28.

ON A RULE FOR ABBREVIATING THE CALCULATION OF THE NUMBER OF IN- OR CO-VARIANTS OF A GIVEN ORDER AND WEIGHT IN THE COEFFICIENTS OF A BINARY QUANTIC OF A GIVEN DEGREE.

[*Messenger of Mathematics*, VIII. (1879), pp. 1—8.]

IF i is the degree of a quantic we know now by *apodictic* reasoning that the number of its in- or co-variants of order j and of weight w in the coefficients is $(w: i, j) - \{(w-1): i, j\}$, where in general $(x: i, j)$ denotes the number of modes of composing x with j numbers each having any value from 0 to i (both inclusive) or (what is the same thing) with i numbers each having any value from 0 to j. The object of this note is to show how to calculate the *difference* between the two denumerants above given without calculating each of them separately, whereby the actual amount of calculation required will be reduced to a small fraction of what it would otherwise be. I shall not stop to draw theoretical consequences from this theorem, but present it to the readers of the *Messenger* in the way it has occurred to me as a rule for abbreviating labour.

It is founded on the exhaustive method of representing partition systems by following a dictionary order of sequence, and it will be best understood by beginning with an example.

Suppose then that $w = 7$, $i = 5$, $j = 4$, we may find $(7: 5, 4)$ by setting out and counting the arrangements where 4 is the number of parts and 5 the limit to each part, namely, 5 . 2, 5 . 1 . 1, 4 . 3, 4 . 2 . 1, 4 . 1 . 1 . 1, 3 . 3 . 1, 3 . 2 . 2, 3 . 2 . 1 . 1, 2 . 2 . 2 . 1.

For brevity the zeros required to fill up the number of parts to 4 are omitted in this table.

To find (6 : 5, 4) we may consider

(1) Those arrangements which begin with 5.

(2) Those arrangements which begin with a number less than 5.

To obtain the latter also arranged in dictionary order of sequence, we may (subject to an exception to be stated immediately below) proceed by diminishing each initial number in the above table by unity.

The exception to be made is where 2 initial numbers are alike, as in 3 . 3 . 1; 2 . 2 . 2 . 1. These arrangements must not be counted in, as the arrangements 2 . 3 . 1; 1 . 2 . 2 . 1 will already have been obtained from 4 . 2 . 1; 3 . 2 . 1 . 1 respectively.

Hence the number of arrangements in the above table to be preserved is less by 2 than the total number.

On the other hand we shall have the arrangement 5 . 1, to which there is nothing corresponding in the table for (7 : 5, 4). Hence the difference required is

$$2-1, \text{ that is, } (7:\ 5, 4)-(6:\ 5, 4)=1.$$

Let us take as a second example w (the weight) 12, i (the limit to each part) 6, and j (the number of parts) 4.

Let A be the table for (12 : 6, 4) in dictionary order, and let A' be the part of the table for (11 : 6, 4), also arranged in dictionary order, for which 6 is nowhere the initial term. Let A_1 be what A becomes when each initial number is diminished by unity.

Then, by the same reasoning as above, we must have $A'-A_1=6\,.\,6$, 5 . 5 . 2, 5 . 5 . 1 . 1, 4 . 4 . 4, 4 . 4 . 3 . 1, 4 . 4 . 2 . 2, 3 . 3 . 3 . 3, 7 in number.

Also calling B the part of the table for (11 : 6, 4), beginning with 6 we have $B=6\,.\,5$, 6 . 4 . 1, 6 . 3 . 2, 6 . 3 . 1 . 1, 6 . 2 . 2 . 1, 5 in number.

Hence (12 : 6, 4) – (11 : 6, 4) = 7 – 5, = 2.

To verify this, let us interchange the values 6 and 4, this by a well-known theorem leaves the value of each denumerant unaltered.

We have now $A'-A_1=4\,.\,4\,.\,4$, 4 . 4 . 3 . 1, 4 . 4 . 2 . 2, 4 . 4 . 2 . 1 . 1, 4 . 4 . 1 . 1 . 1 . 1, 3 . 3 . 3 . 3, 3 . 3 . 3 . 2 . 1, 3.3.3.1.1.1, 3.3.2.2.2, 3.3.2.2.1.1, 2 . 2 . 2 . 2 . 2 . 2, number is 11.

Also $B=4\,.\,4\,.\,3$, 4 . 4 . 2 . 1, 4 . 4 . 1 . 1 . 1, 4.3.3.1, 4.3.2.2, 4.3.2.1.1, 4.3.1.1.1.1, 4.2.2.2.1, 4.2.2.1.1.1, number is 9, and thus

$$(12:\ 4, 6)-(11:\ 4, 6)=11-9=2$$

as before. Evidently this identity between the two forms of

$$(w:\ i, 5)-\{(w-1):\ i, 5\},$$

given by this method, and also the incapability of this difference becoming negative when w is not greater than $\frac{1}{2}ij$, which I have elsewhere demonstrated, may be made to yield arithmetical properties of a new kind, and not unlikely to prove very valuable in certain parts of the theory of numbers; but what has impressed itself on my mind is the *enormous saving* of labour in the actual business of calculating invariantive formulæ, which this method confers. The existence of a perfectly definite table exhibiting an exhaustive arrangement of *ruled partitions* (as I call partitions subject to the two indices i, j) in itself constitutes a theorem (however simple), and the method above given is a further and more recondite theorem deduced from it, combined of course with other *intuitional* propositions.

Let us take as another example $w = 20$, $i = 13$, $j = 3$.

Here $A' - A_1 = 10.10,\ 9.9.2,\ 8.8.4,\ 7.7.6.$ $B = 13.6,\ 13.5.1,\ 13.4.2,\ 13.3.3.$ Therefore $(20:13, 3) - (19:13, 3) = 0$.

Again let us calculate $(40:20, 4) - (39:20, 4)$.

Here $A' - A_1 = 20.20,\ 19.19.2,\ 19.19.1.1,\ 18.18.4,\ 18.18.3.1,\ 18.18.2.2,\ 17.17.6,\ 17.17.5.1,\ 17.17.4.2,\ 17.17.3.3$, and similarly 16.16 with 5 duads, 15.15 with 6 duads, 14.14 with 7 duads. Also 13.13 with $13.1,\ 12.2,\ 11.3,\ 10.4,\ 9.5,\ 8.6,\ 7.7$, 12.12 with $12.4,\ 11.5,\ 10.6,\ 9.7,\ 8.8$, 11.11 with $11.7,\ 10.8,\ 9.9$, $10.10.10.10$. Thus the number of terms in $A' - A_1$ is

$$(1+2+3+4+5+6+7) + (7+5+3+1) = 44.$$

And B is composed of arrangements containing 20, together with the number of triads into which $39 - 20$, that is, 19 can be decomposed, none greater than 20, that is, the number of terms in B is $19:20, 3$, which is the same as the absolute number of modes of resolving 19 into 3 parts or fewer, which is

$$1+1+2+2+3+3+4+4+(10:9, 2)+(9:10, 2)$$
$$+(8:11, 2)+(7:12, 2) = 25+5+5+3+2 = 40.$$

Thus
$$(40:20, 4) - (39:20, 4) = 44 - 40 = 4,$$

which is easily verified, for the difference between the above two denumerants is the number of linearly independent invariants of the 20th order to a quartic, that is, is the number of ways of composing 20 with 2 and 3 (the orders of the fundamental invariants) which is 4 as found above.

The method thus simply and almost intuitively deduced, may be expressed in the form of a theorem as follows:

$$\Sigma_{q=0}^{q=i}(w-2q:q, j-2) - (w-i-1:i, j-1) = (w:i, j) - (w-1:i, j)$$
$$= \Sigma_{q=0}^{q=j}(w-2q:q, i-2) - (w-j-1:j, i-1).$$

The inferior unit is taken zero for the purpose of theoretical simplicity. Let the effective value of this limit be called $[q]$, and consider the first of the above three equals.

The value of $[q]$ is given by the condition that

$$w - 2[q] \text{ shall be not greater than } (j-2)[q],$$

that is, $$[q] \text{ not less than } \frac{w}{j},$$

that is, $$[q] \text{ is } \frac{w}{j} \text{ if } \frac{w}{j} \text{ is an integer, } \frac{w}{j}+1 \text{ if } \frac{w}{j} \text{ is fractional,}$$

that is, $$[q] = E\,\frac{w+j-1}{j};$$

(E standing as usual for the integer part of the quantity which it precedes).

The number of actual terms differing from zero under the sign of summation is therefore

$$i+1-E\,\frac{w+j-1}{j}, \text{ that is } 1+E\,\frac{ij-w}{j},$$

similarly the number of terms under the sign of summation in the conjugate form will be $1+E\,\dfrac{ij-w}{i}$.

Thus the first or second expression will be the best to employ, according as j is greater or less than i.

Again, since $$(w:i,j) = (ij-w:i,j),$$

we may in place of

$$(w:i,j)-(w-1:i,j),$$

employ $$(w':i,j)-(w'+1:i,j),$$

which is $$-[(w'+1:i,j)-(w':i,j)].$$

Hence, we may always secure in the application of this method, that the numerator in $E\,\dfrac{ij-w}{i}$ or in $E\,\dfrac{ij-w}{j}$ shall not be greater than $\frac{1}{2}ij$. Supposing j to be greater or not less than i, so that the first formula is applied, it will be found most convenient, so long as q is less or not greater than $j-2$, to consider q the number of the parts in any of the quantities

$$(w-2q:q, j-2),$$

and $j-2$ the limit to the magnitude of each part, and until q becomes equal to $i-1$, this hypothesis will always be the case. When $q=i$ or when $q=i$ and $q=i-1$ in the respective cases of j being only one unit greater than i or equal to i, the two indices $q:j-2$ may with advantage be reversed. For any other values of $j-i$, the order of the indices need not be disturbed. It may be worth while to call attention to the *two* independent theorems of reciprocity made use of in the preceding discussion, indicated by the equations

$$\begin{aligned}&(w : i, j)\\ &= (w : j, i)\\ &= (ij - w : i, j)\\ &= (ij - w : j, i),\end{aligned}$$

both of them of importance in the theory of invariants after the English method.

Addition.

Notwithstanding what has been stated above as to the choice between the two formulæ representing $\Delta(w : i, j)$, the advantage of diminishing the smaller of the two indices i, j, will simplify the calculations to a degree that far more than outweighs the disadvantage of increasing the number of terms under the sign of summation. Let us suppose then that j is less than w, and that $\Delta(w : i, j)$ is positive, representing in fact indifferently the number of linearly independent covariants of order i to a quantic of degree j, or of order j to a quantic of degree i. Then, unless these covariants are invariants, we must have $w < \frac{1}{2}ij$.

Consequently, the best formula to apply in such case will be obtained by writing

$$\begin{aligned}\Delta(w : i, j) &= (ij - w : i, j) - (ij - w + 1 : i, j)\\ &= -\Delta(ij - w + 1 : i, j)\\ &= (ij - i - w : i, j - 1) - \Sigma_{q=0}^{q=i}(ij - w + 1 - 2q : q, j - 2).\end{aligned}$$

The number of terms other than zero under the sign of summation will then be $1 + E\,\dfrac{w}{j}$.

For the case of invariants we may with at least equal advantage use the formula

$$\Sigma_{q=0}^{q=i}(\tfrac{1}{2}ij - 2q : q, j - 2) - (\tfrac{1}{2}ij - 1 - i : i, j - 1).$$

Let us apply this to the case of finding

$$\Delta\left(\frac{18 \,.\, 5}{2} : 18, 5\right), \text{ that is } (45 : 18, 5).$$

In the work below I use, whenever useful, the formula of transformation

$$(x : i, j) = (ij - x : i, j),$$

and employ $\dfrac{\mu}{3}$ to denote the number of ways of breaking up μ into three or fewer parts, which we know is the nearest integer to $\frac{(\mu+3)^2}{12}$; and in like manner $\frac{\nu}{2}$ for the number of ways of breaking up ν into two parts: also in place of $(x: k, 3)$, whenever k is at least as great as x, I use the obviously equivalent value $\frac{x}{3}$.

Let us then first calculate

$$\Sigma_{q=9}^{q=18}\,[45-2q:q,\,3], \text{ say } S.$$

The values of q inferior to 9 will give quantities in which $3q < 45 - 2q$, and which will therefore be zero.

We have thus

$$\begin{aligned} S &= (9:18,3)+(11:17,3)+(13:16,3)+(15:15,3) \\ &\qquad +(17:14,3)+(19:13,3)+(21:12,3)+(23:11,3) \\ &\qquad +(25:10,3)+(27:9,3) \\ &= \tfrac{9}{3}+\tfrac{11}{3}+\tfrac{13}{3}+\tfrac{15}{3}+(17:14,3)+(19:13,3)+(15:12,3) \\ &\qquad +(10:11,3)+(5:10,3)+(0:9,3). \end{aligned}$$

Also

$$\begin{aligned} (17:14,3) &= (17:17,3)-\tfrac{1}{2}-\tfrac{2}{2}-\tfrac{3}{2} = \tfrac{17}{3}-1-2-2 = \tfrac{17}{3}-5, \\ (19:13,3) &= (19:19,3)-\tfrac{1}{2}-\tfrac{2}{2}-\tfrac{3}{2}-\tfrac{4}{2}-\tfrac{5}{2}-\tfrac{6}{2} = (19:19,3)-15, \\ (15:12,3) &= (15:15,3)-\tfrac{1}{2}-\tfrac{2}{2}-\tfrac{3}{2} = \tfrac{15}{3}-5. \end{aligned}$$

Thus

$$\begin{aligned} S &= \tfrac{9}{3}+\tfrac{11}{3}+\tfrac{13}{3}+\tfrac{15}{3}+\tfrac{17}{3}+\tfrac{19}{3}+\tfrac{15}{3}+\tfrac{10}{3}+\tfrac{5}{3}+\tfrac{0}{3}-25 \\ &= \tfrac{0}{3}+\tfrac{5}{3}+\tfrac{9}{3}+\tfrac{10}{3}+\tfrac{11}{3}+\tfrac{13}{3}+2\,.\,\tfrac{15}{3}+\tfrac{17}{3}+\tfrac{19}{3}-25. \end{aligned}$$

Again let $S' = (44-18:18,4) = (26:18,4)$.

Then

$$\begin{aligned} S' &= (8:18,3)+(9:17,3)+(10:16,3)+(11:15,3) \\ &\qquad +(12:14,3)+(13:13,3)+(14:12,3)+(15:11,3) \\ &\qquad +(16:10,3)+(17:9,3)+(18:8,3)+(19:7,3) \\ &= \tfrac{8}{3}+\tfrac{9}{3}+\tfrac{10}{3}+\tfrac{11}{3}+\tfrac{12}{3}+\tfrac{13}{3}+(\tfrac{14}{3}-3)+(\tfrac{15}{3}-8) \\ &\qquad +(\tfrac{14}{3}-8)+(\tfrac{10}{3}-1)+\tfrac{6}{3}+\tfrac{2}{3}-20 \\ &= \tfrac{2}{3}+\tfrac{6}{3}+\tfrac{8}{3}+\tfrac{9}{3}+2\,.\,\tfrac{10}{3}+\tfrac{11}{3}+\tfrac{12}{3}+\tfrac{13}{3}+2\,.\,\tfrac{14}{3}+\tfrac{15}{3}-20; \end{aligned}$$

therefore

$$\begin{aligned} S-S' &= \tfrac{0}{3}-\tfrac{2}{3}+\tfrac{5}{3}-\tfrac{6}{3}-\tfrac{8}{3}-\tfrac{10}{3}-\tfrac{12}{3}-2\,.\,\tfrac{14}{3}+\tfrac{15}{3}+\tfrac{17}{3}+\tfrac{19}{3}-5 \\ &= 1-2+5-7-10-14-19-48+27+33+40-5 \\ &= 106-105 = 1, \end{aligned}$$

which is right, there being just one invariant to the quantic of the eighteenth order in the coefficients, so that $\Delta(45:18,5)=1$.

It appears from the tables given in M. Faà de Bruno's valuable *Théorie des Formes Binaires*, Turin, 1877, that this invariant contains 848 terms. Therefore the value of (18 : 8, 5) is very considerably greater* than 848.

Thus, by the direct method of calculating $\Delta(45:18, 5)$, many more than 1695 terms would have required setting out.

There is one case which deserves special consideration, namely, when one of the indices i or j becomes infinite.

The function $\Delta(w:\mu, \infty)$ then represents the total number of in- and co-variants of weight w of any given order not less than w to a quantic of the μth degree.

The two formulæ for this case become respectively

$$\Sigma_{q=0}^{q=\infty}\,[w-2q : q, \mu],$$

and

$$\Sigma_{q=0}^{q=\mu}\,[w-2q : q, \infty] - [w-\mu-1 : \mu, \infty],$$

or if we agree to understand in all cases by $\frac{n}{m}$ the number of ways of making up n with the integers $0, 1, 2, 3 \ldots m$, or, what is the same, the number of ways of breaking up n into m or fewer parts, the second formula becomes

$$\Sigma_{q=0}^{q=\mu}\,\frac{w-2q}{q} - \frac{w-\mu-1}{\mu};$$

of these two the first is by far the most expeditious.

Let us take as an example $\Delta(20:6, \infty)$, that is $\frac{20}{6} - \frac{19}{6}$.

The first formula {neglecting the values of q which make $w-2q$ negative and those which make $4q < (w-2q)$}, will give for the value of Δ

$$\begin{array}{lcl} (0:10, 4) & = & (0) \\ +(2:9, 4) & & +(2) \\ +(4:8, 4) & & +(4) \\ +(6:7, 4) & & +\frac{6}{4} \\ +(8:6, 4) & & +(8:6, 4) \\ +(10:5, 4) & & +(10:5, 4) \\ +(12:4, 4) & & +(4:4, 4)\text{, that is } (4), \end{array}$$

* I say very considerably greater than, because only a certain number of the terms which satisfy the required conditions of order and weight actually appear in the octodecimal invariant in question. Thus, for example, there is no f^9, no f^8, and of the (10 : 11, 5) that is $\frac{10}{5}$ terms which might contain f^7, only six, namely the terms contained in $a\,(ac-b^2)^5$ actually make their appearance in it.

where in general (m) means *all* the modes of breaking up m into parts. The value of $(10:5,\ 4)$ will be easily found to be 9, of $(8:6,\ 4)$ 12 and of $\frac{6}{4}$, 9, also of (4) is 5. The value of $\frac{20}{6}-\frac{19}{6}$ thus becomes

$$1+2+5+9+12+9+5=43.$$

By the second formula the value of the same quantity would be

$$\tfrac{8}{6}+\tfrac{10}{5}+\tfrac{12}{4}+\tfrac{14}{3}+\tfrac{16}{2}+\tfrac{18}{1}-\tfrac{13}{5},$$

which would be exceedingly tedious to calculate.

In like manner if w is odd we shall have a series of denumerants of the form

$$\left(1:\frac{w-1}{2},\ \mu\right),\quad \left(3:\frac{w-3}{2},\ \mu\right),\quad \left(5:\frac{w-5}{2},\ \mu\right),\ \text{\&c.}$$

Thus, for example, $\frac{11}{6}-\frac{10}{6}$ (that is, the number of in- and co-variants to a sextic of weight 11 and of any given order not inferior to 11, or, if we please to vary the expression, the number of in- and co-variants of weight 11 and the sixth order to any quantic of a degree not inferior to 11) will be

$$\left.\begin{array}{r}(1:5,\ 4)\\+(3:4,\ 4)\\+(5:3,\ 4)\\+(7:2,\ 4)\end{array}\right\}=\left\{\begin{array}{l}(1)\\+(3)\\+(5:3,\ 4)\\+(1:2,\ 4)\ \text{that is }(1)\end{array}\right.$$

$$=1+3+4+1=9.$$