41, 71, 104, 37, 77, 44, 9, 73, 76, 49, 155 ; ...
163	2, 3 ; 7, 43, 14, 20, 55, 37, 29, 161 ; ...
181	2, 3, 7, 43, 178, 13, 157, 58, 49, 0.
193	2 ; 3, 7, 43, 70, 6, 31, 159, 33, 92, 74, 192 ; ...
199	2, 3 ; 7, 43, 16, 42, 131, 116, 8, 57, 9, 73, 83, 41, 49, 164, 67, 45, 190, 91, 32, 197 ; ...