40.

TABLES OF THE GENERATING FUNCTIONS AND GROUND-FORMS FOR SIMULTANEOUS BINARY QUANTICS OF THE FIRST FOUR ORDERS, TAKEN TWO AND TWO TOGETHER.

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IN the Generating Functions given below, the exponents of the letters a, b, c, d, refer to degree in the coefficients of the quantics of the 1st, 2nd, 3rd and 4th orders respectively; the exponents of the letter x to order in the variables. Where the system consists of two quantics of the same order, the Latin letter and the corresponding Greek letter have been used. In the tabulated numerators, the *minus* sign has been placed *over* the number which it affects.

In each of the systems considered in this paper, with the exception of that consisting of a cubic and a quartic, it is found that there is never more than one groundform of any given type (that is, of a given order in the variables and given degrees in the coefficients of the quantics); where, therefore, in the enumeration of the groundforms, the *type* alone is given, the *number* of groundforms of the type is to be understood to be 1. The symbol (λ, μ) is used to indicate a form of the degrees λ and μ in the coefficients of the two quantics, the number placed first always relating to the quantic of lower order, when the orders are different. In the last three cases, the numbers, as well as the types, of the groundforms are given in tables, which require no explanation.

SYSTEM OF TWO LINEARS*.

G. F. for differentiants, $\dfrac{1}{(1-a)(1-\alpha)(1-a\alpha)}$.

G. F. for covariants, $\dfrac{1}{(1-a\alpha)(1-ax)(1-\alpha x)}$.

Groundforms:

Of order 0 .... (1, 1).

„ „ 1 .... (0, 1), (1, 0).

* "Linear" is here used as a noun, in conformity with the use of the words quadric, cubic, &c.

System of Linear and Quadric.

G. F. for differentiants, $\dfrac{1+ab}{(1-a)(1-b)(1-b^2)(1-a^2b)}$.

G. F. for covariants, $\dfrac{1+abx}{(1-b^2)(1-a^2b)(1-ax)(1-bx^2)}$.

Groundforms:

Of order 0..........(0, 2), (2, 1).
" " 1..........(1, 0), (1, 1).
" " 2..........(0, 1).

System of Linear and Cubic.

G. F. for differentiants, $\dfrac{1+a^2c+(a-a^3)c^2+(1-a^2)c^3-ac^4-a^3c^5}{(1-a)(1-c)(1-c^2)(1-c^4)(1-ac)(1-a^3c)}$.

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-ac)(1-a^3c)(1-ax)(1-cx)(1-cx^3)$.

Numerator: $1-ac+a^2c^2+\{(-1+a^2)c+(2a-a^3)c^2-a^2c^3\}x$
$+\{ac+(1-2a^2)c^2+(-a+a^3)c^3\}x^2+\{-ac^2+a^2c^3-a^3c^4\}x^3$.

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-a^3c)(1-a^2c^2)(1-ax)(1-c^2x^2)(1-cx^3)$.

Numerator: $1+a^3c^3+\{a^2c+ac^2+(a^2-a^4)c^3\}x+\{ac+(a-a^3)c^3-a^3c^5\}x^2$
$+\{(1-a^2)c^3-a^3c^4-a^2c^5\}x^3+\{-ac^3-a^4c^6\}x^4$.

Groundforms:

Of order 0..........(0, 4), (2, 2), (3, 1), (3, 3).
" " 1..........(1, 0), (1, 2), (2, 1), (2, 3).
" " 2..........(0, 2), (1, 1), (1, 3).
" " 3..........(0, 1), (0, 3).

System of Linear and Quartic.

G. F. for differentiants,

$$\frac{1+(a+a^3)d+(a+a^2-a^5)d^2+(1-a^3-a^4)d^3+(-a^2-a^4)d^4-a^5d^5}{(1-a)(1-d)(1-d^2)^2(1-d^3)(1-a^2d)(1-a^4d)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-d^2)(1-d^3)(1-a^2d)(1-a^4d)(1-ax)(1-dx^2)(1-dx^4)$.

Numerator: $1-a^2d+a^4d^2+\{a^3d+(a^3-a^5)d^2\}x+\{(-1+a^2)d$
$+(2a^2-a^4)d^2-a^4d^3\}x^2+\{ad+(a-2a^3)d^2+(-a^3+a^5)d^3\}x^3$
$+\{(1-a^2)d^2-a^2d^3\}x^4+\{-ad^2+a^3d^3-a^5d^4\}x^5$.

G. F. for covariants, representative form,

Denominator: $(1-d^2)(1-d^3)(1-a^4d)(1-a^4d^2)(1-ax)(1-dx^4)(1-d^2x^4)$.

Numerator: $1+a^6d^3+\{a^3d+a^3d^2+(a^5-a^7)d^3\}x+\{a^2d+a^2d^2+(a^4-a^6)d^3\}x^2$
$+\{ad+ad^2+(a^3-a^5)d^3\}x^3+\{(a^2-a^4)d^3-a^6d^4-a^6d^5\}x^4$
$+\{(a-a^3)d^3-a^5d^4-a^5d^5\}x^5+\{(1-a^2)d^3-a^4d^4-a^4d^5\}x^6$
$+\{-ad^3-a^7d^6\}x^7.$

Groundforms:

Of order 0.................................(0, 2), (0, 3), (4, 1), (4, 2), (6, 3).
„ „ 1.................................(1, 0), (3, 1), (3, 2), (5, 3).
„ „ 2.................................(2, 1), (2, 2), (4, 3).
„ „ 3.................................(1, 1), (1, 2), (3, 3).
„ „ 4.................................(0, 1), (0, 2), (2, 3).
„ „ 5.................................(1, 3).
„ „ 6.................................(0, 3).

SYSTEM OF TWO QUADRICS.

G. F. for differentiants, $\dfrac{1+b\beta}{(1-b)(1-b^2)(1-\beta)(1-\beta^2)(1-\beta b)}$.

G. F. for covariants, $\dfrac{1+b\beta x^2}{(1-b^2)(1-\beta^2)(1-b\beta)(1-bx^2)(1-\beta x^2)}$.

Groundforms:

Of order 0.................................(0, 2), (1, 1), (2, 0).
„ „ 2.................................(0, 1), (1, 0), (1, 1).

SYSTEM OF QUADRIC AND CUBIC.

G. F. for differentiants,

$$\frac{1+(2b+b^2)c+(b+b^2+b^3)c^2+c^3-b^4c^4+(-b-b^2-b^3)c^5+(-b^2-2b^3)c^6-b^4c^7}{(1-b)(1-b^2)(1-c)(1-c^2)(1-c^4)(1-bc^2)(1-b^3c^2)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-b^2)(1-c^4)(1-bc^2)(1-b^3c^2)(1-bx^2)(1-cx)(1-cx^3)$.

Numerator: $1+b^3c^4+\{(-1+b+b^2)c+(b+b^2)c^3-b^3c^5\}x$
$+\{(1+b^3)c^2+(-b-b^4)c^4\}x^2+\{bc+(-b^2-b^3)c^3$
$+(-b^2-b^3+b^4)c^5\}x^3+\{-bc^2-b^4c^6\}x^4.$

G. F. for covariants, representative form,

Denominator: $(1-b^2)(1-c^4)(1-bc^2)(1-b^3c^2)(1-bx^2)(1-c^2x^2)(1-cx^3)$.

Numerator: $1+b^3c^4+\{(b+b^2)c+(b+b^2)c^3\}x+\{(b+b^2+b^3)c^2$
$+(b^2-b^4)c^4-b^3c^6\}x^2+\{bc+(1-b^2)c^3+(-b-b^2-b^3)c^5\}x^3$
$+\{(-b^2-b^3)c^4+(-b^2-b^3)c^6\}x^4+\{-bc^3-b^4c^7\}x^5.$

Groundforms:

Of order 0.................................(0, 4), (1, 2), (2, 0), (3, 2), (3, 4).
„ „ 1.................................(1, 1), (1, 3), (2, 1), (2, 3).
„ „ 2.................................(0, 2), (1, 0), (1, 2).
„ „ 3.................................(0, 1), (0, 3), (1, 1).

System of Quadric and Quartic.

G. F. for differentiants,

$$\frac{1+(b+b^2)\,d+(2b-b^3)\,d^2+(1-2b^2)\,d^3+(-b-b^2)\,d^4-b^3d^5}{(1-b)(1-b^2)(1-d)(1-d^2)^2(1-d^3)(1-bd)(1-b^2d)}.$$

G. F. for covariants, reduced form,

Denominator: $(1-b^2)(1-d^2)(1-d^3)(1-bd)(1-b^2d)(1-bx^2)(1-dx^2)$
$(1-dx^4)$.

Numerator: $1-bd+b^2d^2+\{(-1+b+b^2)\,d+(2b-b^3)\,d^2-b^2d^3\}\,x^2$
$+\{bd+(1-2b^2)\,d^2+(-b-b^2+b^3)\,d^3\}\,x^4$
$+\{-bd^2+b^2d^3-b^3d^4\}\,x^6$.

G. F. for covariants, representative form,

Denominator: $(1-b^2)(1-d^2)(1-d^3)(1-b^2d)(1-b^2d^2)(1-bx^2)(1-dx^4)$
$(1-d^2x^4)$.

Numerator: $1+b^3d^3+\{(b+b^2)\,d+(b+b^2)\,d^2+(b^2-b^4)\,d^3\}\,x^2+\{bd+bd^2$
$+(b-b^3)\,d^3-b^3d^4-b^3d^5\}\,x^4+\{(1-b^2)\,d^3+(-b^2-b^3)\,d^4$
$+(-b^2-b^3)\,d^5\}\,x^6+\{-bd^3-b^4d^6\}\,x^8$.

Groundforms:

Of order 0............................(0, 2), (0, 3), (2, 0), (2, 1), (2, 2), (3, 3).
" " 2............................(1, 0), (1, 1), (1, 2), (2, 1), (2, 2), (2, 3).
" " 4............................(0, 1), (0, 2), (1, 1), (1, 2), (1, 3).
" " 6............................(0, 3).

System of Two Cubics.

G. F. for differentiants,

Denominator: $(1-c)(1-c^2)(1-c^4)(1-\gamma)(1-\gamma^2)(1-\gamma^4)(1-c\gamma)$
$(1-c^3\gamma)(1-c\gamma^3)$.

Numerator: $1+c^3+(2c+2c^2-c^5-c^6)\,\gamma+(2c+2c^2-c^4-c^5-c^6-c^7)\,\gamma^2$
$+(1+2c^3-c^4-2c^5-c^6-c^7)\,\gamma^3+(-c^2-c^3-c^5-c^6)\,\gamma^4$
$+(-c-c^2-2c^3-c^4+2c^5+c^8)\,\gamma^5+(-c-c^2-c^3-c^4+2c^6+2c^7)\,\gamma^6$
$+(-c^2-c^3+2c^6+2c^7)\,\gamma^7+(c^5+c^8)\,\gamma^8$.

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-\gamma^4)(1-c\gamma)(1-c^3\gamma)(1-c\gamma^3)(1-cx)(1-cx^3)$
$(1-\gamma x)(1-\gamma x^3)$.

Numerator:

		γ^0	γ^1	γ^2	γ^3	γ^4	γ^5	γ^6
x^0	c^0	1						
	c^2			1				
	c^3				1			
	c^5						1	
x^1	c^0		$\bar{1}$					
	c^1	$\bar{1}$		1		1		
	c^2		1					
	c^4		1					
	c^5							$\bar{1}$
	c^6						$\bar{1}$	
x^2	c^0			1				
	c^1		2				$\bar{1}$	
	c^2	1		$\bar{1}$		$\bar{1}$		
	c^4			$\bar{1}$				
	c^5		$\bar{1}$					
	c^6							1

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^6	c^2		1					
	c^4				1			
	c^5					1		
	c^7							1
x^5	c^1		$\bar{1}$					
	c^2	$\bar{1}$						
	c^3						1	
	c^5						1	
	c^6			1		1		$\bar{1}$
	c^7						$\bar{1}$	
x^4	c^1	1						
	c^2						$\bar{1}$	
	c^3					$\bar{1}$		
	c^5			$\bar{1}$		$\bar{1}$		1
	c^6		$\bar{1}$				2	
	c^7					1		

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6
x^3	c^1		$\bar{1}$		$\bar{1}$		
	c^2	$\bar{1}$				1	
	c^3				$\bar{1}$		$\bar{1}$
	c^4	$\bar{1}$		$\bar{1}$			
	c^5		1				$\bar{1}$
	c^6			$\bar{1}$		$\bar{1}$	

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-\gamma^4)(1-c\gamma)(1-c^3\gamma)(1-c\gamma^3)(1-c^2x^2)(1-cx^3)$
$(1-\gamma^2x^2)(1-\gamma x^3)$.

Numerator :

		γ^0	γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^0	c^0	1							
	c^2			1					
	c^3				1				
	c^5						1		
x^1	c^1			1		1			
	c^2		1		1				
	c^3			1		1			
	c^4		1		1				
x^2	c^1		1		1				
	c^2			1					
	c^3		1		1				
	c^4					1			
	c^5								$\bar{1}$
	c^7						$\bar{1}$		
x^3	c^0				1				
	c^1			1		$\bar{1}$		$\bar{1}$	
	c^2		1						
	c^3	1				$\bar{2}$		$\bar{1}$	
	c^4		$\bar{1}$		$\bar{2}$				
	c^5							$\bar{1}$	
	c^6		$\bar{1}$		$\bar{1}$		$\bar{1}$		

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7	γ^8
x^8	c^3			1					
	c^5					1			
	c^6						1		
	c^8								1
x^7	c^4					1		1	
	c^5				1		1		
	c^6					1		1	
	c^7				1		1		
x^6	c^1			$\bar{1}$					
	c^3	$\bar{1}$							
	c^4				1				
	c^5					1		1	
	c^6						1		
	c^7					1		1	
x^5	c^2			$\bar{1}$		$\bar{1}$		$\bar{1}$	
	c^3		$\bar{1}$						
	c^4					$\bar{2}$		$\bar{1}$	
	c^5		$\bar{1}$		$\bar{2}$				1
	c^6							1	
	c^7		$\bar{1}$		$\bar{1}$		1		
	c^8					1			

		γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7
x^4	c^1	1				$\bar{1}$		
	c^2				$\bar{1}$		$\bar{1}$	
	c^3			$\bar{1}$		$\bar{2}$		$\bar{1}$
	c^4		$\bar{1}$		$\bar{2}$		$\bar{1}$	
	c^5	$\bar{1}$		$\bar{2}$		$\bar{1}$		
	c^6		$\bar{1}$		$\bar{1}$			
	c^7			$\bar{1}$				1

*Table of Groundforms**.

Order in the Variables.	Deg. in coeff's of 2d cubic.	Deg. in coeff's of 1st cubic. 0	1	2	3	4
0	0					1
	1		1		1	
	2			1		
	3		1		1	
	4	1				
1	1			1		1
	2		1		1	
	3			1		
	4		1			

Order in the Variables	Deg. in coeff's of 2d cubic.	Deg. in coeff's of 1st cubic. 0	1	2	3
2	0			1	
	1		1		1
	2	1		1	
	3		1		
3	0		1		1
	1	1		1	
	2		1		
	3	1			
4	1		1		

System of Cubic and Quartic.

G. F. for differentiants,

Denominator: $(1-c)(1-c^2)(1-c^4)(1-d)(1-d^2)^2(1-d^3)(1-c^2d)$
$(1-c^4d)(1-c^2d^3)(1-c^4d^3)$.

Numerator: $1 + c^3 + (3c + 2c^2 + 2c^3 + c^4 - 2c^5 - c^6 - c^7)\,d$
$+ (3c + 5c^2 + 2c^3 + 2c^4 - 3c^5 - 4c^6 - 2c^7 - 2c^8 + c^9)\,d^2$
$+ (1 + 3c^2 + 3c^3 + c^4 - c^5 - 6c^6 - 5c^7 - 4c^8 + 2c^{10})\,d^3$
$+ (-c^2 + c^3 - c^4 - 2c^5 - 5c^6 - 6c^7 - 3c^8 - c^9 + 3c^{10} + 2c^{11} + c^{12})\,d^4$
$+ (-2c^2 - 3c^3 - 3c^4 - 3c^5 - 2c^6 - 2c^7 - c^8 + 2c^{10} + 4c^{11} + 3c^{12} + c^{13})\,d^5$
$+ (-c^2 - 3c^3 - 4c^4 - 2c^5 + c^7 + 2c^8 + 2c^9 + 3c^{10} + 3c^{11} + 3c^{12} + 2c^{13})\,d^6$
$+ (-c^3 - 2c^4 - 3c^5 + c^6 + 3c^7 + 6c^8 + 5c^9 + 2c^{10} + c^{11} - c^{12} + c^{13})\,d^7$
$+ (-2c^5 + 4c^7 + 5c^8 + 6c^9 + c^{10} - c^{11} - 3c^{12} - 3c^{13} - c^{15})\,d^8$
$+ (-c^6 + 2c^7 + 2c^8 + 4c^9 + 3c^{10} - 2c^{11} - 2c^{12} - 5c^{13} - 3c^{14})\,d^9$
$+ (c^8 + c^9 + 2c^{10} - c^{11} - 2c^{12} - 2c^{13} - 3c^{14})\,d^{10} + (-c^{12} - c^{15})\,d^{11}$.

G. F. for covariants, reduced form,

Denominator: $(1-c^4)(1-d^2)(1-d^3)(1-c^2d)(1-c^4d)(1-c^2d^3)(1-c^4d^3)$
$(1-cx)(1-cx^3)(1-dx^2)(1-dx^4)$.

* The forms of ord. 1, deg. 3, 4 and of ord. 1, deg. 4, 3 given by Clebsch and Gordan, do not appear in this table, and it has been proved by the author that no fundamental forms of either of these types exist. [See page 409, below.]

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9
x^0	c^0	1									
	c^2		$\bar{1}$								
	c^4			2	2	2	1				
	c^6			1	1		$\bar{1}$	$\bar{1}$			
	c^8				$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$			
	c^{10}								1		
	c^{12}									$\bar{1}$	
x^1	c^1	$\bar{1}$		1							
	c^3		3	2	1	1					
	c^5		1	$\bar{2}$	$\bar{1}$	$\bar{1}$	$\bar{1}$	1			
	c^7			$\bar{2}$	$\bar{1}$	$\bar{1}$		1	$\bar{1}$		
	c^9				1	1	1		$\bar{2}$		
	c^{11}							$\bar{1}$	$\bar{2}$		
	c^{13}									1	
x^2	c^0		$\bar{1}$								
	c^2	1	1	3	2	1					
	c^4		$\bar{1}$		$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^6		$\bar{1}$		$\bar{2}$	$\bar{2}$			1		
	c^8			1		1	1		2		
	c^{10}						$\bar{1}$	$\bar{1}$		$\bar{2}$	
	c^{12}							1	1		1
x^3	c^1		2								
	c^3			$\bar{3}$							
	c^5		$\bar{1}$	$\bar{2}$	1		1		$\bar{1}$		
	c^7					$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	1	
	c^9						$\bar{1}$			1	
	c^{11}					1	1	2	1	2	
	c^{13}									1	$\bar{1}$

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}
x^8	c^2		1								
	c^4			$\bar{1}$							
	c^6				2	2	2	1			
	c^8				1	1		$\bar{1}$	$\bar{1}$		
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$		
	c^{12}									1	
	c^{14}										$\bar{1}$
x^7	c^1		$\bar{1}$								
	c^3			2	1						
	c^5			2		$\bar{1}$	$\bar{1}$	$\bar{1}$			
	c^7			1	$\bar{1}$		1	1	2		
	c^9				$\bar{1}$	1	1	1	2	$\bar{1}$	
	c^{11}						$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	
	c^{13}								$\bar{1}$	$\bar{1}$	1
x^6	c^2	$\bar{1}$		$\bar{1}$	$\bar{1}$						
	c^4		2		1	1					
	c^6			$\bar{2}$		$\bar{1}$	$\bar{1}$		$\bar{1}$		
	c^8			$\bar{1}$			2	2		1	
	c^{10}				1	2	2	1		1	
	c^{12}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$	$\bar{1}$
	c^{14}									1	
x^5	c^1	1	$\bar{1}$								
	c^3		$\bar{2}$	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$				
	c^5		$\bar{1}$			1					
	c^7		$\bar{1}$	1	2	1	1				
	c^9			1		$\bar{1}$		$\bar{1}$	2	1	
	c^{11}								3		
	c^{13}									$\bar{2}$	

Numerator—(*Continued.*)

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9
x^4	c^0		1							
	c^2	$\bar{1}$		$\bar{1}$	$\bar{1}$	$\bar{1}$				
	c^4	$\bar{1}$		$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^6		1	$\bar{2}$	$\bar{1}$			1		
	c^8			$\bar{1}$			1	2	$\bar{1}$	
	c^{10}					1	2	2		1
	c^{12}					1	1	1		1
	c^{14}								$\bar{1}$	

G. F. for covariants, representative form,

Denominator: $(1-c^4)(1-d^2)(1-d^3)(1-c^4d)(1-c^4d^2)(1-c^2d^3)(1-c^4d^3)$

$(1-cx^3)(1-c^2x^2)(1-dx^4)(1-d^2x^4)$.

Numerator:

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}
x^0	c^0	1											
	c^4			1	2	2	1						
	c^6			1	3	2	1						
	c^8					$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$				
	c^{10}					$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^{14}										$\bar{1}$		
x^1	c^1		1	1									
	c^3		2	3	2	1							
	c^5		1	2	3	2	1	1					
	c^9				$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$			
	c^{11}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$			
	c^{13}								$\bar{1}$	$\bar{1}$			

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}	d^{12}
x^{11}	c^3			1									
	c^7					1	2	2	1				
	c^9					1	3	2	1				
	c^{11}							$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		
	c^{13}							$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{17}												$\bar{1}$
x^{10}	c^4				1	1							
	c^6				2	3	2	1					
	c^8				1	2	3	2	1	1			
	c^{12}						$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
	c^{14}								$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	
	c^{16}										$\bar{1}$	$\bar{1}$	

Numerator—(*Continued.*)

		d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}
x^2	c^2		2	3	2	1							
	c^4		2	4	5	3	1						
	c^8			$\bar{1}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$				
	c^{10}						$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$			
	c^{12}					1	1		$\bar{2}$	$\bar{2}$	$\bar{1}$		
	c^{16}										1		
x^3	c^1		1	1	1								
	c^3	1	1	3	5	3	1						
	c^5		$\bar{1}$		2	3	2						
	c^7		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{1}$				
	c^9					$\bar{2}$	$\bar{4}$	$\bar{3}$	$\bar{3}$	$\bar{1}$			
	c^{11}				1	1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$		
	c^{13}						1	1	2	1	$\bar{1}$		
	c^{15}								1	1	1		
x^4	c^2		1	2	3	1							
	c^4		$\bar{1}$		2	1	$\bar{1}$	$\bar{2}$	$\bar{1}$				
	c^6		$\bar{1}$	$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$				
	c^8			$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{5}$	$\bar{2}$	1	2	1		
	c^{10}					$\bar{1}$	$\bar{1}$	1	1	1			
	c^{12}					1	3	4	4	2			
	c^{14}						1	2	3	2	1	1	1
x^5	c^1		1	1									
	c^3				$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$					
	c^5		$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^7			$\bar{2}$	$\bar{4}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	$\bar{2}$	$\bar{1}$			
	c^9					$\bar{1}$	$\bar{1}$	$\bar{1}$	1	2	2	1	
	c^{11}					1	2	2	4	5	4	2	
	c^{13}						1	3	4	4	4	1	
	c^{15}							1	1	1	1		
	c^{17}										$\bar{1}$		

		d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	d^9	d^{10}	d^{11}	d^{12}
x^9	c^1			$\bar{1}$									
	c^5			1	2	2		$\bar{1}$	$\bar{1}$				
	c^7				2	3	2	1					
	c^9					1	2	3	3	3	1		
	c^{13}							$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	
	c^{15}								$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	
x^8	c^2			$\bar{1}$	$\bar{1}$	$\bar{1}$							
	c^4			1	$\bar{1}$	$\bar{2}$	$\bar{1}$	$\bar{1}$					
	c^6			1	1	1		$\bar{1}$	$\bar{1}$	$\bar{1}$			
	c^8				1	3	3	4	2				
	c^{10}					1	3	3	3	4	2	1	
	c^{12}							$\bar{2}$	$\bar{3}$	$\bar{2}$		1	
	c^{14}							$\bar{1}$	$\bar{3}$	$\bar{5}$	$\bar{3}$	$\bar{1}$	$\bar{1}$
	c^{16}									$\bar{1}$	$\bar{1}$	$\bar{1}$	
x^7	c^3	$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{2}$	$\bar{1}$					
	c^5				$\bar{2}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$				
	c^7				$\bar{1}$	$\bar{1}$	$\bar{1}$	1	1				
	c^9			$\bar{1}$	$\bar{2}$	$\bar{1}$	2	5	5	3	1		
	c^{11}					1	3	4	4	4	2	1	
	c^{13}					1	2	1	$\bar{1}$	$\bar{2}$		1	
	c^{15}								$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
x^6	c^0			1									
	c^2			$\bar{1}$	$\bar{1}$	$\bar{1}$	$\bar{1}$						
	c^4		$\bar{1}$	$\bar{4}$	$\bar{4}$	$\bar{4}$	$\bar{3}$	$\bar{1}$					
	c^6		$\bar{2}$	$\bar{4}$	$\bar{5}$	$\bar{4}$	$\bar{2}$	$\bar{2}$	$\bar{1}$				
	c^8		$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{1}$	1	1	1				
	c^{10}				1	2	2	4	5	4	2		
	c^{12}				1	2	2	3	4	5	3	1	
	c^{14}						1	1	1	1			
	c^{16}										$\bar{1}$	$\bar{1}$	

*Table of Groundforms**.

Order in the Variables.	Deg. in coeff's of cubic.	Deg. in coeff's of quartic. 0	1	2	3	4	5
0	0			1	1		
	2				1		
	4	1	1	2	3	2	1
	6			1	3	2	1
1	1		1	1			
	3		2	3	2	1	
	5		1	2	2		

Order in the Variables.	Deg. in coeff's of cubic.	Deg. in coeff's of quartic. 0	1	2	3
2	2	1	2	2	1
	4		2	2	
3	1	1	1	1	1
	3	1	1	1	1
4	0		1	1	
	2		1	1	1
5	1		1	1	
6	0				1

System of Two Quartics.

G. F. for differentiants,

Denominator: $(1-d)(1-d^2)^2(1-d^3)(1-\delta)(1-\delta^2)^2(1-\delta^3)(1-d\delta)(1-d^2\delta)$
$(1-d\delta^2)$.

Numerator: $1+d^3+(3d+3d^2-d^4-d^5)\delta+(3d+4d^2-d^3-3d^4-2d^5-d^6)\delta^2$
$+(1-d^2-2d^4-3d^5-d^6)\delta^3+(-d-3d^2-2d^3-d^5+d^7)\delta^4$
$+(-d-2d^2-3d^3-d^4+4d^5+3d^6)\delta^5+(-d^2-d^3+3d^5+3d^6)\delta^6$
$+(d^4+d^7)\delta^7$.

G. F. for covariants, reduced form,

Denominator: $(1-d^2)(1-d^3)(1-\delta^2)(1-\delta^3)(1-d\delta)(1-d^2\delta)(1-d\delta^2)$
$(1-dx^2)(1-dx^4)(1-\delta x^2)(1-\delta x^4)$.

* The form of ord. 1, deg. 5, 4, and the two forms of ord. 2, deg. 4, 3, given by Gundelfinger, do not appear in this table, and it has been proved by the author that no fundamental forms of either of these types exist. [See below, p. 409.]

Numerator :

		δ^0	δ^1	δ^2	δ^3	δ^4	δ^5
x^0	d^0	1					
	d^2			1			
	d^4					1	
x^2	d^0		$\bar{1}$				
	d^1	$\bar{1}$	1	1	1		
	d^2		1	1			
	d^3		1		1		
	d^4						$\bar{1}$
	d^5					$\bar{1}$	
x^4	d^0			1			
	d^1		2		$\bar{1}$	$\bar{1}$	
	d^2	1		$\bar{1}$	$\bar{2}$		
	d^3		$\bar{1}$	$\bar{2}$			
	d^4		$\bar{1}$				$\bar{1}$
	d^5					$\bar{1}$	1

		δ^1	δ^2	δ^3	δ^4	δ^5	δ^6
x^{10}	d^2		1				
	d^4				1		
	d^6						1
x^8	d^1		$\bar{1}$				
	d^2	$\bar{1}$					
	d^3			1		1	
	d^4				1	1	
	d^5			1	1	1	$\bar{1}$
	d^6					$\bar{1}$	
x^6	d^1	1	$\bar{1}$				
	d^2	$\bar{1}$				1	
	d^3				$\bar{2}$	$\bar{1}$	
	d^4			$\bar{2}$	$\bar{1}$		1
	d^5		$\bar{1}$	$\bar{1}$		2	
	d^6				1		

G. F. for covariants, representative form,

Denominator: $(1-d^2)(1-d^3)(1-\delta^2)(1-\delta^3)(1-d\delta)(1-d^2\delta)(1-d\delta^2)$
$(1-dx^4)(1-d^2x^4)(1-\delta x^4)(1-\delta^2 x^4)$.

Numerator:

		δ^0	δ^1	δ^2	δ^3	δ^4	δ^5	δ^6
x^0	d^0	1						
	d^2			1				
	d^4					1		
x^2	d^1		1	1	1			
	d^2		1	1	1			
	d^3		1	1	1			
x^4	d^1		1	1				
	d^2		1	1				
	d^3				1	1		
	d^4				1		$\bar{1}$	$\bar{1}$
	d^5					$\bar{1}$		
	d^6					$\bar{1}$		
x^6	d^0				1			
	d^1		1	1		$\bar{1}$	$\bar{1}$	
	d^2		1	1	$\bar{1}$	$\bar{2}$	$\bar{1}$	
	d^3	1		$\bar{1}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	
	d^4		$\bar{1}$	$\bar{2}$	$\bar{2}$			
	d^5		$\bar{1}$	$\bar{1}$	$\bar{1}$			

		δ^1	δ^2	δ^3	δ^4	δ^5	δ^6	δ^7
x^{14}	d^3			1				
	d^5					1		
	d^7							1
x^{12}	d^4				1	1	1	
	d^5				1	1	1	
	d^6				1	1	1	
x^{10}	d^1			$\bar{1}$				
	d^2			$\bar{1}$				
	d^3	$\bar{1}$	$\bar{1}$		1			
	d^4			1	1			
	d^5					1	1	
	d^6					1	1	
x^8	d^2				$\bar{1}$	$\bar{1}$	$\bar{1}$	
	d^3				$\bar{2}$	$\bar{2}$	$\bar{1}$	
	d^4		$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{1}$		1
	d^5		$\bar{1}$	$\bar{2}$	$\bar{1}$	1	1	
	d^6		$\bar{1}$	$\bar{1}$		1	1	
	d^7				1			

*Table of Groundforms *.*

Order in the Variables.	Deg. in coeff's of 2d quartic.	Deg. in coeff's of 1st quartic. 0	1	2	3
0	0			1	1
	1		1	1	
	2	1	1	1	
	3	1			
2	1		1	1	1
	2		1	1	1
	3		1	1	

Order in the Variables.	Deg. in coeff's of 2d quartic.	Deg. in coeff's of 1st quartic. 0	1	2	3
4	0		1	1	
	1	1	1	1	
	2	1	1		
6	0				1
	1		1	1	
	2		1		
	3	1			

The following table exhibits the total numbers of groundforms; the quantics themselves and the absolute constant are included in the numbers†.

Order of Quantic.	Order of Quantic. 0	1	2	3	4
0	1	2	3	5	6
1		4	6	14	21
2			7	16	19
3				27	62
4					29

* The forms of ord. 4, deg. 2, 2, and of ord. 6, deg. 2, 2, given by Gordan, do not appear in this table, and have been proved by the author to be compound forms. [See below, p. 409.]

† Some remarks on the preceding tables (to save delay in going to press) have been made the subject of a separate article in this number. [p. 406, below.]

REMARKS ON THE TABLES FOR BINARY QUANTICS.

The valuable idea of using different roman letters, *a*, *b*, *c*, *d*, to correspond to the coefficients of quantics of different orders, is due to Mr Franklin. Had it occurred previously I should have employed it in the tables of the generating functions and groundforms of single quantics. The *n*th letter of the alphabet, say θ, will in this way symbolize the $(n+1)$ coefficients $\theta_0, \theta_1, \theta_2, \dots \theta_n$ and so x regarded as a new point of departure in the alphabet will symbolize x_0, x_1.

I pass on to a remark of greater importance referring to the separation of the Parallelopiped which may be imagined to represent the complete tabulation of the representative G.F. to a system of two simultaneous quantics, and its use in simplifying the process of tamisage.

To fix the ideas, let us take the case of a Cubic and Quartic. Then, to represent the collected signification of the rectangles at pp. [400, 401, above]*, we may suppose a parallelopiped 12 inches in length, 17 in breadth, and 11 in depth, 12, 17, 11 being the highest exponents which appear in such rectangles of *d*, *c*, *x*, respectively, and confine our attention to the sign proper to each of the 12.17.11 cubical spaces (inch cubes) which may be either + or − or vacancy, if sign that may be called where sign is none. We may, if we please, imagine these cubes or cells to be filled with positive, negative or neutral electricity.

According to the chorographical law (foot-note, p. [310, above]), it ought to and would be found that the occupied portions of this parallelopiped would separate into a certain number of distinct blocks of positive and negative signs. Let us limit our attention to the first of these blocks†. The tamisage, according to the principle laid down in the remarks at the end of the preceding paper, may be limited to this block, although, as a matter of fact (and for greater assurance) in deducing the tables of

* The vacant lines and columns suppressed in the rectangular tables referred to, are supposed to be supplied.

† Planes passing through that angle of the parallelopiped at which is situated the absolute constant, may be termed the planes of reference.

In order to determine whether or not a given space or cell (as we may term it) belongs to the first block, the following is the rule to be observed: (1) If its sign is negative, it is to be rejected. (2) If three lines be drawn through its centre parallel to the edges of the parallelopiped *towards* the planes of reference, and any of these passes through a negative cell, it is to be rejected. (3) In every other case, the cell (or term which occupies it) forms a part of the primary block. So to obtain the second block required for determining the syzygants of the first species, (and notice that under a general point of view groundforms may be regarded as syzygants of species zero or on the other hand and preferably syzygants of the *i*th may be regarded as groundforms of the $(i+1)$th species) we may take any negative cell such that the three lines drawn through it parallel to the edges and towards the plane of reference shall not pass through any positive one. The *ensemble* of such constitute the second block. Then for the third block we may take the groundforms, it was actually applied to all the positive terms in the 11 rectangles.

An inspection of the rectangle affected with x^7 and x^8, p. [401, above] will show that they may be omitted as forming no part of the first positive block. In the rectangle affected with x^9, it will be found that the only terms subject to examination, that is, the only terms with positive coefficients which are not *preceded* vertically or horizontally by terms with negative coefficients, are

$$\begin{array}{lll} 2c^5d^4x^9 & 2c^5d^5x^9 & \\ 2c^7d^4x^9 & 3c^7d^5x^9 & 2c^7d^6x^9 \\ & c^9d^5x^9 & 2c^9d^6x^9 \end{array}$$

Calling any one of these terms $kc^\lambda d^\mu x^9$, it will be found, on examining the preceding rectangles, that $c^\lambda d^\mu$ occurs in one or more of them affected with a negative numerical coefficient. Consequently, these terms do not belong to the primary block, and, in like manner, it will be found that the rectangles subsequent to x^9 form no part of it.

The tamisage may therefore be confined to the rectangles belonging to x^0, x^1, x^2, x^3, x^4, x^5, x^6 and the only terms to be retained will be seen to be those exhibited in the following table:

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ensemble of positive cells not included in the first block and such that the lines through any one of them drawn as before shall not pass through a negative cell, and so on until all the cells are distributed into their respective blocks.

It may not be out of place to observe here that groundforms and syzygants may be regarded as existences and privations of existence, and the Fundamental Postulate so often previously quoted (on which the legitimacy of tamisage depends) is analogous to the asertion that free electricities of the two kinds cannot coexist at the same time at the same point of a body. Are there not some phenomena in electricity (certain visible effects at the poles of an electrical machine or at the extremities of the electric arc) which seem to indicate that the two electricities, although mutually quelling, are not absolutely antithetical in the sense that they might be reversed throughout an environment without any change of effect of any kind resulting? Unless this is true the analogy of the relation of Groundforms and Syzygants to Positive and Negative Electricity halts on one foot. But if it be true we may perhaps see foreshadowed in the constitution of the generating function, the possibility of physical research hereafter bringing to light residual phenomena in which freer and rarer kinds of positive and negative electricity in succession will make their appearance.

Their supposed possible prototypes as yet, play no part in any developed algebraical theory, and indeed the consciousness of only a few algebraists is as yet fully awakened to a sense of their existence. If to any one the idea of physical being foreshadowed in algebraical laws should appear extravagant and visionary, let him reflect on the certain fact that the conception of chemical units as molecules composed of atoms and of the new theory of atomicity or *valence* in each essential particular might have been safely inferred as a possible hypothesis, from the ascertained laws of the constitution and mutual actions upon one another of invariantive forms. If we only allow that the so-called laws of nature have their origin in reason and are not merely arbitrary or *fiat* laws, we can very well understand how an unfailing parallelism should exist between the phenomena of the outer world and those phenomena of the pure intelligence with which algebraical science is concerned.

	c^4d^2	$2c^4d^3$	$2c^4d^4$	c^4d^5		
	c^6d^2	$3c^6d^3$	$2c^6d^4$	c^6d^5		
	cdx	cd^2x				
	$2c^3dx$	$3c^3d^2x$	$2c^3d^3x$	c^3d^4x		
	c^5dx	$2c^5d^2x$	$3c^5d^3x$	$2c^5d^4x$	c^5d^5x	c^5d^6x
	$2c^2dx^2$	$3c^2d^2x^2$	$2c^2d^3x^2$	$c^2d^4x^2$		
	$2c^4dx^2$	$4c^4d^2x^2$	$5c^4d^3x^2$	$3c^4d^4x^2$	$c^4d^5x^2$	
	cdx^3	cd^2x^3	cd^3x^3			
c^3x^3	c^3dx^3	$3c^3d^2x^3$	$5c^3d^3x^3$	$3c^3d^4x^3$	$c^3d^5x^3$	
	c^2dx^4	$2c^2d^2x^4$	$3c^2d^3x^4$	$c^2d^4x^4$		
	cdx^5	cd^2x^5				
	d^3x^6					

Thus, it is evident at a glance that the highest order in the variables, the highest degrees in the cubic and quartic coefficients respectively, of any groundform, are 6, 4 and 5 respectively. Prior to all tamisage, 6, 4, 5 are seen to be superior limits to such order and degrees, because no powers of x, d, c figure among the above terms higher than 6, 4, 5, and a slight examination shows that some terms, containing x^6, d^4, c^5, survive the operation of the tamisage.

The number of types submitted to tamisage, it will be seen, is 45, as previously stated.

The number of forms contained under these types is 83.

The number of types absolutely abolished by the operation is 10, bringing down the number to 35; and the reduction in the total number of forms is 33, bringing down the number to 50*.

These remarks have reference *solely to the groundforms* represented by the *numerator* of the Generating Function. The denominator yields 11 groundforms, thus raising the total number to 61, which is the right number when the absolute constant is not counted *in* as the representative of an invariant†.

Possibly, when I may be again able to secure the services of Mr Franklin, without whose intelligent cooperation I believe it would have been impracticable for me to have calculated the tables contained in this and the preceding

* There is every reason to believe that a calculating machine might be constructed without difficulty for performing mechanically the process of *tamisage* whether simple (involving only a single variable) as for invariants of single forms or compound (involving several variables) as for covariants or invariants of systems.

† It should be noticed that some of the entries in the Table of Groundforms, p. [402], are made up partly from the numerator and partly from the denominator, as for example the number 3 in the column headed 3 and in the line marked 4 for the order 0, is made up partly of the 2 in the surviving term $2d^3c^4$ of the numerator and partly of a unit taken from the term $1-d^3c^4$ of the

number of the *Journal*, I shall be able to extend the limit to the order of the combined quantics. At all events, the labour of forming the tables of the combinations of 1, 2, 3, 4, 5, 6 with 6, would probably not exceed the amount which has been incurred in calculating the groundforms of a single quantic of the 9th order. The references to the *Comptes Rendus* made in the foot-notes are to Vol. LXXXIV. 1ier semestre for 1877, p. 1285, for the disproof of the existence of the *two* forms given in the accepted tables belonging to a system of two binary quartics*; to Vol. LXXXVII. 2me semestre for 1878, p. 445, and again p. 477, for the disproof of the existence of the *three* accepted superfluous forms for a system of a binary cubic and quartic†, and to Vol. LXXXIX. 2me semestre for 1879, p. 828, for the disproof of the existence of the *two* superfluous accepted forms belonging to the system of two binary cubics‡. The proof of the Fundamental Theorem is given as a Postscriptum in a paper in *Borchardt's Journal* "Sur les actions mutuelles des formes invariantives," 1878 [p. 232, above], and in a paper entitled "Proof of the hitherto undemonstrated fundamental theorem for Invariants," in the *Philosophical Magazine* for the same year, 1878 [p. 117, above].

The term *Reduced Generating Function* being apt to lead to the erroneous impression that it is obtained by reducing the representative one, whereas the representative is in fact obtained from the reduced G.F. by multiplication of its numerator and denominator by a common factor, it may be well to explain that I use the appellation *reduced* with reference to the *crude form of the generating function*, the former representing that branch, or the totality of those branches, in the development of the crude form which contain no negative powers of x.

I add a few words respecting differentiants which are simply such symmetrical functions of the roots as are complete functions of the differences of the roots of the form or system of forms to which the several tables refer.

In the G.F. for differentiants for a single quantic, the coefficient of a^j represents the total number of linearly independent differentiants of the degree j belonging to a quantic of the order i; that is, the total number of covariants of the degree j in the coefficients and of *all* orders in the variables, belonging to that quantic. The G.F. for differentiants can therefore be obtained from the G.F. for covariants (although not in its simplest form) by putting $x=1$ in the latter. In like manner, for a system of quantics, the

denominator. It is an erroneous and misleading expression into which invariantists (myself included) have fallen of speaking of a definite number, say ν, of groundforms of a certain type. The true idea is that of an unique form of that type with ν parameters. It is, so to say, a single *form* of the νth degree of plasticity or deformability or of ν dimensions in the sense in which we speak of the dimensions of space. I mean that an elastic string, an india-rubber disk and an india-rubber ball may be regarded as symbols of a groundform with one, two and three parameters respectively.

[* p. 63, above.] [† pp. 132, 136, above.] [‡ p. 258, above.]

G. F. for differentiants (or to speak more precisely, its algebraical equivalent) can be obtained from the G. F. for covariants by putting $x=1$.

To obtain the G. F. for differentiants for a single form without previously having the G. F. for covariants, we may make use of the fact that the sum of the quantities

$$(w:i,j)-(w-1:i,j)^*$$

for *all* admissible values of w is equal to the value of $(w:i,j)$ for the *highest* admissible value of w. Now the *order* corresponding to the highest weight is 0 or 1†; hence the number of differentiants of the degree j belonging to a quantic of the order i is the coefficient of a^jx^0 or of a^jx^1 (according as ij is even or odd) in the development of

$$\frac{1}{(1-ax^i)(1-ax^{i-2})\dots(1-ax^{-i+2})(1-ax^{-i})}.$$

The generating function for differentiants is therefore the sum of the multipliers of x^0 and x^1 in the development of the above fraction. (When the quantic is of even order, x^1 does not appear in the development, and the G. F. for differentiants is simply the part independent of x in the development.)

In like manner, for a system of two quantics, the G. F. for differentiants is the sum of the multipliers of x^0 and x^1 in the development of

$$\frac{1}{(1-ax^i)(1-ax^{i-2})\dots(1-ax^{-i})(1-ax^{i'})(1-ax^{i'-2})\dots(1-ax^{-i'})}.$$

And we may proceed in an analogous manner when a system of forms is in question. I need hardly add that a differentiant in respect to either variable, say x, is only another name for any rational integral function of the coefficients of a quantic which, when the coefficient of the highest power of the selected variable (x) in the quantic is made equal to unity, becomes a function of the differences of its $\frac{x}{y}$ roots. Gordan's and Jordan's results concerning symbolical determinants are correlative and coextensive with theorems concerning root-differences, so that the method of differentiants when fully developed would lead to the substitution of actual differences or determinants for symbolical determinants in the Gordan theory, it being borne in mind that to determine the ground-covariants of a quantic or quantic system is the same question as that of determining its ground-differentiants, inasmuch as to every covariant corresponds a single differentiant, and *vice versâ*.

* w is the weight of any covariant, j its degree in the coefficients and i the order of the quantic in the variables; and $(w:i,j)$ denotes the number of modes of composing w with j of the elements 0, 1, 2, 3, ... i or *vice versâ* with i of the elements 0, 1, 2, 3, ... j each any number of times repeated.

† If e is the order of the covariant in the variables $2w=ij-e$.