CHAPTER XVII.

RECTIFICATION (II.).

Central Conic, Limaçon, Lemniscate, Trochoids, etc. Application of Elliptic Functions.

566. We have reserved for a separate chapter the consideration of those curves whose rectification needs the employment of Elliptic Integrals.

567. Rectification of the Ellipse. Arc measured from the End of the Minor Axis.

If θ be the eccentric angle of a point x, y on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

we have $\qquad x=a\cos\theta, \qquad y=b\sin\theta,$

$$dx=-a\sin\theta\,d\theta, \quad dy=b\cos\theta\,d\theta.$$

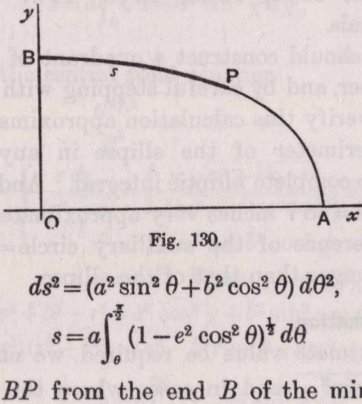

Fig. 130.

Hence $\qquad ds^2=(a^2\sin^2\theta+b^2\cos^2\theta)\,d\theta^2,$

and $\qquad s=a\int_0^{\frac{\pi}{2}}(1-e^2\cos^2\theta)^{\frac{1}{2}}\,d\theta$

gives the arc BP from the end B of the minor axis to any point P on the curve.

Putting $\theta=\frac{\pi}{2}-\chi$,

$$s=a\int_0^\chi \sqrt{1-e^2\sin^2\chi}\,d\chi=aE(\chi, e).$$

(See Chapter XI.)

568. This integral is Legendre's elliptic integral of the second kind, and is not expressible in terms of the ordinary circular or inverse circular functions. But its value can be found for specific values of e and χ from the tables calculated for the function E. Thus, for instance, the tables for E corresponding to $e=\frac{1}{2}$ give

$$\left.\begin{array}{l} E(10^\circ)=\cdot17431\\ E(20^\circ)=\cdot34733\\ E(30^\circ)=\cdot51788\\ E(40^\circ)=\cdot68506\\ E(50^\circ)=\cdot84832\\ E(60^\circ)=1\cdot00756\\ E(70^\circ)=1\cdot16318\\ E(80^\circ)=1\cdot31606\\ E(90^\circ)=1\cdot46746 \end{array}\right\}$$ Values extracted from tables given in Bertrand, *Calc. Intég.*, p. 717.

Hence, taking an ellipse with a 20-inch major axis and eccentricity $\frac{1}{2}$, the arcs for eccentric angles 80°, 70°, 60°, ... 0°, measured from B, the end of the minor axis, are: 1·74, 3·47, 5·18, 6·85, 8·48, 10·08, 11·63, 13·16, 14·67 inches to two places of decimals.

The student should construct a quadrant of such an ellipse on squared paper, and by careful stepping with dividers round the perimeter verify this calculation approximately.

The total perimeter of the ellipse in any case is $4aE_1$, where E_1 is the complete elliptic integral. And in the present case $4\times 14\cdot6746=58\cdot7$ inches very approximately.

The circumference of the auxiliary circle $=20\pi=62\cdot8318$, *i.e.* 4·1 inches longer than that of the ellipse.

569. **Approximation.**

If an approximate value be required, we may expand the radical $\sqrt{1-e^2\sin^2\chi}$, and in cases where the eccentricity is small the series is rapidly convergent.

We then have

$$s=a\int_0^\chi\left(1-\frac{1}{2}e^2\sin^2\chi-\frac{1}{2}\cdot\frac{1}{4}e^4\sin^4\chi-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}e^6\sin^6\chi-\dots\right)d\chi.$$

For a quadrant the limits are 0 and $\frac{\pi}{2}$, and the arc of the quadrant

$$=a\left(\frac{\pi}{2}-\frac{1}{2}e^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{2}\cdot\frac{1}{4}e^4\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}e^6\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-\dots\right)$$

$$=\frac{\pi a}{2}\left(1-\frac{1}{2^2}e^2-\frac{1^2.3}{2^2.4^2}e^4-\frac{1^2.3^2.5}{2^2.4^2.6^2}e^6-\dots\text{ to }\infty\right).$$

The first three terms give for the above ellipse a perimeter of 58·7 approximately.

570. **Other modes of procedure** may be adopted.

Cartesians.

Keeping x for the independent variable, we have

$$\frac{dy}{dx}=-\frac{b^2}{a^2}\frac{x}{y};$$

$$\therefore\ \left(\frac{ds}{dx}\right)^2=1+\frac{b^2}{a^2}\frac{x^2}{a^2-x^2}=1+(1-e^2)\frac{x^2}{a^2-x^2}=\frac{a^2-e^2x^2}{a^2-x^2}.$$

Hence $$s=\int_0^x\sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}\,dx.$$

If we now put $x=a\sin\chi$, where χ is, as before, the complement of the eccentric angle, this reduces at once to

$$s=a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi,$$

as before.

571. Taking the **central pedal equation**

$$\frac{a^2b^2}{p^2}=a^2+b^2-r^2,$$

we get $$s=\int\frac{r\,dr}{\sqrt{r^2-p^2}}=\int\frac{r\sqrt{a^2+b^2-r^2}}{\sqrt{(a^2-r^2)(r^2-b^2)}}\,dr.$$

Putting
$$r^2=a^2\sin^2\chi+b^2\cos^2\chi,$$
$$r\,dr=(a^2-b^2)\sin\chi\cos\chi\,d\chi,$$
$$a^2+b^2-r^2=a^2\cos^2\chi+b^2\sin^2\chi=a^2(1-e^2\sin^2\chi),$$
and
$$(a^2-r^2)(r^2-b^2)=(a^2-b^2)^2\sin^2\chi\cos^2\chi;$$

$$\therefore\ s=a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi.$$

572. Taking the **focal** p-r **equation**

$$\frac{b^2}{p^2}=\frac{2a}{r}-1,$$

$$s=\int\frac{r\,dr}{\sqrt{r^2-\frac{b^2r}{2a-r}}}=\int\frac{r\sqrt{2a-r}\,dr}{\sqrt{2ar^2-b^2r-r^3}}=\int\frac{\sqrt{2ar-r^2}}{\sqrt{2ar-b^2-r^2}}dr.$$

Putting $r=a(1+e\sin\chi)$ this reduces at once to

$$s=a\int_0^{\chi}\sqrt{1-e^2\sin^2\chi}\,d\chi,$$

as before.

573. It appears then that $aE(\chi, e)$, *i.e.*

$$a\int_0^{\chi}\sqrt{1-e^2\sin^2\chi}\,d\chi,$$

represents the length of the arc of an ellipse measured from the end of the minor axis to a point, on the curve, whose eccentric angle is $\frac{\pi}{2}-\chi$, the semi-major axis being a and the eccentricity e. (See Art. 567.)

This may be written as

$$\int_0^{\chi}\sqrt{a^2\cos^2\chi+b^2\sin^2\chi}\,d\chi,$$

or as

$$\int_0^{\chi}\sqrt{l^2+2lm\cos 2\chi+m^2}\,d\chi,$$

where $l+m=a$ and $l-m=b$. And it is useful to be able to recognise these forms at once, when they appear, as representing an arc of an ellipse. They occur in many other rectifications.

574. **March of the Second Elliptic Function.**

The form $$s=a\int_0^{\chi}\sqrt{1-e^2\sin^2\chi}\,d\chi$$

for an ellipse gives a very clear idea of the "march" of the "second elliptic function" corresponding to any given modulus e, and it is easy to construct a graph of the relation between χ and s by measuring off ordinates equal to the arc of the ellipse and abscissae proportional to the complement of the eccentric angle.

Taking $a=1$, the figure (Fig. 131) shows the march of the function for the values $e=0$, which gives a straight line, viz. $s=\chi$;

$$e=\tfrac{1}{2}, \text{ which gives } s=\int_0^{\chi}\sqrt{1-\tfrac{1}{4}\sin^2\chi}\,d\chi=E(\chi, \tfrac{1}{2}),$$

and $e=1$, which gives $s=\sin\chi$, the curve of sines.

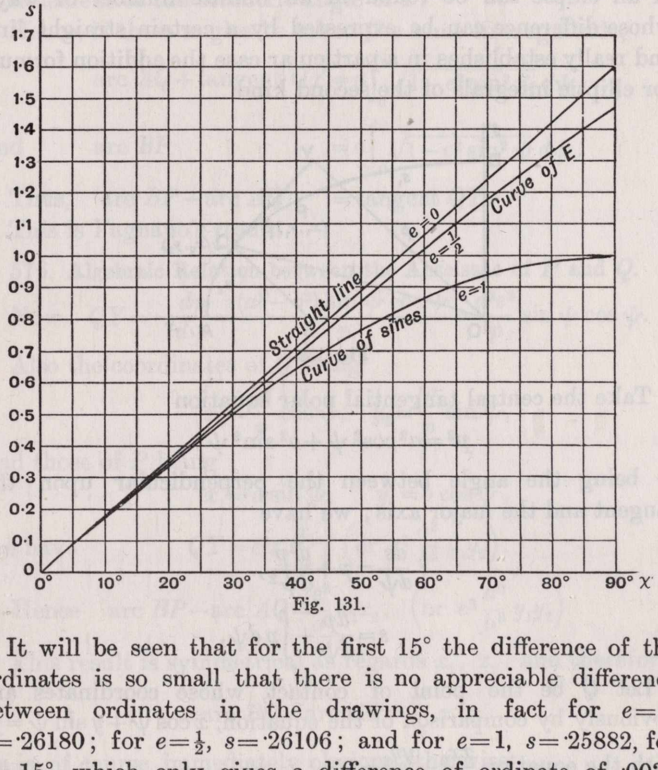

Fig. 131.

It will be seen that for the first 15° the difference of the ordinates is so small that there is no appreciable difference between ordinates in the drawings, in fact for $e=0$, $s=\cdot 26180$; for $e=\tfrac{1}{2}$, $s=\cdot 26106$; and for $e=1$, $s=\cdot 25882$, for $\chi=15°$, which only gives a difference of ordinate of ·0030 between the greatest and least, and the curve $s=E(\chi)$ lies between these extremes. There is much more rapid deviation of $s=E\left(\chi, \sin\frac{\pi}{6}\right)$ from the curve $s=\sin\chi$ after $\chi=\frac{\pi}{4}$.

575. **Arc measured from the End of the** MAJOR AXIS. FAGNANO'S THEOREM.

Another method of proceeding gives the length of the arc AQ measured from the end of the *major axis*, and incidentally a comparison of the two methods establishes a remarkable result with regard to the difference of two arcs, one measured from A, the other from B. This theorem is known as Fagnano's theorem, being discovered by Giulio, Count de Fagnano (1682-1760).* It shows that two arcs of an ellipse can be found in an infinite number of ways, whose difference can be expressed by a certain straight line, and really establishes in a particular case the addition formula for elliptic integrals of the second kind.

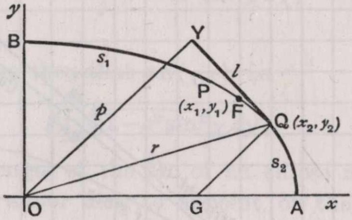

Fig. 132.

Take the central tangential polar equation

$$p^2 = a^2 \cos^2 \psi + b^2 \sin^2 \psi,$$

ψ being the angle between the perpendicular upon the tangent and the major axis; we have

$$\frac{ds}{d\psi} = p + \frac{d^2p}{d\psi^2},$$

i.e.

$$s = \frac{dp}{d\psi} + \int p \, d\psi.$$

Let Q be the point of contact, whose coordinates are obviously by comparison of the equation, $x \cos \psi + y \sin \psi = p$, with the equation $\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1$,

$$x_2 = \frac{a^2 \cos \psi}{p}, \quad y_2 = \frac{b^2 \sin \psi}{p}.$$

Also $\frac{dp}{d\psi} = -QY$, the negative sign occurring, because in this case Y is on the "forward drawn" tangent from Q, and p is diminishing as ψ is increasing.

* Cajori, *History of Mathematics*, p. 241.

Also

$$\int p\,d\psi = \int \sqrt{a^2\cos^2\psi + b^2\sin^2\psi}\,d\psi = a\int_0^{\psi}\sqrt{1-e^2\sin^2\psi}\,d\psi,$$

which is the same integral as obtained in Art. 567 for the arc BP, ψ being in that case a different angle, viz. the complement of the eccentric angle of P.

Hence, if these angles be taken the same in magnitude,

$$\text{arc } AQ + \text{tangent } QY = a\int_0^{\psi}\sqrt{1-e^2\sin^2\psi}\,d\psi,$$

and $$\text{arc } BP = a\int_0^{\psi}\sqrt{1-e^2\sin^2\psi}\,d\psi.$$

Thus, $\text{arc } BP - \text{arc } AQ = \text{tangent } QY.$

This is Fagnano's result.

576. **Algebraic Relation between the Abscissae of P and Q.**

Now $$QY = -\frac{dp}{d\psi} = \frac{(a^2-b^2)\sin\psi\cos\psi}{p} = \frac{a^2e^2}{p}\sin\psi\cos\psi.$$

Also the coordinates of Q being

$$x_2 = \frac{a^2}{p}\cos\psi, \quad y_2 = \frac{b^2}{p}\sin\psi,$$

and those of P being

$$x_1 = a\sin\psi, \qquad y_1 = b\cos\psi,$$

we have $$QY = e^2 x_2\frac{x_1}{a}, \left(\text{or } e^2\frac{a^2}{b^3}y_1y_2\right).$$

Hence $$\text{arc } BP - \text{arc } AQ = \frac{e^2}{a}x_1x_2, \quad \left(\text{or } e^2\frac{a^2}{b^3}y_1y_2\right)$$

This result is symmetrical as regards x_1, x_2, and therefore

$$\text{arc } BQ - \text{arc } AP = \frac{e^2}{a}x_1x_2,$$

as is, of course, immediately obvious otherwise.

Also $\frac{e^2}{a}x_1x_2 = $ tangent PY', if OY' be the perpendicular on the tangent at P from O. Hence $QY = PY'$.

Again, $$(a^2-x_1{}^2)(a^2-x_2{}^2) = (a^2-a^2\sin^2\psi)\left(a^2 - \frac{a^4\cos^2\psi}{p^2}\right)$$

$$= \frac{a^4\cos^2\psi}{p^2}b^2\sin^2\psi = (1-e^2)x_1{}^2x_2{}^2;$$

$$\therefore\ e^2x_1{}^2x_2{}^2 - a^2(x_1{}^2+x_2{}^2) + a^4 = 0.$$

577. The corresponding relation between y_1 and y_2 is

$$a^2e^2y_1{}^2y_2{}^2+b^4(y_1{}^2+y_2{}^2)-b^6=0,$$

that is
$$e'^2y_1{}^2y_2{}^2-b^2(y_1{}^2+y_2{}^2)+b^4=0,$$

where
$$\frac{1}{e^2}+\frac{1}{e'^2}=1,$$

e' being the "imaginary" eccentricity.

578. THE FAGNANO POINTS.

It will be noticed also that

$$\frac{x_1x_2}{a^3}=\frac{y_1y_2}{b^3}.$$

Hence, at the point F on the arc AB at which P and Q coincide when ϕ is suitably chosen,

$$\frac{x^2}{a^3}=\frac{y^2}{b^3}=\frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}}{a+b}=\frac{1}{a+b},$$

and the coordinates of the point are therefore

$$x=\sqrt{\frac{a^3}{a+b}},\quad y=\sqrt{\frac{b^3}{a+b}},$$

and this is called the "Fagnano Point,"* for the first quadrant.

579. **Properties.**

At this point F,

$$\text{arc } BF-\text{arc } AF=\frac{e^2x^2}{a}=\frac{a^2-b^2}{a^3}\cdot\frac{a^3}{a+b}=a-b$$

$=$ the difference of the semiaxes.

And the length of the projection of the radius vector OF on the tangent at F is also $=a-b$.

580. The expression for QY, viz. $\dfrac{a^2e^2\sin\psi\cos\psi}{\sqrt{a^2\cos^2\psi+b^2\sin^2\psi}}$, may be written as

$$\frac{a^2-b^2}{\sqrt{a^2\operatorname{cosec}^2\psi+b^2\sec^2\psi}},\quad i.e.\ \frac{a^2-b^2}{\sqrt{(a+b)^2+(a\cot\psi-b\tan\psi)^2}},$$

and therefore QY attains its maximum when $\tan\psi=\sqrt{\dfrac{a}{b}}$, viz. $a-b$. The Fagnano point is therefore the point for which QY has a maximum value. QY varies continuously from zero to $a-b$ in travelling from B or A to F.

* Greenhill's *Elliptic Functions*, p. 178 onward.

581. If we seek for a point Q upon the quadrantal arc AB of an ellipse such that QY, the projection of OQ upon the tangent at Q, is of given length l, where $0<l<a-b$, there will be two solutions, viz. the points P and Q, whose positions are given by the equations

$$r^2=l^2+p^2 \quad \text{and} \quad \frac{a^2b^2}{p^2}=a^2+b^2-r^2,$$

r being the radius vector to either of the required points, viz. OP or OQ.

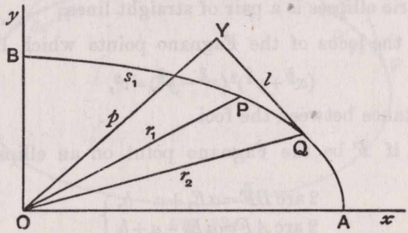

Fig. 133.

Eliminating p we have

$$(r^2-a^2-b^2)(r^2-l^2)+a^2b^2=0,$$

$$r^4-(a^2+b^2+l^2)r^2+l^2(a^2+b^2)+a^2b^2=0, \quad \text{............(1)}$$

with roots ${r_1}^2$, ${r_2}^2$, such that

$${r_1}^2+{r_2}^2=a^2+b^2+l^2, \quad \text{......................(2)}$$

and equal roots when $l=a-b$ and $r^2=a^2-ab+b^2$.

If we differentiate equation (2),

$$r_1\,dr_1+r_2\,dr_2=l\,dl.*$$

If we call BP, s_1, and BQ, s_2, and remember that

$$r\frac{dr}{ds}=\text{projection of radius vector on the tangent,}$$

viz. l in both cases,

$$ds_1+ds_2=dl,$$

i.e. $$s_1+s_2=l+C, \quad \text{......................(3)}$$

where C is a constant.

Taking the case when $r_1=b$, that is P at B, we have ${r_2}^2=a^2+l^2$, and therefore r_2 must $=a$ and $l=0$, for $r_2 \not> a$, so that Q is at A; then $s_1=0$, $s_2=\text{arc } AB$, $l=0$ simultaneously;

$$\therefore\ C=\text{arc } AB;$$

$$\therefore\ \text{arc } BP+\text{arc } BQ=l+\text{arc } BA, \quad \textit{i.e. } \text{arc } BP-\text{arc } AQ=l,$$

* See Bertrand, *Calc. Intég.*, p. 380.

which is Fagnano's result, and the points P, Q, in which the arc AP must be divided to give a definite value l for QY, are determined by equation (1).

EXAMPLES.

1. Show that if coaxial ellipses be drawn with a given centre such that the areas enclosed between them and their respective director circles is constant, the locus of the Fagnano points is a circle of the same area.

2. Show that the locus of the Fagnano points for similar and similarly situated concentric ellipses is a pair of straight lines.

3. Show that the locus of the Fagnano points which lie on confocal ellipses is

$$(x^{\frac{2}{3}}+y^{\frac{2}{3}})^2(x^{\frac{2}{3}}-y^{\frac{2}{3}})=c^2,$$

$2c$ being the distance between the foci.

4. Show that if F be the Fagnano point on an ellipse of semiaxes $OA=a$, $OB=b$,

$$\left.\begin{aligned}2\,\text{arc}\,BF&=aE_1+a-b,\\ 2\,\text{arc}\,AF&=aE_1-a+b,\end{aligned}\right\}$$

where E_1 is the complete elliptic integral of the second kind

$$\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\sin^2\phi}\,d\phi.$$

5. Show that the central perpendicular upon the tangent at a Fagnano point is a geometric mean between the semiaxes, and equal to the semidiameter conjugate to the radius to the Fagnano point. Further, that the radius of curvature at this point is also equal to the perpendicular, and that the normals at the corresponding point on the evolute pass through the centre. Finally, that the arc of the evolute is at such a point divided in the ratio

$$b^{\frac{3}{2}}:a^{\frac{3}{2}}.$$

6. Show that if a straight rod LM of length $a+b$ slides with its ends on two axes Ox, Oy at right angles and carries a point F whose distance from L and M are respectively a and b, which thus describes an ellipse, then at the instant when LM is tangential to the path of F, F is a Fagnano point on the described ellipse, and the circle on LM for diameter passes through the point on the normal at F where that normal touches the evolute.

7. Show that the tangents at the points $P(x_1, y_1)$, $Q(x_2, y_2)$ on an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which are related to each other so that $\frac{x_1x_2}{a^3}=\frac{y_1y_2}{b^3}$, intersect on a confocal hyperbola which passes through the Fagnano points.

[Many properties of these points will be found in Greenhill's *Elliptic Functions*, pages 182, 183.]

582. Properties of the Locus traced by a Pointer which pulls taut an Inextensible String passing round a given Oval.

Taking the case of any oval curve, let A be the point from which s is measured; PQ, $P'Q'$, the tangents at contiguous

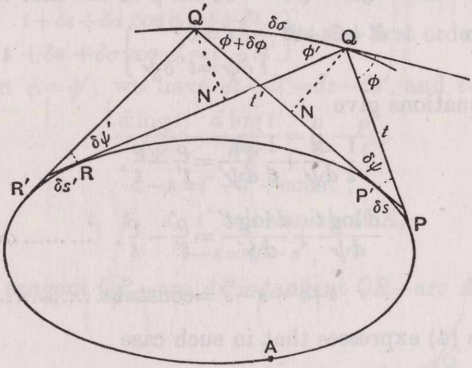

Fig. 134.

points (s, ψ) $(s+\delta s, \psi+\delta\psi)$ of the oval; and let a length $PQ=t$ be measured upon the forward drawn tangent at P, $P'Q'=t+\delta t$ upon the tangent at P'. Let the tangent to the locus of Q make an angle ϕ with the tangent at P to the oval. Draw QN perpendicular to $P'Q'$, and let the arc $QQ'=\delta\sigma$.

Then, to the first order,

$$QN=t\,\delta\psi, \quad Q'N=\delta\sigma\cos\phi,$$

and
$$t+\delta t+\delta s=t\cos\delta\psi+NQ'$$
$$=t+\delta\sigma\cos\phi;$$

$$\therefore\ \delta t+\delta s=\cos\phi\,\delta\sigma. \quad\text{.............................(1)}$$

If QR, $Q'R'$ of lengths t', $t'+\delta t'$ be the other tangents from Q, Q' which can be drawn to the oval, and s', $s'+\delta s'$ be the arcs APR, APR' respectively, and if ϕ' be the angle which QR makes with the tangent QQ' to the Q-locus and $\delta\psi'$ the difference of the angles of contingence at R, R', we have in the same way, $Q'N'$ being the perpendicular upon QR,

$$Q'N'=t'\delta\psi', \quad QN'=\delta\sigma\cos\phi',$$

$$t'+\delta s'=\delta\sigma\cos\phi'+t'+\delta t',$$

to the first order;

$$\therefore\ \delta t'-\delta s'=-\cos\phi'\,\delta\sigma. \quad\text{.........................(2)}$$

If the Q-locus be such that the tangent at Q always bisects the exterior angle between the tangents from Q to the oval,

$$\phi=\phi' \quad\text{and}\quad QN=Q'N'=\delta\sigma\sin\phi \text{ to the first order.}$$

Therefore
$$\left.\begin{aligned}\delta t+\delta s+\delta t'-\delta s'&=0,\\ t\,\delta\psi&=t'\,\delta\psi'\end{aligned}\right\}$$
and

These equations give

$$\frac{1}{t}\frac{dt}{d\psi}+\frac{1}{t'}\frac{dt'}{d\psi'}=\frac{\rho'}{t'}-\frac{\rho}{t},$$

i.e.
$$\frac{d\log t}{d\psi}+\frac{d\log t'}{d\psi'}=\frac{\rho'}{t'}-\frac{\rho}{t}, \quad\text{.....................(3)}$$

and also
$$t+t'+s-s'=\text{constant.} \quad\text{......................(4)}$$

Equation (4) expresses that in such case

$$QP+QR-\text{arc } PR=\text{constant},$$

i.e.
$$QP+QR+\text{arc } PAR=\text{constant.}$$

In this case the Q-locus is an oval traced by a pencil at Q which draws taut a loop of string placed round the original oval.

583. Dr. Graves's Theorem.

The case when the original oval is an ellipse and the Q-locus is a confocal, when the necessary property holds, viz. that the tangent to the Q-locus bisects the exterior angle between QP, QR, gives the well-known theorem due to Dr. Graves, viz.

If two tangents be drawn to an ellipse from any point of a confocal ellipse, the excess of the sum of these two tangents over the intercepted arc is constant.*

Incidentally, we have a method of drawing an ellipse confocal to a given one.

584. If the Q-locus be such that its tangent bisects the *interior* angle between the tangents QP, QR, as it would do in the case of an ellipse and a confocal hyperbola, and if we measure s and s' in opposite directions from the

* Salmon's *Conic Sections*, p. 357 ; Graves's *Translation of Chasles's Memoirs*.

point A, where the Q-locus meets the oval, we have, in the same way,

$$QN=\delta\sigma\sin\phi=t\,d\psi, \quad QN'=\delta\sigma\sin\phi'=t'\,d\psi',$$
$$NQ'=\delta\sigma\cos\phi, \quad N'Q'=\delta\sigma\cos\phi';$$

and
$$\left.\begin{aligned} t+\delta s+\delta\sigma\cos\phi&=t+\delta t,\\ t'+\delta s'+\delta\sigma\cos\phi'&=t'+\delta t',\end{aligned}\right\}\text{to the first order;}$$

and when $\phi=\phi'$, we have $dt-dt'=ds-ds'$, and $t\,d\psi=t'\,d\psi'$,

so that
$$\frac{d\log t}{d\psi}-\frac{d\log t'}{d\psi}=\frac{\rho}{t}-\frac{\rho'}{t'},$$

and also
$$t-s=t'-s'+\text{const.};$$

also, as $t,\ t',\ s,\ s'$ all vanish at A,

$$t-s=t'-s',$$

i.e. tangent QP − arc AP = tangent QR − arc AR.

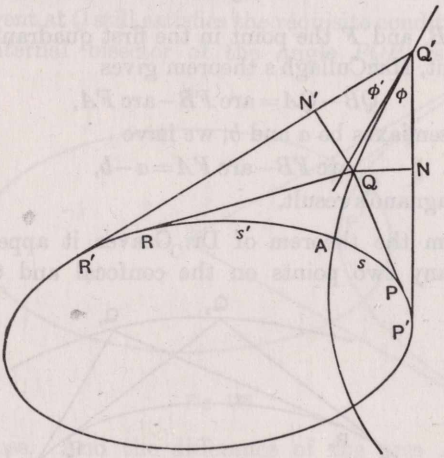

Fig. 135.

MACCULLAGH'S THEOREM.

For the case of the ellipse and the confocal hyperbola, where the condition $\phi=\phi'$ is necessarily satisfied, we have the following result.

If tangents QP, QR be drawn from a point Q on a hyperbola to a confocal ellipse cutting the hyperbola at A, the difference of the tangents is equal to the difference of the arcs AP, AR. This theorem is due to MacCullagh.*

* Salmon's *Conic Sections*, p. 358; Chasles, *Comptes Rendus*, Tom. xvii.

585. **Deductions.**

If we draw tangents to the ellipse at the extremities of the axes, the particular confocal to the ellipse which passes through the corners of the rectangle formed cuts the ellipse in the Fagnano points, and if Q be the intersection of tangents

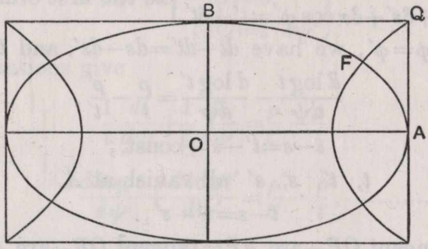

Fig. 136.

at A and B, and F the point in the first quadrant where the confocals cut, MacCullagh's theorem gives

$$QB - QA = \text{arc } FB - \text{arc } FA,$$

and if the semiaxes be a and b, we have

$$\text{arc } FB - \text{arc } FA = a - b,$$

which is Fagnano's result.

586. From the theorem of Dr. Graves it appears that if Q_1, Q_2 be any two points on the confocal and Q_1P_1, Q_1R_1;

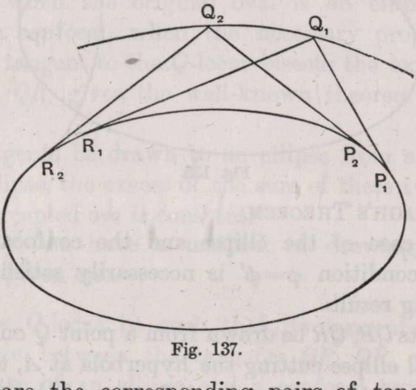

Fig. 137.

Q_2P_2, Q_2R_2 are the corresponding pairs of tangents to the original ellipse,

$$Q_1P_1 + Q_1R_1 - \text{arc } P_1R_1 = Q_2P_2 + Q_2R_2 - \text{arc } P_2R_2;$$

and therefore that the difference of the arcs P_1R_1, P_2R_2 is

$$(Q_1P_1+Q_1R_1)-(Q_2P_2+Q_2R_2)$$

and is therefore rectifiable in terms of known lines.

The particular value of the constant to which

$$QP+QR-\text{arc } PR$$

is equal may be found by taking Q at a specified point on the confocal, *e.g.* where it cuts the conjugate axis.

And a similar result follows also from MacCullagh's theorem.

587. Exactly in the same way, if Q be a point on the ellipse and QP, QP' be tangents to the same branch of the hyperbola, it will be clear that

$$QP-\text{arc } AP=QP'-\text{arc } AP',$$

for the tangent at Q still satisfies the requisite condition, namely that the internal bisector of the angle PQP' is a tangent

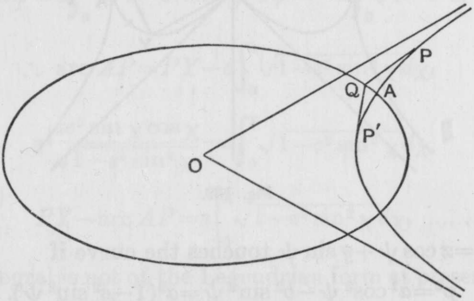

Fig. 138.

to the ellipse. And the difference of the arcs AP, AP' is therefore expressible as the difference of two straight lines and is rectifiable. Moreover, if Q_1 be another point on the ellipse, such that tangents Q_1P_1, Q_1P_1' can be drawn to the same branch of the confocal hyperbola, the difference of the arcs PP_1, $P'P_1'$ is rectifiable. In order that the point Q should be such that tangents can be drawn to the same branch of the hyperbola, such point must obviously lie in one of the regions between the asymptotes in which the hyperbola lies. In the limiting case in which QP is an asymptote, the difference of the infinite portion of the asymptote QP and the infinite arc AP is finite and equal to the difference of QP' and the arc AP', Q being now at the point of intersection of the asymptote with the ellipse.

588. Rectification of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

Let C be the centre, CA the semimajor axis, s the length of an arc AP measured from A in the first quadrant, CY the perpendicular p upon the tangent at P.

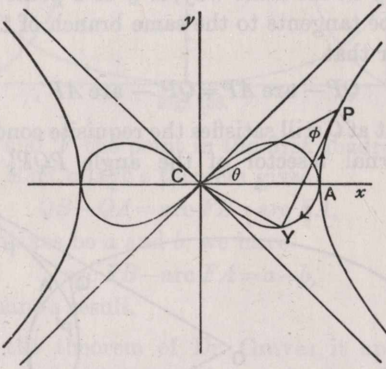

Fig. 139.

Then $p=x\cos\psi+y\sin\psi$ touches the curve if

$$p^2=a^2\cos^2\psi-b^2\sin^2\psi=a^2(1-e^2\sin^2\psi).$$

In the case of the hyperbola, when P lies in the first quadrant, ψ is the angle xCY and is negative, and as s increases from 0 to ∞ whilst P travels along the arc from A, Y travels from A towards C along the first positive pedal curve $r^2=a^2\cos^2\theta-b^2\sin^2\theta$, which becomes a Lemniscate of Bernoulli when $b=a$, *i.e.* when the hyperbola is rectangular. The angle ψ therefore remains negative, and as its actual magnitude is increasing ψ is algebraically decreasing and an increment $d\psi$ is negative. When P has travelled to ∞ along this branch of the curve the limiting position of YP is an asymptote. The tangents at the node of the pedal are therefore the perpendiculars to the asymptotes of the hyperbola, coinciding with them in the case of the rectangular hyperbola and its pedal $r^2=a^2\cos 2\theta$.

Let us find the length of the arc AP from A to a point P for which $\psi=-\chi$.

We have $$\frac{ds}{d\psi}=p+\frac{d^2p}{d\psi^2} \quad \text{and} \quad t=\frac{dp}{d\psi};$$

therefore, integrating,

$$s-t=\int_0^\psi p\,d\psi.$$

Now $t\equiv\dfrac{dp}{d\psi}$ is the projection of the radius vector OP upon the tangent $=PY$, and is positive.

$$\therefore\ PY=\frac{-ae^2\sin\psi\cos\psi}{\sqrt{1-e^2\sin^2\psi}}=\frac{ae^2\sin\chi\cos\chi}{\sqrt{1-e^2\sin^2\chi}},$$

and $$\int_0^\psi p\,d\psi=a\int_0^{-\chi}\sqrt{1-e^2\sin^2\psi}\,d\psi=-a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi;$$

$$\therefore\ \text{arc } AP=PY-a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi,$$

i.e. $$=\frac{ae^2\sin\chi\cos\chi}{\sqrt{1-e^2\sin^2\chi}}-a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi,$$

or $$PY-\text{arc } AP=a\int_0^\chi\sqrt{1-e^2\sin^2\chi}\,d\chi. \quad \text{.................(1)}$$

This integral is not of the Legendrian form at present, e being essentially greater than unity.

If P be allowed to travel to ∞, χ ultimately becomes

$$\tan^{-1}\frac{a}{b}\quad\left(i.e.\ \frac{\pi}{2}-\tan^{-1}\frac{b}{a}\right).$$

Hence the excess of the infinite asymptote $C\infty$ over the infinite arc $A\infty$ is

$$a\int_0^{\tan^{-1}\frac{a}{b}}\sqrt{1-e^2\sin^2\chi}\,d\chi.$$

It is easy to reduce the integral in equation (1) to two integrals of Legendre's standard form.

Let $e\sin\chi=\sin\omega$.

Then $e\cos\chi\, d\chi = \cos\omega\, d\omega$, and

$$\int_0^\chi \sqrt{1-e^2\sin^2\chi}\, d\chi$$

$$=\frac{1}{e}\int_0^\omega \frac{\cos^2\omega\, d\omega}{\sqrt{1-\frac{1}{e^2}\sin^2\omega}}$$

$$=e\int_0^\omega \frac{\left(\frac{1}{e^2}-1\right)+\left(1-\frac{1}{e^2}\sin^2\omega\right)}{\sqrt{1-\frac{1}{e^2}\sin^2\omega}}\, d\omega$$

$$=e\left[-\frac{e^2-1}{e^2}\int_0^\omega \frac{d\omega}{\sqrt{1-\frac{1}{e^2}\sin^2\omega}}+\int_0^\omega \sqrt{1-\frac{1}{e^2}\sin^2\omega\, d\omega}\right]$$

$$=e\left(-\cos^2\alpha\int_0^\omega \frac{d\omega}{\sqrt{1-\sin^2\alpha\sin^2\omega}}+\int_0^\omega \sqrt{1-\sin^2\alpha\sin^2\omega}\, d\omega\right),$$

where $\qquad \cot\alpha=\frac{b}{a},\quad$ *i.e.* $e^2=\frac{a^2+b^2}{a^2}=\operatorname{cosec}^2\alpha,$

and α is the complement of the half angle between the asymptotes.

Hence,

$$\text{Arc } AP = PY + ae[\cos^2\alpha\, F(\omega, \sin\alpha) - E(\omega, \sin\alpha)],$$

F and E being the Legendrian standard integrals of the first and second species, whose values are tabulated for particular values of the modulus $\sin\alpha$, ω being $\sin^{-1}\left(\frac{\sin\chi}{\sin\alpha}\right)$ in the upper limit and PY, written in terms of ω, being

$$\frac{a}{\sin\alpha}\tan\omega\sqrt{1-\sin^2\alpha\sin^2\omega} \equiv ae\tan\omega\,\Delta\ \left(\text{Mod. } \frac{1}{e}\right),$$

where $\qquad \Delta=\sqrt{1-\frac{1}{e^2}\sin^2\omega},$

i.e. $\qquad \text{Arc} = ae\{\tan\omega\Delta + \cos^2\alpha\, F(\omega, \sin\alpha) - E(\omega, \sin\alpha)\}.$ ......(2)

589. In a rectangular hyperbola $\alpha=\frac{\pi}{4}$, $e=\sqrt{2}$, and we have

$$\text{Arc} = a\sqrt{2}\left[\tan\omega\sqrt{1-\tfrac{1}{2}\sin^2\omega}+\tfrac{1}{2}F\left(\omega, \frac{1}{\sqrt{2}}\right)-E\left(\omega, \frac{1}{\sqrt{2}}\right)\right]$$

EXAMPLES.

1. In the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, put $a=b\tan\alpha$, $\Delta=\sqrt{1-\sin^2\alpha\sin^2\phi}$, and show that we may take $x=b\tan\alpha\sec\phi\,\Delta$, $y=b\cos\alpha\tan\phi$, and that

$$\frac{ds}{d\phi}=\frac{b\cos\alpha}{\Delta\cos^2\phi},\quad t=\frac{b}{\cos\alpha}\Delta\tan\phi,$$

and $$s=b\sec\alpha\tan\phi\Delta+b\cos\alpha\,F(\phi,\sin\alpha)-b\sec\alpha\,E(\phi,\sin\alpha).$$

2. From the polar equation $r^2=a^2\sec 2\theta$ deduce the rectification of the rectangular hyperbola, viz.

$$s=a\sqrt{2}[\Delta\tan\omega+\tfrac{1}{2}F-E].$$

3. If PQ be a chord of one branch of a hyperbola, touching a confocal ellipse at F, and the confocal cutting that branch of the hyperbola at A and B, and if PR, QS be the other tangents from P and Q to the ellipse, show that the elliptic arcs AR, BS exceed the elliptic arc AFB by the excess of the tangents PR, QS over the chord PQ, *i.e.* that

$$\text{arc } AR+\text{arc } BS-\text{arc } AFB$$

is rectifiable in terms of known lines.

In particular, examine what happens:

(1) When F is the vertex of the confocal ellipse.

(2) When F is at B.

(3) When PR and QS are at right angles to PQ and F the vertex of the ellipse.

590. Another Method of Treatment for the Central Conics. Use of Hyperbolic Functions.

In the case of the central conics it is instructive to consider another mode of treatment of the rectification.

The relation $$x+\iota y=c\sin(u+\iota v)$$
gives $$x=c\sin u\cosh v,\quad y=c\cos u\sinh v$$

Then $v=$const. is the equation to the ellipse

$$\frac{x^2}{c^2\cosh^2 v}+\frac{y^2}{c^2\sinh^2 v}=1,$$

and $u=$const. is the equation to the hyperbola

$$\frac{x^2}{c^2\sin^2 u}-\frac{y^2}{c^2\cos^2 u}=1,$$

and different constant values of v and u give confocal ellipses and hyperbolae.

Now $\frac{dx}{c}=\cos u \cosh v\, du+\sin u \sinh v\, dv,$

$$\frac{dy}{c}=-\sin u \sinh v\, du+\cos u \cosh v\, dv.$$

Hence

$$\frac{ds^2}{c^2}=(\cos^2 u \cosh^2 v+\sin^2 u \sinh^2 v)(du^2+dv^2)$$

$$=\{(1-\sin^2 u)\cosh^2 v+\sin^2 u(\cosh^2 v-1)\}(du^2+dv^2)$$

$$=(\cosh^2 v-\sin^2 u)(du^2+\ v^2).$$

Hence, for any of the family of the ellipses $v=$const.,

$$\frac{ds}{c}=\sqrt{\cosh^2 v-\sin^2 u}\, du \quad (\ =\text{const.});$$

and for any of the family of hyperbolae $u=$const.,

$$\frac{ds}{c}=\sqrt{\cosh^2 v-\sin^2 u}\, dv\ (u=\text{const.}).$$

591. **In the case of the ellipse** $x^2/a^2+y^2/b^2=1$,

$$a=c\cosh v,\quad b=c\sinh v,\quad c^2=a^2-b^2=a^2e^2,$$

where e is the eccentricity, and $\therefore\ e=\operatorname{sech} v$.

And $ds=a\sqrt{1-e^2\sin^2 u}\, du,$

$$s=a\int_0^u \sqrt{1-e^2\sin^2 u}\, du=aE(u,e).$$

In the case of the hyperbola $x^2/a^2-y^2/b^2=1$,

$$a=c\sin u,\quad b=c\cos u,\quad \text{and}\quad c^2=a^2+b^2=a^2e^2,\quad e=\operatorname{cosec} u.$$

With the notation of Art. 589, in which

$$\psi=-\chi,\quad \sin\chi=\sin u \sin\omega,$$

we have

$$\cos\chi=\sqrt{1-\sin^2 u \sin^2\omega}=\Delta \quad \text{and} \quad t\equiv PY=c\tan\omega\,\Delta.$$

The line $x\cos\psi+y\sin\psi=p$ is tangential, provided that

$$p^2=a^2\cos^2\psi-b^2\sin^2\psi$$

$$=c^2\sin^2 u\,\Delta^2-c^2\cos^2 u \sin^2 u \sin^2\omega=c^2\sin^2 u\cos^2\omega;$$

$$\therefore\ p=c\sin u\cos\omega.$$

The point of contact P is given by

$$x=\frac{a^2\cos\psi}{p}=c\sin u\,\Delta\sec\omega,\quad y=-\frac{b^2\sin\psi}{p}=c\cos^2 u\tan\omega,$$

and, as these are to be $c\sin u\cosh v$, $c\cos u\sinh v$, we have

$$\cosh v=\Delta\sec\omega,\quad \sinh v=\cos u\tan\omega.$$

It follows that $\cosh v\,dv=\cos u\sec^2\omega\,d\omega$,

i.e. $$dv=\frac{\cos u\,d\omega}{\Delta\cos\omega}.$$

Again,

$$\sqrt{\cosh^2 v-\sin^2 u}=\sqrt{\Delta^2\sec^2\omega-\sin^2 u}=\cos u\sec\omega.$$

Hence $$\frac{s}{c}=\int\sqrt{\cosh^2 v-\sin^2 u}\,dv$$

$$=\cos^2 u\int\frac{\sec^2\omega}{\Delta}d\omega$$

$$=\Delta\tan\omega+\cos^2 uF-E\quad(\text{mod.}\sin u)$$

by Legendre's fourth formula, p. 399;

$$\therefore\ \text{Arc}=PY+ae\left(1-\frac{1}{e^2}\right)F(\omega,\sin u)-aeE(\omega,\sin u),$$

the same result as before.

592. **The Lemniscate.**

The equation is $r^2=a^2\cos 2\theta$;

we have at once $$\frac{dr}{r\,d\theta}=-\tan 2\theta;$$

whence $$\frac{ds}{d\theta}=r\sec 2\theta=\frac{a}{\sqrt{\cos 2\theta}},$$

$$s=a\int_0^\theta\frac{d\theta}{\sqrt{\cos 2\theta}}.$$

Put $\cos 2\theta=\cos^2\phi$; $\therefore\ d\theta=\dfrac{\sin\phi\cos\phi\,d\phi}{\sin 2\theta}$

$$s=a\int_0^\phi\frac{\sin\phi\cos\phi\,d\phi}{\cos\phi\sqrt{1-\cos^4\phi}}=a\int_0^\phi\frac{d\phi}{\sqrt{2-\sin^2\phi}}$$

$$=\frac{a}{\sqrt{2}}\int_0^\phi\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}=\frac{a}{\sqrt{2}}F\left(\phi,\frac{1}{\sqrt{2}}\right),$$

or $$=\frac{a}{\sqrt{2}}\,\text{am}^{-1}\phi.$$

Hence $$\operatorname{am}\frac{s\sqrt{2}}{a}=\phi,$$

$$\operatorname{cn}\frac{s\sqrt{2}}{a}=\cos\phi=\frac{r}{a}.$$

Hence $$s=\frac{a}{\sqrt{2}}\operatorname{cn}^{-1}\frac{r}{a},\ \text{mod.}\ \frac{1}{\sqrt{2}}.$$

Here s is measured from the vertex.

We might have expressed θ from the beginning in terms of r, and then
$$\theta=\frac{1}{2}\cos^{-1}\frac{r^2}{a^2},$$
$$\frac{d\theta}{dr}=-\frac{r}{\sqrt{a^4-r^4}},$$
$$\frac{ds}{dr}=-\frac{a^2}{\sqrt{a^4-r^4}}\qquad s=a^2\int_r^a\frac{dr}{\sqrt{a^4-r^4}};$$
then putting $r=a\cos\phi$ the work proceeds as before.

For the whole length of the arc, we have
$$\frac{4a}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}=2a\sqrt{2}\,F_1,\quad \text{mod.}\ \frac{1}{\sqrt{2}}.$$

The tables for F_1 (Bertrand, *C.I.* p. 716) give $F_1=1{\cdot}85407$, whence $$\text{whole arc}=2a\sqrt{2}\times1{\cdot}85407=a\times5{\cdot}2441.$$

We might, however, proceed as follows:
$$s=4a\int_0^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{\cos2\theta}}.$$

Putting $2\theta=\omega$, we have
$$s=2a\int_0^{\frac{\pi}{2}}(\cos\omega)^{-\frac{1}{2}}\,d\omega=2a\frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{2\Gamma(\frac{3}{4})}.$$

It will be shown later (Art. 872) that
$$\Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi},$$
where n is less than unity. Borrowing this theorem for present purposes,
$$\Gamma(\tfrac{3}{4})\Gamma(\tfrac{1}{4})=\frac{\pi}{\sin\frac{\pi}{4}}=\pi\sqrt{2};$$

$$\therefore\ \text{Perimeter}=2a\frac{[\Gamma(\frac{1}{4})]^2\sqrt{\pi}}{2\pi\sqrt{2}}=\frac{a}{\sqrt{2\pi}}[\Gamma(\tfrac{1}{4})]^2=ka,\ \text{say}.$$

The values of the Γ functions are calculated. Tables of these values are given in Bertrand's *Calcul Intégral*, pages 285, 286, to seven places of decimals from $\operatorname{Log}\Gamma(1)$ to $\operatorname{Log}\Gamma(2)$. As the values of $\Gamma(x)$ from $\Gamma(1)$ to $\Gamma(2)$ are all fractional, 10 is added to their ordinary logarithms for convenience of tabulation, as is usual in tables of logarithms of sines and cosines. (See Chambers's *Mathematical Tables.*)

Now $$\Gamma(\tfrac{5}{4})=\tfrac{1}{4}\Gamma(\tfrac{1}{4}),$$
and $$L\,\Gamma(\tfrac{1}{4})=L\,\Gamma(\tfrac{5}{4})+\log 4,$$
where L denotes the tabular logarithm,

$= 9{\cdot}9573211$ from the tables of $L\,\Gamma(x)$.
$+\ \cdot6020600$ $\qquad \log 2 = \cdot3010300$
$10{\cdot}5593811$ $\qquad \log\pi = \cdot4971499$

$2\log\Gamma(\tfrac{1}{4}) = 1{\cdot}1187622$ $\qquad \log 2\pi = \cdot7981799$
$\log\sqrt{2\pi} = \cdot3990899$ $\qquad \log\sqrt{2\pi} = \cdot3990899$

$\log k = \cdot7196723$
$\log 5{\cdot}2441 = \cdot7196710$
13

Difference for $1 = 8$
50
50

Hence $k=5{\cdot}244116$.

Hence the whole perimeter of $r^2=a^2\cos 2\theta$ is, as before,
$$5{\cdot}244116\times a.$$

593. Incidentally, it may be remarked that the equation
$$r=a\,\operatorname{cn}\frac{s\sqrt{2}}{a}$$

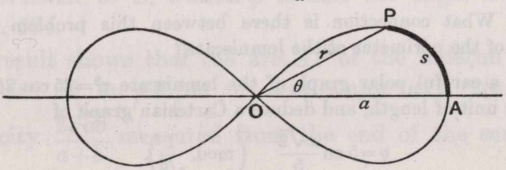

Fig. 140.

for a lemniscate gives a very good idea of the graph of the functions cn and cn^{-1} for the case mod. $\dfrac{1}{\sqrt{2}}$, and we can readily draw a graph, taking, for instance, as unit length $\frac{a}{\sqrt{2}}$ on the x-axis, and any convenient unit on the y-axis, say a, and constructing the curve with abscissa s and ordinate r.

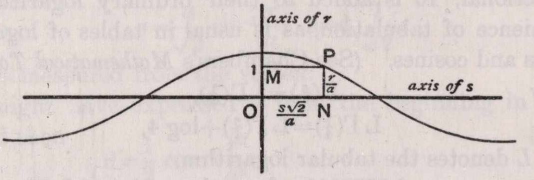

Fig. 141.

The ordinate shows the march of the function $\operatorname{cn} x$, the abscissa the march of $\operatorname{cn}^{-1}x$.

EXAMPLES.

1. Find the length of the arc of a lemniscate $r^2=a^2\cos 2\theta$ from $\theta=0$ to $\theta=\frac{\pi}{6}$.

Here

$$s=\frac{a}{\sqrt{2}}\int_0^{\phi}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}},\quad \text{and } k=\frac{1}{\sqrt{2}},\quad \cos^2\phi=\cos\frac{\pi}{3}=\tfrac{1}{2},\quad \phi=\frac{\pi}{4},$$

and from the tables for $F\left(\phi, \frac{1}{\sqrt{2}}\right)$, (Bertrand, *Calcul Intégral*, p. 716.)

$$\int_0^{\frac{\pi}{4}}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}=\cdot 82602\,;$$

$$\therefore\ s=a\sqrt{2}\times\cdot 41301$$

$$=\cdot 5841a.$$

2. Find the area of the curve $y^2=\frac{1}{1-x^4}$ for the portion in the first quadrant. What connection is there between this problem and the evaluation of the perimeter of the lemniscate?

3. Draw a careful polar graph of the lemniscate $r^2=25\cos 2\theta$, taking one inch as unit of length, and deduce a Cartesian graph of

$$y=5\operatorname{cn}\frac{x\sqrt{2}}{5}\quad\left(\text{mod. }\frac{1}{\sqrt{2}}\right).$$

4. Show that the difference between the lengths of the asymptote and the infinite arc of the hyperbola $x^2/a^2-y^2/b^2=1$ in the first quadrant is

$$t-s=\frac{\pi a}{2}\left[\frac{1}{2}\cdot\frac{1}{e}+\frac{1\,.\,1^2}{2^2\,.\,4}\cdot\frac{1}{e^3}+\frac{1\,.\,1^2\,.\,3^2}{2^2\,.\,4^2\,.\,6}\,\frac{1}{e^5}+\frac{1\,.\,1^2\,.\,3^2\,.\,5^2}{2^2\,.\,4^2\,.\,6^2\,.\,8}\cdot\frac{1}{e^7}+\dots\right].$$

594. **The Limaçon** $r=a+b\cos\theta$.

Here $\dfrac{dr}{d\theta}=-b\sin\theta$ and $\left(\dfrac{ds}{d\theta}\right)^2=a^2+2ab\cos\theta+b^2$;

$$\begin{aligned}\therefore\quad s&=\int_0^\theta\sqrt{a^2+2ab\cos\theta+b^2}\,d\theta\\&=\int_0^\theta\sqrt{(a+b)^2-4ab\sin^2\frac{\theta}{2}}\,d\theta\qquad(\text{Let }\theta=2\phi.)\\&=2(a+b)\int_0^\phi\sqrt{1-k^2\sin^2\phi}\,d\phi,\ \text{where}\ k^2=\frac{4ab}{(a+b)^2},\\&=2(a+b)\,E\left(\phi,\ \frac{2\sqrt{ab}}{a+b}\right).\end{aligned}$$

An obvious modification will be necessary if a and b be of opposite sign.

This curve very well illustrates the march of the second elliptic integral E. The arc AP measured from the vertex

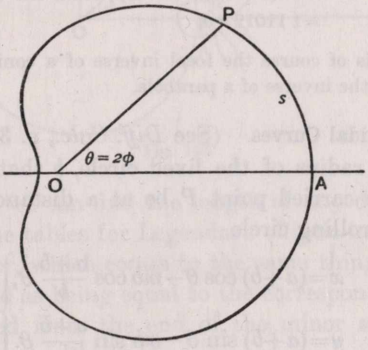

Fig. 142. For the case $a>b$.

is proportional to E, whilst ϕ is half the angle AOP. See also Art. 574.

The result shows that the arc AP of the limaçon is equal to the arc of an ellipse of semi-major axis $2(a+b)$ and eccentricity $\dfrac{2\sqrt{ab}}{a+b}$, measured from the end of the semi-minor axis to a point on the ellipse for which the complement of the eccentric angle is $\dfrac{\theta}{2}$ (compare Art. 573). The semiaxes of the ellipse in question are then $2(a+b)$ and $2(a-b)$.

This would also be evident upon writing

$$\int_0^\theta \sqrt{a^2+2ab\cos\theta+b^2}\,d\theta$$

as
$$\int_0^\theta \sqrt{(a+b)^2\cos^2\frac{\theta}{2}+(a-b)^2\sin^2\frac{\theta}{2}}\,d\theta$$

$$=\int_0^\phi \sqrt{(2a+b)^2\cos^2\phi+(2a-b)^2\sin^2\phi}\,d\phi, \quad \text{where} \quad \theta=2\phi.$$

595. Ex. Consider the case of the limaçon in which $\frac{a}{b}=\frac{2+\sqrt{3}}{2-\sqrt{3}}$ for the portion from $\theta=0$ to $\theta=\frac{\pi}{3}$.

Here $\frac{a+b}{a-b}=\frac{2}{\sqrt{3}}$, and $k^2=\frac{4ab}{(a+b)^2}=\frac{1}{4}$, $k=\frac{1}{2}=\sin\frac{\pi}{6}$,

$$s=2(a+b)\int_0^{\frac{\pi}{6}}\sqrt{1-\tfrac{1}{4}\sin^2\phi}\,d\phi$$
$$=8a(2-\sqrt{3})\times\cdot 51788, \quad \text{from the tables for } E(\phi,\tfrac{1}{2}),$$
$$=1\cdot 11012\times a.$$

The limaçon is of course the focal inverse of a conic, and when $a=b$ the cardioide is the inverse of a parabola.

596. **Trochoidal Curves.** (See *Diff. Calc.*, p. 344.)

If a be the radius of the fixed circle, b that of the rolling circle and the carried point P be at a distance mb from the centre of the rolling circle,

$$\left.\begin{aligned} x&=(a+b)\cos\theta-mb\cos\frac{a+b}{b}\theta,\\ y&=(a+b)\sin\theta-mb\sin\frac{a+b}{b}\theta.\end{aligned}\right\}$$

Hence
$$\left.\begin{aligned}\frac{dx}{d\theta}&=-(a+b)\sin\theta+m(a+b)\sin\frac{a+b}{b}\theta,\\ \frac{dy}{d\theta}&=\ \ (a+b)\cos\theta-m(a+b)\cos\frac{a+b}{b}\theta;\end{aligned}\right\}$$

$$\therefore \left(\frac{ds}{d\theta}\right)^2=(a+b)^2(1+m^2)-2m(a+b)^2\cos\frac{a\theta}{b}$$

$$=(a+b)^2(1+m)^2\left[1-\frac{4m}{(1+m)^2}\cos^2\frac{a\theta}{2b}\right].$$

Let
$$\frac{a\theta}{2b}=\frac{\pi}{2}+\chi.$$

Then $s=\frac{2b}{a}(a+b)(1+m)\int_0^\chi \sqrt{1-k^2\sin^2\chi}\,d\chi$, where $k=\frac{2\sqrt{m}}{1+m}$,

$$=\frac{2b}{a}(a+b)(1+m)E(\chi, k),$$

where s is measured from the point at which $\chi=0$, *i.e.* $\theta=\frac{b\pi}{a}$, *i.e.* from a vertex V, as in the case of the epicycloid (Art. 540).

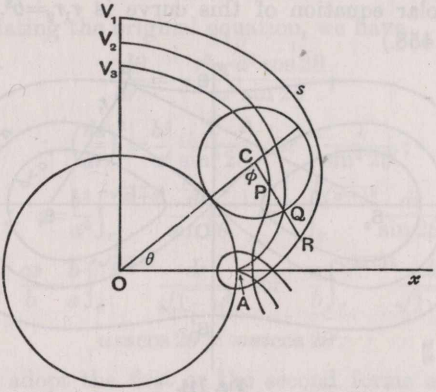

Fig. 143.

Hence again we can find the length of any desired portion by means of the tables for Legendre's elliptic integrals of the second form; or, which comes to the same thing, such length can be expressed as being equal to the corresponding arc of an ellipse, measured from the end of the minor axis, the semi-major axis being $\frac{2b}{a}(a+b)(1+m)$, the eccentricity being $a=\frac{2\sqrt{m}}{1+m}$, and χ being the complement of the eccentric angle at the end of the elliptic arc.

For a circle, when $m=0$,

$$s=\frac{2b}{a}(a+b)\,\chi=(a+b)\left(\theta-\frac{\pi b}{a}\right)+\text{const.}$$

For the epicycloid, when $m=1$.

$$s=\frac{4b}{a}(a+b)\sin\chi=\frac{-4b}{a}(a+b)\cos\frac{a\theta}{2b}+\text{const.}$$

which agrees with the result of Art. 540.

We might use this curve, like the ellipse and the limaçon, to construct a graph showing the march of

$$\int_0^{\chi}\sqrt{1-k^2\sin^2\chi}\,d\chi$$

for any modulus $k=\dfrac{2\sqrt{m}}{1+m}$.

597. **The Cassinian Oval.**

The bipolar equation of this curve is $r_1 r_2 = b^2$. (See *Diff. Calc.*, Art. 458.)

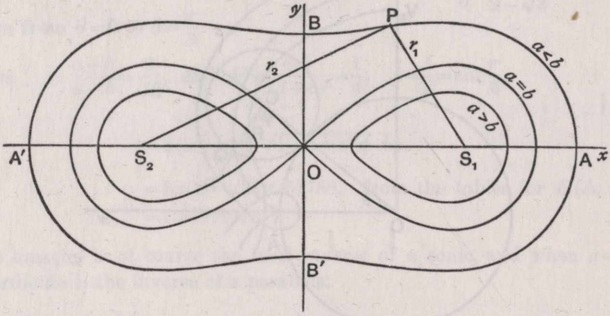

Fig. 144.

If S_1, S_2 be the foci, $S_1S_2=2a$, and if the line of foci be taken as x-axis and its centre O as origin, the equivalent polar equation is

$$r^4-2a^2r^2\cos 2\theta+a^4=b^4.$$

Three cases arise:

(1) $a>b$, two separate twin ovals with vertices distant $\sqrt{a^2+b^2}$, $\sqrt{a^2-b^2}$ from O.

(2) $a=b$, reducing to Bernoulli's lemniscate.

(3) $a<b$, one single oval lying outside the lemniscate, which may or may not possess inflexions.

The equation may be written

$$r^2+\frac{a^4-b^4}{r^2}=2a^2\cos 2\theta.$$

Take an auxiliary angle θ' such that

$$r^2+\frac{b^4-a^4}{r^2}=2b^2\cos 2\theta'.$$

Then $$r^2=a^2\cos 2\theta+b^2\cos 2\theta',$$

$$\frac{a^4-b^4}{r^2}=a^2\cos 2\theta-b^2\cos 2\theta';$$

$$\therefore\ a^4-b^4=a^4\cos^2 2\theta-b^4\cos^2 2\theta',$$

or $$a^4\sin^2 2\theta=b^4\sin^2 2\theta',$$

i.e. the auxiliary angle θ' is such that

$$a^2\sin 2\theta=b^2\sin 2\theta'.$$

Differentiating the original equation, we have

$$\frac{r\,d\theta}{dr}=-\frac{r^2-a^2\cos 2\theta}{a^2\sin 2\theta};$$

$$\therefore\ \left(\frac{ds}{dr}\right)^2=\frac{b^4}{a^4}\frac{1}{\sin^2 2\theta}\quad\text{or}\quad\frac{1}{\sin^2 2\theta'};$$

$$\therefore\ s=\frac{b^2}{a^2}\int_r^{\sqrt{a^2+b^2}}\frac{dr}{\sin 2\theta}\quad\text{or}\quad\int_r^{\sqrt{a^2+b^2}}\frac{dr}{\sin 2\theta'};$$

$$\therefore\ \frac{as}{b}=\frac{b}{a}\int_r^{\sqrt{a^2+b^2}}\frac{dr}{\sqrt{1-u^2}}\quad\text{or}\quad\frac{a}{b}\int_r^{\sqrt{a^2+b^2}}\frac{dr}{\sqrt{1-v^2}},$$

where $$u\equiv\cos 2\theta,\quad v\equiv\cos 2\theta'.$$

We shall adopt the first or the second forms according as a is $>$ or $<$ than b.

Let $$\lambda=\frac{\sqrt{a^4-b^4}}{a^2},\ (a \nless b);\ =\cos 2\alpha,\ \text{where}\ \frac{b^2}{a^2}=\sin 2\alpha;$$

$$\mu=\frac{\sqrt{b^4-a^4}}{b^2},\ (a \ngtr b);\ =\cos 2\beta,\ \text{where}\ \frac{a^2}{b^2}=\sin 2\beta.$$

In the case $a \nless b$, $u\equiv\cos 2\theta\equiv\dfrac{r^4+a^4-b^4}{2a^2r^2}$,

so $$\frac{r^2}{a^2}+\lambda^2\frac{a^2}{r^2}=2u;$$

$$\therefore\ \frac{r}{a}+\lambda\frac{a}{r}=\sqrt{2}\sqrt{u+\lambda},$$

$$\frac{r}{a}-\lambda\frac{a}{r}=\sqrt{2}\sqrt{u-\lambda};$$

$$\therefore\ r=\frac{a}{\sqrt{2}}(\sqrt{u+\lambda}+\sqrt{u-\lambda}),$$

$$dr=\frac{a}{2\sqrt{2}}\left(\frac{du}{\sqrt{u+\lambda}}+\frac{du}{\sqrt{u-\lambda}}\right),$$

$$\therefore \frac{a}{b}s=\frac{b}{2\sqrt{2}}\left[\int_u^1\frac{du}{\sqrt{(1-u^2)(u+\lambda)}}+\int_u^1\frac{du}{\sqrt{(1-u^2)(u-\lambda)}}\right]$$

$$=\frac{b}{2}\left[\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\cos\alpha},\ \cos\alpha\right)+\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\sin\alpha},\ \sin\alpha\right)\right]$$

where $\sin 2\alpha=\frac{b^2}{a^2}$ (Art. 388, 4).

In the case $a \ngtr b$, $v\equiv\cos 2\theta'=\frac{r^4+b^4-a^4}{2b^2r^2}$,

$$\frac{r^2}{b^2}+\mu^2\frac{b^2}{r^2}=2v,$$

and the work proceeds precisely as before, interchanging a and b, u and v, θ and θ', λ and μ, α and β, on the right-hand side of the values of $\frac{as}{b}$.

$$\therefore \frac{a}{b}s=\frac{a}{2}\left[\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\cos\beta},\ \cos\beta\right)+\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\sin\beta},\ \sin\beta\right)\right],$$

where $\theta'=\frac{1}{2}\sin^{-1}(\sin 2\beta\sin 2\theta)$ and $\sin 2\beta=\frac{a^2}{b^2}$.

The arc is in both cases measured from the vertex, where

$$r=\sqrt{a^2+b^2}.$$

598. In the case of the **Lemniscate**,

$$a=b,\quad r^2=2a^2\cos 2\theta=c^2\cos 2\theta, \text{ say};$$

then $\theta=\theta'$, and either case gives

$$s=\frac{a}{2}\cdot 2\operatorname{sn}^{-1}\left(\sqrt{2}\sin\theta,\ \frac{1}{\sqrt{2}}\right)$$

$$=a\operatorname{cn}^{-1}\left(\sqrt{1-2\sin^2\theta},\ \frac{1}{\sqrt{2}}\right)=\frac{c}{\sqrt{2}}\operatorname{cn}^{-1}\left(\sqrt{\cos 2\theta},\ \frac{1}{\sqrt{2}}\right)$$

$$=\frac{c}{\sqrt{2}}\operatorname{cn}^{-1}\left(\frac{r}{c},\ \frac{1}{\sqrt{2}}\right),\quad \text{as in Art. 592.}$$

599. It is a very instructive process to perform the same rectification first expressing θ in terms of r. We have

$$\sin^2 2\theta=1-\left(\frac{r^4+a^4-b^4}{2a^2r^2}\right)^2$$

$$=(r^2+a^2+b^2)(-r^2+a^2+b^2)(r^2-a^2+b^2)(r^2+a^2-b^2)/4a^4r^4$$

$$=[(a^2+b^2)^2-r^4][r^4-(a^2\sim b^2)^2]/4a^4r^4.$$

Let $$r=\sqrt{a^2+b^2}\,u \quad\text{and}\quad \lambda^2=\frac{a^2\sim b^2}{a^2+b^2},$$
the positive value to be taken.
$$\sin 2\theta=(a^2+b^2)\sqrt{(1-u^4)(u^4-\lambda^4)}/2a^2u^2,$$
and $$dr=\sqrt{a^2+b^2}\,du\,;$$
$$\therefore\ s=\frac{2b^2}{\sqrt{a^2+b^2}}\int_u^1\frac{u^2du}{\sqrt{(1-u^4)(u^4-\lambda^4)}}.$$

Again,
$$\begin{aligned}(1-u^4)(u^4-\lambda^4)&=[(1-u^2)(u^2-\lambda^2)][(1+u^2)(u^2+\lambda^2)]\\&=[(1+\lambda^2)u^2-(u^4+\lambda^2)][(1+\lambda^2)u^2+(u^4+\lambda^2)]\\&=(1+\lambda^2)^2u^4-(u^4+\lambda^2)^2\\&=(1+\lambda^2)^2u^4(1-v^2),\end{aligned}$$
where $$\frac{u^4+\lambda^2}{u^2}=(1+\lambda^2)v.$$

This transformation gives
$$u^2+\frac{\lambda^2}{u^2}=(1+\lambda^2)v\,;$$
$$\therefore\ u+\frac{\lambda}{u}=\sqrt{(1+\lambda^2)v+2\lambda},$$
$$u-\frac{\lambda}{u}=\sqrt{(1+\lambda^2)v-2\lambda},$$
$$2u=\sqrt{(1+\lambda^2)v+2\lambda}+\sqrt{(1+\lambda^2)v-2\lambda},$$
$$\frac{4du}{\sqrt{1+\lambda^2}}=\frac{dv}{\sqrt{v+\dfrac{2\lambda}{1+\lambda^2}}}+\frac{dv}{\sqrt{v-\dfrac{2\lambda}{1+\lambda^2}}}\,;$$
$$\therefore\ s=\frac{2b^2}{\sqrt{a^2+b^2}}\cdot\int_v^1\frac{1}{(1+\lambda^2)\sqrt{1-v^2}}\cdot\frac{\sqrt{1+\lambda^2}}{4}\times\left\{\frac{1}{\sqrt{v+\dfrac{2\lambda}{1+\lambda^2}}}+\frac{1}{\sqrt{v-\dfrac{2\lambda}{1+\lambda^2}}}\right\}dv\,;$$
$$\therefore\ s=\frac{b^2}{2\sqrt{a^2+b^2}\sqrt{1+\lambda^2}}\times\left\{\int_v^1\frac{dv}{\sqrt{(1-v^2)\left(v+\dfrac{2\lambda}{1+\lambda^2}\right)}}+\int_v^1\frac{dv}{\sqrt{(1-v^2)\left(v-\dfrac{2\lambda}{1+\lambda^2}\right)}}\right\}.$$

Now an integral of form $I=\displaystyle\int_v^1\frac{dv}{\sqrt{(1-v^2)(v+c)}}$ can be converted at once into the standard Legendrian form as follows (Art. 388, 4):

Put $v+c=(1+c)\cos^2\phi$.

Then

$$I=\int_\phi^0 \frac{-2(1+c)\sin\phi\cos\phi\,d\phi}{\sqrt{\{(1+c)-(1+c)\cos^2\phi\}\{(1-c)+(1+c)\cos^2\phi\}(1+c)\cos^2\phi}}$$

$$=2\int_0^\phi \frac{d\phi}{\sqrt{2-(1+c)\sin^2\phi}}$$

$$=\sqrt{2}\int_0^\phi \frac{d\phi}{\sqrt{1-\frac{1+c}{2}\sin^2\phi}},$$

and as in our case $c=\pm\frac{2\lambda}{1+\lambda^2}$, it is numerically less than unity and $\frac{1+c}{2}$ is positive and less than unity;

$$\therefore\ \phi=\operatorname{am}(I/\sqrt{2}),\ \text{mod.}\ \sqrt{\frac{1+c}{2}},$$

$$\cos\phi=\operatorname{cn}(I/\sqrt{2})\quad\text{and}\quad I=\sqrt{2}\operatorname{cn}^{-1}\sqrt{\frac{v+c}{1+c}}.$$

Hence, finally, we have

$$s=\frac{b^2}{\sqrt{2}\sqrt{a^2+b^2}\sqrt{1+\lambda^2}}\left\{\operatorname{cn}^{-1}\left(\sqrt{\frac{v+\frac{2\lambda}{1+\lambda^2}}{1+\frac{2\lambda}{1+\lambda^2}}},\ \sqrt{\frac{1}{2}\left(1+\frac{2\lambda}{1+\lambda^2}\right)}\right)\right.$$
$$\left.+\operatorname{cn}^{-1}\left(\sqrt{\frac{v-\frac{2\lambda}{1+\lambda^2}}{1-\frac{2\lambda}{1+\lambda^2}}},\ \sqrt{\frac{1}{2}\left(1-\frac{2\lambda}{1+\lambda^2}\right)}\right)\right\}$$

$$=\frac{b^2}{\sqrt{2}\sqrt{(a^2+b^2)+(a^2\sim b^2)}}$$
$$\times\left\{\operatorname{cn}^{-1}\left(\frac{u+\frac{\lambda}{u}}{1+\lambda},\ \frac{1+\lambda}{\sqrt{2(1+\lambda^2)}}\right)+\operatorname{cn}^{-1}\left(\frac{u-\frac{\lambda}{u}}{1-\lambda},\ \frac{1-\lambda}{\sqrt{2(1+\lambda^2)}}\right)\right\}$$

$$=\frac{b^2}{\sqrt{2}\sqrt{(a^2+b^2)+(a^2\sim b^2)}}$$
$$\times\left\{\operatorname{cn}^{-1}\left(\frac{r+\frac{\sqrt{a^4\sim b^4}}{r}}{\sqrt{(a^2+b^2)}+\sqrt{(a^2\sim b^2)}}\right)+\operatorname{cn}^{-1}\left(\frac{r-\frac{\sqrt{a^4\sim b^4}}{r}}{\sqrt{(a^2+b^2)}-\sqrt{(a^2\sim b^2)}}\right)\right\},$$

the respective moduli being

$$\frac{\sqrt{a^2+b^2}+\sqrt{a^2\sim b^2}}{\sqrt{2}\sqrt{(a^2+b^2)+(a^2\sim b^2)}}\quad\text{and}\quad\frac{\sqrt{a^2+b^2}-\sqrt{a^2\sim b^2}}{\sqrt{2}\sqrt{(a^2+b^2)+(a^2\sim b^2)}}.$$

For the twin-loop curve $a>b$,

$$s=\frac{b^2}{2a}\left\{\operatorname{cn}^{-1}\frac{r+\frac{\sqrt{a^4-b^4}}{r}}{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}+\operatorname{cn}^{-1}\frac{r-\frac{\sqrt{a^4-b^4}}{r}}{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}\right\},$$

with respective moduli

$$\frac{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}{2a},\quad \frac{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}{2a}.$$

For the single-loop curve $a<b$,

$$s=\frac{b}{2}\left\{\mathrm{cn}^{-1}\frac{r+\dfrac{\sqrt{b^4-a^4}}{r}}{\sqrt{b^2+a^2+\sqrt{b^2-a^2}}}+\mathrm{cn}^{-1}\frac{r-\dfrac{\sqrt{b^4-a^4}}{r}}{\sqrt{b^2+a^2-\sqrt{b^2-a^2}}}\right\},$$

with respective moduli

$$\frac{\sqrt{b^2+a^2}+\sqrt{b^2-a^2}}{2b},\quad \frac{\sqrt{b^2+a^2}-\sqrt{b^2-a^2}}{2b}.$$

600. The expressions written in this rectification are less simple than when written in terms of θ, as in Art. 597, but can readily be reduced.

In the case $a>b$, let $\sin 2\alpha=\dfrac{b^2}{a^2}$; then $r^4-2a^2r^2\cos 2\theta+a^4\cos^2 2\alpha=0$.

Also $\quad \cos 2\alpha=\sqrt{1-\dfrac{b^4}{a^4}},\quad \sin\alpha=\dfrac{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}{2a},$

$$\cos\alpha=\frac{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}{2a},$$

and

$$\mathrm{cn}^{-1}\left(\frac{r+\dfrac{\sqrt{a^4-b^4}}{r}}{\sqrt{a^2+b^2+\sqrt{a^2-b^2}}}\right)=\mathrm{cn}^{-1}\frac{r+\dfrac{a^2\cos 2\alpha}{r}}{2a\cos\alpha}$$

$$=\mathrm{cn}^{-1}\left(\frac{\sqrt{\cos 2\alpha+\dfrac{r^4+a^4\cos^2 2\alpha}{2a^2r^2}}}{\sqrt{2}\cos\alpha}\right)$$

$$=\mathrm{cn}^{-1}\frac{\sqrt{\cos 2\alpha+\cos 2\theta}}{\sqrt{2}\cos\alpha}$$

$$=\mathrm{cn}^{-1}\sqrt{\frac{\cos^2\alpha-\sin^2\theta}{\cos^2\alpha}}=\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\cos\alpha}\right).$$

Similarly,

$$\mathrm{cn}^{-1}\left(\frac{r-\dfrac{\sqrt{a^4-b^4}}{r}}{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}\right)=\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\sin\alpha}\right).$$

Hence $a>b$,

$$s=\frac{b^2}{2a}\left[\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\cos\alpha},\ \cos\alpha\right)+\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\sin\alpha},\ \sin\alpha\right)\right],$$

as before.

Also for the case $a<b$, since

$$r^2+\frac{b^4-a^4}{r^2}=2b^2\cos 2\theta' \quad \text{(Art. 597)},$$

$$r^2+\frac{b^4\cos^2 2\beta}{r^2}=2b^2(1-2\sin^2\theta');$$

$$\therefore\ \left(r+\frac{b^2}{r}\cos 2\beta\right)^2 = 2b^2(1-2\sin^2\theta')+2b^2\cos 2\beta$$
$$= 4b^2(\cos^2\beta - \sin^2\theta');$$

$$\therefore\ \frac{r+\dfrac{b^2\cos 2\beta}{r}}{2b\cos\beta} = \sqrt{1-\frac{\sin^2\theta'}{\cos^2\beta}};$$

$$\therefore\ \mathrm{cn}^{-1}\frac{r^2+b^2\cos 2\beta}{2rb\cos\beta} = \mathrm{sn}^{-1}\left(\frac{\sin\theta'}{\cos\beta}, \cos\beta\right).$$

Similarly, $\mathrm{cn}^{-1}\dfrac{r^2-b^2\cos 2\beta}{2rb\sin\beta} = \mathrm{sn}^{-1}\left(\dfrac{\sin\theta'}{\sin\beta}, \sin\beta\right);$

$\therefore\ a<b,\quad s=\dfrac{b}{2}\left[\mathrm{sn}^{-1}\left(\dfrac{\sin\theta'}{\cos\beta}, \cos\beta\right)+\mathrm{sn}^{-1}\left(\dfrac{\sin\theta'}{\sin\beta}, \sin\beta\right)\right],$

where $\theta' = \frac{1}{2}\sin^{-1}(\sin 2\beta\sin 2\theta)$, the result of Art. 597.

601. Serret's Method of Rectification of a Cassinian.

A different method of rectification of a Cassinian Oval is given by Serret* connecting two arcs measured from different vertices of the curve, and expressing these arcs directly in terms of θ.

In the twin-oval case $a > b$, let A and B be the vertices of one of the ovals, and let a radius vector OQP be drawn

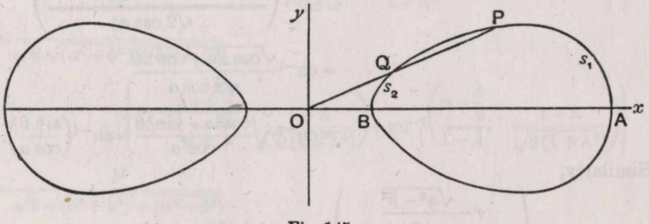

Fig. 145.

cutting that oval in Q and P. Let the vertex A be the one furthest from the centre O. Let arcs AP, BQ be called s_1, s_2 respectively. Let $b^2 = a^2\sin 2\alpha$.

Then $\qquad r^4 - 2a^2r^2\cos 2\theta + a^4 = b^4.$

Solving, $\qquad r^2 = a^2\cos 2\theta \pm a^2\sqrt{\cos^2 2\theta - \cos^2 2\alpha},$

the upper sign giving OP^2, the lower OQ^2.

* *Calcul Intégral*, p. 265.

Now, as before, $\frac{ds_1}{dr} = -\frac{b^2}{a^2}\frac{1}{\sin 2\theta}$,

and $$\frac{ds_1}{r\,d\theta} = +\frac{b^2}{r^2 - a^2\cos 2\theta} = +\frac{b^2}{a^2\sqrt{\cos^2 2\theta - \cos^2 2a}};$$

$$\therefore \frac{ds_1}{d\theta} = \frac{b^2}{a}\frac{\sqrt{\cos\ \theta + \sqrt{\cos^2 2\theta - \cos^2 2a}}}{\sqrt{\cos^2\ \theta - \cos^2 2a}},$$

the positive sign being taken as s_1 increases with θ.

Similarly $$\frac{ds_2}{d\theta} = \frac{b^2}{a}\frac{\sqrt{\cos 2\theta - \sqrt{\cos^2 2\theta - \cos^2 2a}}}{\sqrt{\cos^2 2\theta - \cos^2 2a}};$$

$$\therefore \left(\frac{ds_1}{d\theta} + \frac{ds_2}{d\theta}\right)^2 = \frac{b^4}{a^2}\frac{2(\cos 2\theta + \cos 2a)}{\cos^2 2\theta - \cos^2 2a} = \frac{b^4}{a^2}\frac{1}{\cos 2\theta - \cos 2a}$$

and $$\left(\frac{ds_1}{d\theta} - \frac{ds_2}{d\theta}\right)^2 = \frac{b^4}{a^2}\frac{2(\cos 2\theta - \cos 2a)}{\cos^2 2\theta - \cos^2 2a} = \frac{2b^4}{a^2}\frac{1}{\cos 2\theta + \cos 2a}.$$

Hence

$$s_1 + s_2 = \frac{b^2}{a}\sqrt{2}\int_0^\theta \frac{d\theta}{\sqrt{\cos 2\theta - \cos 2a}} = \frac{b^2}{a}\int_0^\theta \frac{d\theta}{\sqrt{\sin^2 a - \sin^2\theta}},$$

$$s_1 - s_2 = \frac{b^2}{a}\sqrt{2}\int_0^\theta \frac{d\theta}{\sqrt{\cos 2\theta + \cos 2a}} = \frac{b^2}{a}\int_0^\theta \frac{d\theta}{\sqrt{\cos^2 a - \sin^2\theta}}.$$

In these integrals put $\sin\theta = \sin a \sin\phi$ and $\sin\theta = \cos a \sin\psi$ respectively.

Then $$s_1 + s_2 = \frac{b^2}{a}\int_0^\phi \frac{d\phi}{\sqrt{1 - \sin^2 a \sin^2\phi}},$$

$$s_1 - s_2 = \frac{b^2}{a}\int_0^\psi \frac{d\phi}{\sqrt{1 - \cos^2 a \sin^2\psi}},$$

i.e. $$\phi = \mathrm{am}\,\frac{a}{b^2}(s_1 + s_2),\ \mathrm{mod.}\ \sin a,$$

$$\psi = \mathrm{am}\,\frac{a}{b^2}(s_1 - s_2),\ \mathrm{mod.}\ \cos a;$$

$$\therefore \frac{\sin\theta}{\sin a} = \mathrm{sn}\,\frac{a}{b^2}(s_1 + s_2);\ \frac{\sin\theta}{\cos a} = \mathrm{sn}\,\frac{a}{b^2}(s_1 - s_2);$$

$$\therefore s_1 + s_2 = \frac{b^2}{a}\,\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\sin a}, \sin a\right),$$

$$s_1 - s_2 = \frac{b^2}{a}\,\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\cos a}, \cos a\right);$$

$$\therefore\ s_1=\frac{b^2}{2a}\left[\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\sin\alpha},\sin\alpha\right)+\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\cos\alpha},\cos\alpha\right)\right],$$

$$s_2=\frac{b^2}{2a}\left[\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\sin\alpha},\sin\alpha\right)-\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\cos\alpha},\cos\alpha\right)\right],$$

the former of these being the result previously obtained.

Reducing in the case of Bernoulli's Lemniscate, we have

$$\alpha=\frac{\pi}{4},\ r^2=2a^2\cos 2\theta,$$

$$s_1=a\operatorname{sn}^{-1}\sqrt{2}\sin\theta$$

$$=a\operatorname{cn}^{-1}\sqrt{\cos 2\theta},\ \text{mod.}\ \frac{1}{\sqrt{2}},$$

$$=a\operatorname{cn}^{-1}\frac{r}{a\sqrt{2}},\ \text{as in Art. 598.}$$

602. **The Single-loop Case.**

In the one-loop case $a<b$, the same method cannot be adopted, and M. Serret considers the arcs traversed by a pair of perpendicular radii vectores OP, OQ, starting from the ends

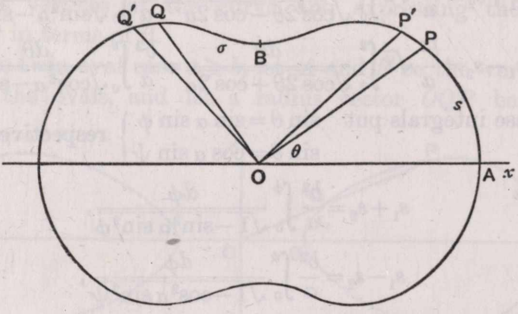

Fig. 146.

A, B of the two perpendicular axes. Let the arcs AP, BQ be respectively s and σ, and let $a^2=b^2\sin 2\beta$. Then, solving as before,

$$r^4-2a^2r^2\cos 2\theta+a^4\cos^2 2\theta=a^4(\cos^2 2\theta+\cot^2 2\beta)$$

and
$$r^2=a^2\cos 2\theta\pm a^2\sqrt{\cos^2 2\theta+\cot^2 2\beta},$$

and the positive sign must now be taken.

Also, as before,

$$\frac{ds}{r\,d\theta}=\frac{b^2}{r^2-a^2\cos 2\theta},\quad \frac{ds}{d\theta}=\frac{b^2}{a}\frac{\sqrt{\cos 2\theta+\sqrt{\cos^2 2\theta+\cot^2 2\beta}}}{\sqrt{\cos^2 2\theta+\cot^2 2\beta}}.$$

Writing $\theta+\frac{\pi}{2}$ for θ,

$$\frac{d\sigma}{d\theta}=\frac{b^2}{a}\frac{\sqrt{-\cos 2\theta+\sqrt{\cos^2 2\theta+\cot^2 2\beta}}}{\sqrt{\cos^2 2\theta+\cot^2 2\beta}};$$

$$\therefore \left(\frac{ds}{d\theta}+\frac{d\sigma}{d\theta}\right)^2=\frac{2b^4}{a^2}\frac{\sqrt{\cos^2 2\theta+\cot^2 2\beta}+\cot 2\beta}{\cos^2 2\theta+\cot^2 2\beta},$$

and
$$\left(\frac{ds}{d\theta}-\frac{d\sigma}{d\theta}\right)^2=\frac{2b^4}{a^2}\frac{\sqrt{\cos^2 2\theta+\cot^2 2\beta}-\cot 2\beta}{\cos^2 2\theta+\cot^2 2\beta}.$$

In each of these change the variable to θ', where

$$\sin 2\theta=\frac{\sin 2\theta'}{\sin 2\beta}, \text{ and therefore } \cos 2\theta\, d\theta=\frac{\cos 2\theta'\, d\theta'}{\sin 2\beta}.$$

Then

$$\cos^2 2\theta+\cot^2 2\beta=1+\cot^2 2\beta-\frac{\sin^2 2\theta'}{\sin^2 2\beta}=\frac{\cos^2 2\theta'}{\sin^2 2\beta}.$$

Then

$$\left(\frac{ds}{d\theta'}+\frac{d\sigma}{d\theta'}\right)^2=\frac{2b^4}{a^2}\frac{\cos 2\theta'+\cos 2\beta}{\cos^2 2\theta'}\frac{\cos^2 2\theta'}{\sin 2\beta}\frac{1}{1-\frac{\sin^2 2\theta'}{\sin^2 2\beta}}$$

$$=\frac{2b^4}{a^2}\frac{\cos 2\theta'+\cos 2\beta}{\sin^2 2\beta-\sin^2 2\theta'}\sin 2\beta$$

$$=\frac{2b^4}{a^2}\frac{\sin 2\beta}{\cos 2\theta'-\cos 2\beta}=\frac{b^4}{a^2}\frac{\sin 2\beta}{\sin^2\beta-\sin^2\theta'}.$$

Similarly

$$\left(\frac{ds}{d\theta'}-\frac{d\sigma}{d\theta'}\right)^2=\frac{2b^4}{a^2}\frac{\sin 2\beta}{\cos 2\theta'+\cos 2\beta}=\frac{b^4}{a^2}\frac{\sin 2\beta}{\cos^2\beta-\sin^2\theta'},$$

i.e.
$$s+\sigma=\frac{b^2}{a}\sqrt{\sin 2\beta}\int\frac{d\theta'}{\sqrt{\sin^2\beta-\sin^2\theta'}},$$

$$s-\sigma=\frac{b^2}{a}\sqrt{\sin 2\beta}\int\frac{d\theta'}{\sqrt{\cos^2\beta-\sin^2\theta'}}.$$

In these integrals put respectively

$$\sin\theta'=\sin\beta\sin\phi \quad\text{and}\quad \sin\theta'=\cos\beta\sin\psi,$$

and remembering that $\sin 2\beta=\frac{a^2}{b^2}$,

$$s+\sigma=b\int_0^\phi\frac{d\phi}{\sqrt{1-\sin^2\beta\sin^2\phi}},$$

$$s-\sigma=b\int_0^\psi\frac{d\psi}{\sqrt{1-\cos^2\beta\sin^2\psi}};$$

$$\therefore\ \phi = \operatorname{am}\frac{s+\sigma}{b}, \quad \psi = \operatorname{am}\frac{s-\sigma}{b};$$

$$\therefore\ \frac{\sin\theta'}{\sin\beta} = \operatorname{sn}\frac{s+\sigma}{b}, \quad \frac{\sin\theta'}{\cos\beta} = \operatorname{sn}\frac{s-\sigma}{b};$$

$$s+\sigma = b\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\sin\beta}, \sin\beta\right); \quad s-\sigma = b\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\cos\beta}, \cos\beta\right);$$

whence

$$s = \frac{b}{2}\left[\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\sin\beta}, \sin\beta\right) + \operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\cos\beta}, \cos\beta\right)\right],$$

$$\sigma = \frac{b}{2}\left[\operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\sin\beta}, \sin\beta\right) - \operatorname{sn}^{-1}\left(\frac{\sin\theta'}{\cos\beta}, \cos\beta\right)\right],$$

where $\theta' = \frac{1}{2}\sin^{-1}(\sin 2\beta \sin 2\theta)$.

The first of these was established in Art. 597.

603. The Elastica or Lintearia.

This curve is of considerable importance in various branches of Physics. It is (1) the form assumed by a uniform originally straight elastic rod bent into a bow by a bow-string, or by equal thrusts at its extremities, *i.e.* it may take the form *ABC* or

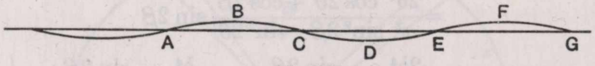

Fig. 147.

ABCDE, etc., according as the string is tied at *A* and *C*, *A* and *E*, etc. This is called an undulating elastica. When the bending is slight, the form is approximately the curve of cosines (E. J. Routh, *Anal. Statics*, vol. ii. p. 281, "Bending of Rods").

(2) It is the form assumed by a flexible thin rectangular sheet, two of whose opposite edges are fixed horizontally at

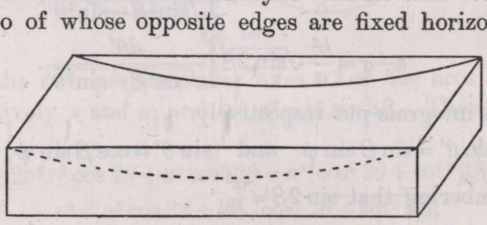

Fig. 148.

the same height, the flexible rectangular sheet forming the base of a rectangular box with vertical sides into which water is poured, the material being supposed impermeable for water and the base fitting the sides so closely as to prevent appreciable escape of water. From this property the second name arises (*lintearius* = made of linen).

(3) The curve also occurs in the case of water drawn up by capillary action against a partially immersed vertical plate.

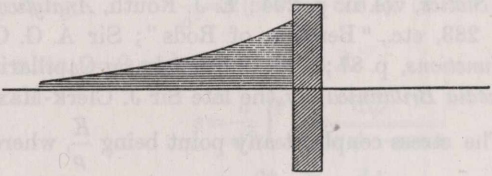

Fig. 149.

The curve may assume various shapes according to the physical circumstances occurring. It may undulate, or there may be any number of complete convolutions forming loops and nodes. Such cases are exhibited in the accompanying figures.

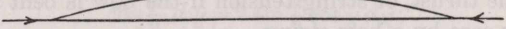

Fig. 150.

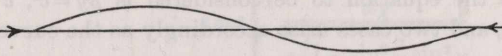

Fig. 151.

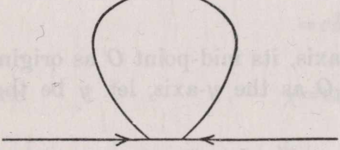

Fig. 152.

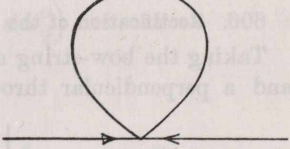

Fig. 153.

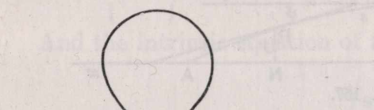

Fig. 154.

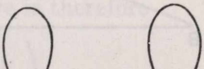

Fig. 155.

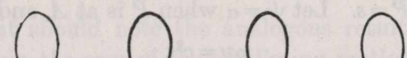

Fig. 156.

604. The determination of the nature of this curve is due to James Bernoulli (1654-1705).

For much detailed information as to the curve and its physical properties, the student may consult W. H. Besant, *Hydromechanics*, pages 168-171, p. 194, p. 201, etc.; G. M. Minchin, *Statics*, vol. ii. p. 204; E. J. Routh, *Analytical Statics*, vol. ii. p. 283, etc., "Bending of Rods"; Sir A. G. Greenhill, *Elliptic Functions*, p. 87; and the article on Capillarity in the *Encyclopaedia Britannica*, by the late Sir J. Clerk-Maxwell.

605. The stress couple at any point being $\frac{K}{\rho}$, where ρ is the radius of curvature and K a certain constant called the flexural rigidity, we have as the geometrical property of the curve,

$$\frac{K}{\rho} = Ty,$$

where y is the ordinate from any point to the line of thrust and T the thrust, or string tension if the bow is bent as in the ordinary case by a bow-string.

Hence the equation to be considered is $\rho y = c^2$, c being a constant, and two cases arise accordingly as the curve is

(1) undulating, (2) nodal.

606. **Rectification of the Bow.**

Taking the bow-string as x-axis, its mid-point O as origin, and a perpendicular through O as the y-axis, let y be the

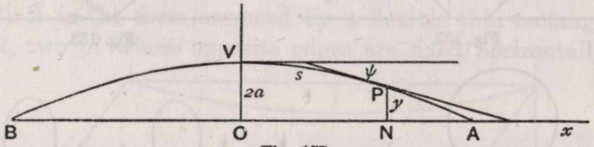

Fig. 157.

ordinate of any point P, and let ψ be the acute angle the tangent makes with the tangent at the vertex V of the arc, and let arc $VP = s$. Let $\psi = a$ when P is at A, and let $OV = 2a$

Then $$\rho y = c^2;$$

$$\therefore \frac{c^2}{\rho} = y.$$

Differentiating, $\frac{c^2}{\rho^2}\frac{d\rho}{d\psi}=-\frac{dy}{d\psi}=+\rho\sin\psi \quad \left(\rho=+\frac{ds}{d\psi}\right)$;

$$\therefore \frac{c^2}{\rho^3}d\rho=\sin\psi\,d\psi,$$

and integrating, $\frac{c^2}{\rho^2}=2(\cos\psi-\cos\alpha)$,

for $\psi=\alpha$ when $y=0$ and $\rho=\infty$, *i.e.* at A.

Hence
$$s=\frac{c}{\sqrt{2}}\int_0^\psi\frac{d\psi}{\sqrt{\cos\psi-\cos\alpha}}$$

$$=\frac{c}{2}\int_0^\psi\frac{d\psi}{\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\psi}{2}}}.$$

Let $\sin\frac{\psi}{2}=\sin\frac{\alpha}{2}\sin\chi$;

$$\therefore \cos\frac{\psi}{2}d\psi=2\sin\frac{\alpha}{2}\cos\chi\,d\chi;$$

$$\therefore s=c\int_0^\chi\frac{d\chi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\chi}}$$

$$=cF\left(\chi,\sin\frac{\alpha}{2}\right)$$

and $\chi=\operatorname{am}\frac{s}{c}$;

$$\therefore \sin\frac{\psi}{2}=\sin\frac{\alpha}{2}\operatorname{sn}\frac{s}{c};\ \text{mod.}\ \sin\frac{\alpha}{2}.$$

And the intrinsic equation of the curve is therefore

$$s=c\operatorname{sn}^{-1}\left(\frac{\sin\frac{\psi}{2}}{\sin\frac{\alpha}{2}},\ \sin\frac{\alpha}{2}\right). \quad \text{..................(1)}$$

The student should note the analogous result in Kinetics in Art. 389, viz. the case of the oscillating motion of a simple circular pendulum. For a comparison of the two results, see Greenhill, *Elliptic Functions*, p. 87.

The ordinate y is given by

$$y=\frac{c^2}{\rho}=2c\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{\psi}{2}}$$

$$=2c\sin\frac{\alpha}{2}\cos\chi=2c\sin\frac{\alpha}{2}\,.\,\mathrm{cn}\frac{s}{c};$$

$$\therefore\ y=2c\sin\frac{\alpha}{2}\mathrm{cn}\left(\frac{s}{c},\sin\frac{\alpha}{2}\right). \qquad\text{.....................(2)}$$

To find the abscissa x, we have

$$\frac{dx}{ds}=\cos\psi;$$

$$\therefore\ \frac{dx}{d\psi}=\cos\psi\frac{ds}{d\psi}$$

and
$$\frac{dx}{d\chi}=\cos\psi\frac{ds}{d\chi}=c\frac{1-2\sin^2\frac{\alpha}{2}\sin^2\chi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\chi}},$$

and adding $\frac{ds}{d\chi}$,

$$\frac{d(x+s)}{d\chi}=2c\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\chi};$$

$$\therefore\ x+s=2c\int_0^\chi\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\chi}\,d\chi,$$

i.e.
$$x=2cE\left(\chi,\sin\frac{\alpha}{2}\right)-s. \qquad\text{.........................(3)}$$

We thus have for the bow, or undulatory elastica, $\rho y=c^2$,

$$\left.\begin{aligned} s&=c\,\mathrm{sn}^{-1}\left(\frac{\sin\frac{\psi}{2}}{\sin\frac{\alpha}{2}},\sin\frac{\alpha}{2}\right),\\ x&=2cE\left(\sin^{-1}\frac{\sin\frac{\psi}{2}}{\sin\frac{\alpha}{2}},\sin\frac{\alpha}{2}\right)-s,\\ y&=2c\sin\frac{\alpha}{2}\mathrm{cn}\left(\frac{s}{c},\sin\frac{\alpha}{2}\right).\end{aligned}\right\}$$

607. **Rectification of the Elastica in the case when there are several Convolutions, viz. the Nodal Elastica.**

Taking the y-axis to pass through a vertex V as before and the line of terminal thrusts as the x-axis and ψ the angle

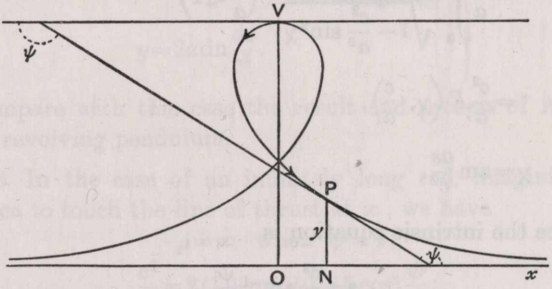

Fig. 158.

which the tangent at P has turned through in passing from V to P, we have again $\frac{c^2}{\rho}=y$.

$$\frac{c^2}{\rho^2}\frac{d\rho}{d\psi}=-\frac{dy}{d\psi}=\rho\sin\psi,$$

$$\frac{c^2}{\rho^3}\,d\rho=\sin\psi\,d\psi,$$

and integrating $\frac{c^2}{\rho^2}=2\cos\psi+\text{a constant}=2\cos\psi+A$, say. We have not, however, in this case, as we had before, any point at which ρ is infinite. Let $2a$ be the ordinate of the vertex. Then at V, $\rho=\frac{c^2}{2a}$.

$$\therefore \text{ putting } \quad \rho=\frac{c^2}{2a}, \quad \text{when} \quad \psi=0, \quad A=\frac{4a^2}{c^2}-2;$$

$$\therefore \frac{c^2}{\rho^2}=\frac{4a^2}{c^2}-2(1-\cos\psi)$$

$$=4\left(\frac{a^2}{c^2}-\sin^2\frac{\psi}{2}\right),$$

$\frac{a}{c}$ being >1, as ρ cannot be ∞ by supposition, and

$$\frac{ds}{d\psi}=\frac{c}{2}\frac{1}{\sqrt{\frac{a^2}{c^2}-\sin^2\frac{\psi}{2}}};$$

$$\therefore\ s=\frac{c^2}{2a}\int_0^{\psi}\frac{d\psi}{\sqrt{1-\frac{c^2}{a^2}\sin^2\frac{\psi}{2}}},\ \text{ or putting } \psi=2\chi,$$

$$=\frac{c^2}{a}\int_0^{\chi}\frac{d\chi}{\sqrt{1-\frac{c^2}{a^2}\sin^2\chi}}\quad\left(\frac{c}{a}<1\right)$$

$$=\frac{c^2}{a}F\left(\chi,\ \frac{c}{a}\right),$$

and $\chi=\operatorname{am}\frac{as}{c^2}$.

Hence the intrinsic equation is

$$s=\frac{c^2}{a}\operatorname{am}^{-1}\frac{\psi}{2}.\ \text{.........................}(1)$$

Also $y=\frac{c^2}{\rho}=2a\sqrt{1-\frac{c^2}{a^2}\sin^2\frac{\psi}{2}}=2a\,\Delta\left(\frac{\psi}{2}\right)=2a\,\Delta(\chi)$;

$$\therefore\ y=2a\operatorname{dn}\frac{as}{c^2}.\ \text{...}(2)$$

Again, $\frac{dx}{ds}=-\cos\psi$,

$$\frac{dx}{d\psi}=-\cos\psi\,\frac{ds}{d\psi},$$

$$\frac{dx}{d\chi}=-\frac{c^2}{a}\frac{(1-2\sin^2\chi)}{\sqrt{1-\frac{c^2}{a^2}\sin^2\chi}}$$

$$=-a\frac{\left[\left(\frac{c^2}{a^2}-2\right)+2\left(1-\frac{c^2}{a^2}\sin^2\chi\right)\right]}{\sqrt{1-\frac{c^2}{a^2}\sin^2\chi}};$$

$$\therefore\ x=a\left(2-\frac{c^2}{a^2}\right)\int_0^{\chi}\frac{d\chi}{\sqrt{1-\frac{c^2}{a^2}\sin^2\chi}}\quad 2a\int_0^{\chi}\sqrt{1-\frac{c^2}{a^2}\sin^2\chi}\,d\chi;$$

$$\therefore\ x=\left(2\frac{a^2}{c^2}-1\right)s-2aE\left(\chi,\ \frac{c}{a}\right).\ \text{..............................}(3)$$

Hence, in the nodal case of $\rho y = c^2$,

$$\left.\begin{aligned} s &= \frac{c^2}{a}\operatorname{am}^{-1}\frac{\psi}{2}, \\ x &= \left(2\frac{a^2}{c^2} - 1\right)s - 2aE\left(\frac{\psi}{2}, \frac{c}{a}\right), \\ y &= 2a\operatorname{dn}\frac{as}{c^2}. \end{aligned}\right\}$$

Compare with this case the result and process of Art. 390 for a revolving pendulum.

608. In the case of an infinitely long rod, imagining the elastica to touch the line of thrust at ∞, we have

$$\rho = \infty \quad \text{when} \quad \psi = \pi,$$

and $$\frac{c^2}{\rho^2} = 2(1 + \cos\psi) = 4\cos^2\frac{\psi}{2};$$

$$\therefore \frac{ds}{d\psi} = \frac{c}{2}\sec\frac{\psi}{2} \quad \text{and} \quad s = c\log\tan\left(\frac{\psi}{4} + \frac{\pi}{4}\right),$$

s being still measured from the vertex.

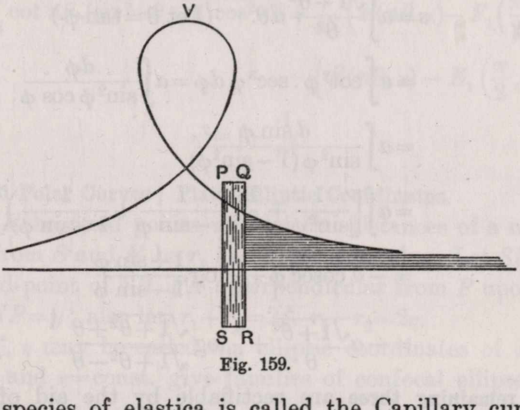

Fig. 159.

This species of elastica is called the Capillary curve (see Besant, *Hydromechanics*, p. 201), the shaded portion in Fig. 159 representing the water raised above the normal level by capillary action due to the presence of a partially immersed vertical plate $PQRS$. In this case $\rho = \frac{c}{2}$ at the vertex, and $c = a$, the modulus of the elliptic functions occurring in the second case becoming unity.

609. **Cotes' Spirals.**

These Spirals are defined by the pedal equation

$$\frac{1}{p^2}=\frac{A}{r^2}+B. \quad \text{(See } \textit{Diff. Calc.}\text{, Art. 454.)}$$

There are five varieties:

(1) $B=0$, an Equiangular Spiral.

(2) $A=1$, in which case B is essentially positive (as $r>p$); the curve is the Reciprocal Spiral (*Diff. Calc.*, Art. 452); and the other three are reducible to the polar forms

$$u=a\sin n\theta, \quad u=a\sinh n\theta \quad \text{and} \quad u=a\cosh n\theta.$$

(1) The rectification of an **equiangular spiral** has been effected in Art. 449, *Diff. Calc.*

(2) In the **reciprocal** spiral $r=\frac{a}{\theta}$, we have $\dot{r}=-\frac{a}{\theta^2}$, and

$$\frac{ds}{d\theta}=\frac{a}{\theta}\sqrt{1+\frac{1}{\theta^2}},$$

giving
$$s=a\int\frac{\sqrt{1+\theta^2}}{\theta^2}d\theta. \quad (\text{Let } \theta=\tan\phi.)$$

$$=a\int\cot^2\phi\,.\sec^3\phi\,d\phi=a\int\frac{d\phi}{\sin^2\phi\cos\phi}$$

$$=a\int\frac{d\sin\phi}{\sin^2\phi(1-\sin^2\phi)}$$

$$=a\int\left[\frac{1}{\sin^2\phi}+\frac{1}{2}\left(\frac{1}{1-\sin\phi}+\frac{1}{1+\sin\phi}\right)\right]d\sin\phi$$

$$=-a\operatorname{cosec}\phi+\frac{a}{2}\log\frac{1+\sin\phi}{1-\sin\phi}$$

$$=-a\frac{\sqrt{1+\theta^2}}{\theta}+\frac{a}{2}\log\frac{\sqrt{1+\theta^2}+\theta}{\sqrt{1+\theta^2}-\theta}.$$

The **remaining three** are rectifiable by the aid of elliptic functions. For instance, take the first, viz. $u=a\sin n\theta$ for the case $n>1$.

$$s=\int\frac{1}{u^2}\sqrt{\left(\frac{du}{d\theta}\right)^2+u^2}\,d\theta \quad \text{(Art. 511)};$$

$$\therefore as=\int_{\frac{\pi}{2n}}^{\theta}\frac{\sqrt{\sin^2 n\theta+n^2\cos^2 n\theta}}{\sin^2 n\theta}\,d\theta,$$

measuring s from the vertex at $\theta=\frac{\pi}{2n}$. (See figure of curve in Art. 387, *Diff. Calc.*)

Let $$n\theta=\phi;$$

$$\therefore\ as=\frac{1}{n}\int_{\frac{\pi}{2}}^{\phi}\frac{\sqrt{n^2-(n^2-1)\sin^2\phi}}{\sin^2\phi}\,d\phi$$

$$=\int_{\frac{\pi}{2}}^{\phi}\frac{\Delta}{\sin^2\phi}\,d\phi,$$

where $\Delta=\sqrt{1-\kappa^2\sin^2\phi}$ and $\kappa^2=\frac{n^2-1}{n^2}$;

$$\therefore\ as=\Big[-\Delta\cot\phi\Big]_{\frac{\pi}{2}}^{\phi}+\int_{\frac{\pi}{2}}^{\phi}\cot\phi\,\frac{-\kappa^2\sin\phi\cos\phi}{\Delta}\,d\phi$$

$$=-\Delta\cot\phi+\int_{\frac{\pi}{2}}^{\phi}\frac{(1-\kappa^2)-(1-\kappa^2\sin^2\phi)}{\Delta}\,d\phi$$

$$=-\Delta\cot\phi+(1-\kappa^2)\int_{\frac{\pi}{2}}^{\phi}\frac{d\phi}{\Delta}-\int_{\frac{\pi}{2}}^{\phi}\Delta\,d\phi$$

$$=-\frac{1}{n}\cot n\theta\sqrt{\sin^2 n\theta+n^2\cos^2 n\theta}+\frac{1}{n^2}\left\{F(n\theta,\kappa)-F_1\left(\frac{\pi}{2},\kappa\right)\right\}$$
$$-\left\{E(n\theta,\kappa)-E_1\left(\frac{\pi}{2},\kappa\right)\right\},$$

where $$\kappa^2=1-\frac{1}{n^2}.$$

610. **Bi-Polar Curves; Plane Elliptic Coordinates.**

Let S, H be fixed points, and let the distances of a moving point P from S and H be r_1 and r_2 respectively. Let $SH=2c$; O the mid-point of SH, PN a perpendicular from P upon SH; $ON=x$, $NP=y$; also let $r_1+r_2=2\xi$, $r_1-r_2=2\eta$.

Then ξ, η may be called the elliptic coordinates of P; for ξ=const. and η=const. give families of confocal ellipses and hyperbolae.

Let Δ be the area of the triangle SPH.

Then

$$16\Delta^2=(2c+r_1+r_2)(-2c+r_1+r_2)(2c-r_1+r_2)(2c+r_1-r_2),$$

i.e. $\Delta^2=(\xi^2-c^2)(c^2-\eta^2)$,

where ξ is necessarily $\not< c$ and $\eta\not> c$.

Hence $$cy=\sqrt{(\xi^2-c^2)(c^2-\eta^2)}.$$

Also, if m be the length of the median OP,

$$m^2+c^2=\frac{r_1{}^2+r_2{}^2}{2}=\xi^2+\eta^2;$$

$$\therefore\ x^2=m^2-y^2=(\xi^2+\eta^2-c^2)-\frac{(\xi^2-c^2)(c^2-\eta^2)}{c^2}=\frac{\xi^2\eta^2}{c^2};$$

$$\therefore\ cx=\xi\eta.$$

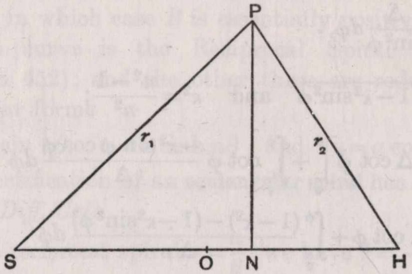

Fig. 160.

Thus the Cartesian coordinates of P are given by

$$cx=\xi\eta,\quad cy=\sqrt{\xi^2-c^2}\sqrt{c^2-\eta^2};\ \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\therefore\ c\,dx=\eta\,d\xi+\xi\,d\eta,$$

$$c\,dy=\xi\sqrt{\frac{c^2-\eta^2}{\xi^2-c^2}}\,d\xi-\eta\sqrt{\frac{\xi^2-c^2}{c^2-\eta^2}}\,d\eta.$$

And therefore, if ds be an element of the arc of the Bi-Polar curve traced by P for any relation between r_1 and r_2,

$$c^2ds^2=\left(\eta^2+\xi^2\frac{c^2-\eta^2}{\xi^2-c^2}\right)d\xi^2+\left(\xi^2+\eta^2\frac{\xi^2-c^2}{c^2-\eta^2}\right)d\eta^2$$

$$=c^2(\xi^2-\eta^2)\left(\frac{d\xi^2}{\xi^2-c^2}+\frac{d\eta^2}{c^2-\eta^2}\right),$$

and $$s=\int\sqrt{\xi^2-\eta^2}\sqrt{\frac{d\xi^2}{\xi^2-c^2}+\frac{d\eta^2}{c^2-\eta^2}}.\ \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

If we put $\qquad \xi=c\cosh v,\quad \eta=c\sin u,$

we have $$s=c\int\sqrt{\cosh^2 v-\sin^2 u}\sqrt{du^2+dv^2}.\ \ldots\ldots\ldots\ldots(3)$$

Moreover, $\qquad x=c\cosh v\sin u,$

$\qquad y=c\sinh v\cos u,$

and $\qquad x+\iota y=c\sin(u+\iota v),$

the transformation used in Art. 590 for the rectification of the central conics.

The (u, v) system and the (ξ, η) system are therefore connected, and either may be regarded as "elliptic" coordinates. Moreover, we have a definite interpretation of u, v as used in Art. 590, viz.

$$v=\cosh^{-1}\frac{r_1+r_2}{2c}, \quad u=\sin^{-1}\frac{r_1-r_2}{2c},$$

and they are thus expressed in terms of the bi-polar determination of a point.

Ex. Employ Formula (3) in the case

$$\sin u = m\cosh v.$$

To what curve does this equation refer?

611. If we wish to express the result of Art. 610 in terms of the original radii vectores r_1, r_2, we have

$$\xi^2-\eta^2=r_1r_2,$$

and
$$\frac{d\xi^2}{\xi^2-c^2}+\frac{d\eta^2}{c^2-\eta^2}=\frac{(dr_1+dr_2)^2}{(r_1+r_2)^2-4c^2}+\frac{(dr_1-dr_2)^2}{4c^2-(r_1-r_2)^2}$$

$$=\frac{(dr_1+dr_2)^2[4c^2-(r_1-r_2)^2]+(dr_1-dr_2)^2[(r_1+r_2)^2-4c^2]}{(2c+r_1+r_2)(-2c+r_1+r_2)(2c-r_1+r_2)(2c+r_1-r_2)}$$

$$=4\,\frac{r_1r_2(dr_1{}^2+dr_2{}^2)+dr_1dr_2(a^2-r_1{}^2-r_2{}^2)}{16\sigma(\sigma-a)(\sigma-r_1)(\sigma-r_2)},$$

where $2c=a$ and $2\sigma=a+r_1+r_2$;

$$\therefore\ s=\frac{1}{2}\int\sqrt{r_1r_2}\,\frac{\sqrt{r_1r_2(dr_1{}^2+dr_2{}^2)+(a^2-r_1{}^2-r_2{}^2)dr_1dr_2}}{\sqrt{\sigma(\sigma-a)(\sigma-r_1)(\sigma-r_2)}}. \quad (4)$$

List of Well-known Bi-Polar Equations.

612. The principal bi-polar cases of well-known curves are:

Name.	Bi-Polar Equation.	Form of Equation in Elliptic Coordinates.
1. Ellipse	$r_1+r_2=2a$	$\xi=a$
2. Hyperbola	$r_1-r_2=2a$	$\eta=a$
3. Cartesian oval	$l_1r_1+l_2r_2=n$	$\frac{\xi}{a}+\frac{\eta}{b}=1$
4. Circle	$r_1=\kappa r_2$	$\eta=m\xi$
5. Circle	$r_1{}^2+r_2{}^2=\kappa^2$	$\xi^2+\eta^2=\frac{\kappa^2}{2}$
6. Straight line	$r_1{}^2-r_2{}^2=\kappa^2$	$\xi\eta=\frac{\kappa^2}{4}$
7. Cassinian oval	$r_1r_2=\kappa^2$	$\xi^2-\eta^2=\kappa^2$

613. Ex. 1. Rectify the ellipse $r_1+r_2=2a$.

Here $$\xi=a,\quad d\xi=0.$$

$$s=\int_0^\eta \sqrt{\frac{a^2-\eta^2}{c^2-\eta^2}}\,d\eta \quad (\eta \text{ increasing}) \quad (\eta<c<a)$$

$$=aE\left(\theta,\frac{c}{a}\right), \text{ where } \eta=c\sin\theta \quad \text{(cf. Art. 567).}$$

Ex. 2. Rectify the hyperbola $r_1-r_2=2a$.

Here $$\eta=a,\quad d\eta=0.$$

$$s=\int_c^\xi \sqrt{\frac{\xi^2-a^2}{\xi^2-c^2}}\,d\xi \quad (\xi>c>a) \quad \text{(cf. Art 388, Case 6),}$$

$$=\tan\omega\sqrt{c^2-a^2\sin^2\omega}+\frac{c^2-a^2}{c}F\left(\omega,\frac{a}{c}\right)-cE\left(\omega,\frac{a}{c}\right),$$

where $$\frac{\xi^2-c^2}{\xi^2-a^2}=\sin^2\omega \quad \text{(cf. Art. 588).}$$

Ex. 3. Consider the case of the Bernoulli's Lemniscate $r_1r_2=c^2$.

Here $$\xi^2-\eta^2=c^2 \quad\text{and}\quad \frac{d\xi}{\eta}=\frac{d\eta}{\xi}.$$

Hence $$\frac{d\xi^2}{\xi^2-c^2}+\frac{d\eta^2}{c^2-\eta^2}=\frac{d\eta^2}{\eta^2+c^2}+\frac{d\eta^2}{c^2-\eta^2}=\frac{2c^2d\eta^2}{c^4-\eta^4};$$

$$\therefore\ s=c^2\sqrt{2}\int_\eta^c \frac{d\eta}{\sqrt{c^4-\eta^4}} \quad \text{(cf. Art. 388, Case 2),}$$

$$=c\,\mathrm{cn}^{-1}\left(\frac{\eta}{c},\frac{1}{\sqrt{2}}\right) \quad \text{(cf. } \textit{Diff. Calc.}\text{, Art. 458, and } \textit{Int. Calc.}\text{, Art. 592).}$$

614. **Use of Bi-Angular Coordinates.**

It is sometimes desirable to express an element of arc of a bi-polar curve in terms of the bi-angular coordinates θ_1, θ_2 which r_2, r_1 respectively make with the line joining the poles.

Let $f(r_1, r_2)=$const. be the bi-polar equation of a curve, c the distance between the poles S, H. Let the angles of the triangle SHP be θ_2, θ_1, θ_3; so that r_1, θ_2 are the polar coordinates of P with SH for initial line, r_2, θ_1 the polar coordinates with HS for initial line. Let the normal PG cut the line SH at G and the circumcircle of SHP at Q. Let

$$S\hat{P}Q=\chi_1,\quad H\hat{P}Q=\chi_2,$$

and let $$SQ=\rho_2,\quad HQ=\rho_1,\quad PQ=N.$$

Then $$r_1\frac{d\theta_2}{ds}=\cos\chi_1=\frac{c^2+\rho_1{}^2-\rho_2{}^2}{2c\rho_1},$$

$$-r_2\frac{d\theta_1}{ds}=\cos\chi_2=\frac{c^2+\rho_2{}^2-\rho_1{}^2}{2c\rho_2}.$$

Hence multiplying by ρ_1, ρ_2 respectively, and then adding and subtracting,

$$\rho_1 r_1 \frac{d\theta_2}{ds} - \rho_2 r_2 \frac{d\theta_1}{ds} = c, \quad \text{..........................(i)}$$

$$\rho_1 r_1 \frac{d\theta_2}{ds} + \rho_2 r_2 \frac{d\theta_1}{ds} = \frac{\rho_1^2 - \rho_2^2}{c}. \quad \text{.................(ii)}$$

Now *PSQH* being cyclic,

$$\rho_1 r_1 + \rho_2 r_2 = Nc.$$

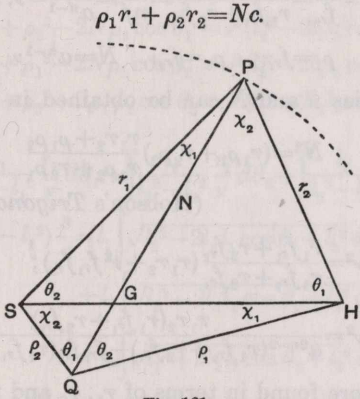

Fig. 161.

Hence these results may be respectively written

$$c\,ds = (Nc - \rho_2 r_2)\,d\theta_2 - (Nc - \rho_1 r_1)\,d\theta_1 \quad \text{...........(iii)}$$
$$= \rho_1 r_1\,d\theta_1 - \rho_2 r_2\,d\theta_2 - Nc(d\theta_1 - d\theta_2),$$

and $$\frac{\rho_2^2 - \rho_1^2}{c} ds = \rho_1 r_1\,d\theta_1 + \rho_2 r_2\,d\theta_2 + Nc\,d\theta_3, \quad \text{.........(iv)}$$

for $$d\theta_1 + d\theta_2 + d\theta_3 = 0.$$

The last equation (iv) is due to Mr. Roberts (*vide* Professor Williamson's *Integral Calculus*, p. 501, for a somewhat different proof).

Again, in travelling along the curve $f(r_1, r_2) = \text{const.}$,

$$f_{r_1}\,dr_1 + f_{r_2}\,dr_2 = 0 \quad \left(\text{where } f_{r_1} \text{ stands for } \frac{\partial f}{\partial r_1}, \text{ etc.}\right),$$

i.e. $$f_{r_1} \sin\chi_1 - f_{r_2} \sin\chi_2 = 0.$$

Hence (*a*) $$\frac{SG}{HG} = \frac{r_1 \sin\chi_1}{r_2 \sin\chi_2} = \frac{r_1 f_{r_2}}{r_2 f_{r_1}}$$

(see *Diff. Calc.*, p. 181, Ex. 32);

(*b*) $$\frac{\rho_2}{\rho_1} = \frac{\sin\chi_1}{\sin\chi_2} = \frac{f_{r_2}}{f_{r_1}}.$$

In cases in which $f(r_1, r_2)$ is homogeneous in r_1 and r_2 and of degree n, and if for convenience we write the constant as

$$c\frac{a^{n-1}}{n}, \quad \text{so that} \quad f(r_1, r_2)=c\frac{a^{n-1}}{n},$$

we have, by the theorems of Ptolemy and Euler,

$$\frac{\rho_1}{f_{r_1}}=\frac{\rho_2}{f_{r_2}}=\frac{r_1\rho_1+r_2\rho_2}{r_1 f_{r_1}+r_2 f_{r_2}}=\frac{Nc}{nf}=\frac{N}{a^{n-1}}=\nu, \text{ say.}$$

Then $\qquad \rho_1=f_{r_1}\nu, \quad \rho_2=f_{r_2}\nu, \quad N=a^{n-1}\nu.$

The quantities ν and N can be obtained in terms of r_1, r_2 as follows:

$$N^2=(r_1\rho_1+r_2\rho_2)\frac{r_1r_2+\rho_1\rho_2}{r_1\rho_2+r_2\rho_1}$$

(Hobson's *Trigonometry*, p. 203);

$$\therefore\ a^{2n-2}\nu^2=\frac{r_1 f_{r_1}+r_2 f_{r_2}}{r_1 f_{r_2}+r_2 f_{r_1}}(r_1r_2+\nu^2 f_{r_1}f_{r_2});$$

$$\therefore\ \nu^2=\frac{r_1r_2(r_1 f_{r_1}+r_2 f_{r_2})}{a^{2n-2}(r_1 f_{r_2}+r_2 f_{r_1})-f_{r_1}f_{r_2}(r_1 f_{r_1}+r_2 f_{r_2})},$$

and ν is therefore found in terms of r_1, r_2 and the constant a.

And as $\qquad \rho_1=\nu f_{r_1}, \quad \rho_2=\nu f_{r_2}, \quad N=a^{n-1}\nu,$

ρ_1, ρ_2, N are also known in terms of r_1, r_2.

Also, since $\qquad \dfrac{r_1}{\sin\theta_1}=\dfrac{r_2}{\sin\theta_2}=\dfrac{c}{\sin(\theta_1+\theta_2)}$

and $\qquad f(r_1, r_2)=c\dfrac{a^{n-1}}{n},$

we have theoretically the means of expressing r_1, r_2, ρ_1, ρ_2 and N either in terms of θ_1 or in terms of θ_2, as required.

Hence the rectification of the curve depends upon the integration of either of the formulae

$$cs=\int\rho_1r_1\,d\theta_2-\int\rho_2r_2\,d\theta_1$$

or
$$\frac{s}{c}=\int\frac{\rho_1r_1}{\rho_1{}^2-\rho_2{}^2}\,d\theta_2+\int\frac{\rho_2r_2}{\rho_1{}^2-\rho_2{}^2}\,d\theta_1,$$

or
$$=\int\frac{\rho_2r_2}{\rho_2{}^2-\rho_1{}^2}\,d\theta_2+\int\frac{\rho_1r_1}{\rho_2{}^2-\rho_1{}^2}\,d\theta_1+\int\frac{Nc}{\rho_2{}^2-\rho_1{}^2}\,d\theta_3.$$

615. **Rectification of a Cartesian Oval. Genocchi's Result.**

The last form was used by Mr. Roberts in a proof of Prof. Angelo Genocchi's Theorem, that the arc of a Cartesian oval can be expressed in terms of three elliptic arcs.

Thus, for this oval, viz. $l_1 r_1 + l_2 r_2 = c l_3$,

we have $$\frac{\rho_1}{l_1} = \frac{\rho_2}{l_2} = \frac{Nc}{cl_3} = \frac{N}{l_3} = \nu, \text{ say,}$$

and $$r_1^2 = N^2 + \rho_2^2 - 2N\rho_2 \cos\theta_1 = \nu^2(l_2^2 - 2l_2 l_3 \cos\theta_1 + l_3^2),$$
$$r_2^2 = N^2 + \rho_1^2 - 2N\rho_1 \cos\theta_2 = \nu^2(l_3^2 - 2l_3 l_1 \cos\theta_2 + l_1^2),$$
$$c^2 = \rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \cos\theta_3 = \nu^2(l_1^2 + 2l_1 l_2 \cos\theta_3 + l_2^2).$$

Hence
$$\frac{s}{c} = \int \frac{l_1}{l_2^2 - l_1^2} \frac{r_1}{\nu} d\theta_1 + \int \frac{l_2}{l_2^2 - l_1^2} \frac{r_2}{\nu} d\theta_2 + \int \frac{l_3}{l_2^2 - l_1^2} \frac{c}{\nu} d\theta_3,$$

and $$(l_2^2 - l_1^2)\frac{s}{c} = l_1 \int \sqrt{l_2^2 - 2l_2 l_3 \cos\theta_1 + l_3^2}\, d\theta_1$$
$$+ l_2 \int \sqrt{l_3^2 - 2l_3 l_1 \cos\theta_2 + l_1^2}\, d\theta_2$$
$$+ l_3 \int \sqrt{l_1^2 + 2l_1 l_2 \cos\theta_3 + l_2^2}\, d\theta_3.$$

And these are the integrations required in the rectification of ellipses. This is Genocchi's result.

For a full description of the elements of these ellipses and for many other important properties of the Cartesian Ovals, the student should consult Professor Williamson's *Differential Calculus*, pp. 375–382, and *Integral Calculus*, pp. 239–243.

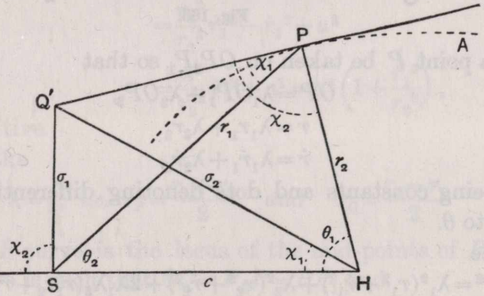

Fig. 162.

616. In a similar manner, if the *tangent* to the curve cut the circumcircle of the triangle SPH at a point Q' whose

bi-polar coordinates are σ_1, σ_2, and T be the length of the tangent PQ', which makes angles χ_1, χ_2 with r_1 and r_2, we have

$$\frac{dr_1}{ds} = -\cos\chi_1 = -\frac{c^2+\sigma_2^{\ 2}-\sigma_1^{\ 2}}{2c\sigma_2}$$

$$\frac{dr_2}{ds} = -\cos\chi_2 = \frac{c^2+\sigma_1^{\ 2}-\sigma_2^{\ 2}}{2c\sigma_1};$$

$$\therefore\ \sigma_2\frac{dr_1}{ds}+\sigma_1\frac{dr_2}{ds}=\frac{\sigma_1^{\ 2}-\sigma_2^{\ 2}}{c} \quad \text{and} \quad \sigma_1\frac{dr_2}{ds}-\sigma_2\frac{dr_1}{ds}=c;$$

$$\left.\begin{aligned} \therefore\ cs &= \int\sigma_1\,dr_2-\int\sigma_2\,dr_1 \\ \text{and}\qquad \frac{s}{c} &= \int\frac{\sigma_2}{\sigma_1^{\ 2}-\sigma_2^{\ 2}}\,dr_1+\int\frac{\sigma_1}{\sigma_1^{\ 2}-\sigma_2^{\ 2}}\,dr_2. \end{aligned}\right\}$$

617. A General Theorem.

Let there be two given curves

$$r_1 = f_1(\theta), \quad r_2 = f_2(\theta),$$

and let OP_2P_1 be a radius vector from the origin cutting these curves at P_2 and P_1.

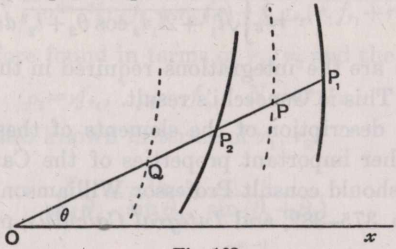

Fig. 163.

Let a point P be taken on OP_2P_1 so that

$$OP = \lambda_1 OP_1 + \lambda_2 OP_2,$$

i.e. $$r = \lambda_1 r_1 + \lambda_2 r_2$$

and $$\dot{r} = \lambda_1\dot{r}_1 + \lambda_2\dot{r}_2,$$

λ_1, λ_2 being constants and dots denoting differentiation with regard to θ.

Hence

$$r^2+\dot{r}^2 = \lambda_1^{\ 2}(r_1^{\ 2}+\dot{r}_1^{\ 2})+\lambda_2^{\ 2}(r_2^{\ 2}+\dot{r}_2^{\ 2})+2\lambda_1\lambda_2(r_1r_2+\dot{r}_1\dot{r}_2). \quad \ldots\ldots(1)$$

Let s_1, s_2, s_P be corresponding arcs of the three curves.

Now $$(r_1r_2+\dot{r}_1\dot{r}_2)^2+(r_1\dot{r}_1-r_2\dot{r}_2)^2 = (r_1^{\ 2}+\dot{r}_2^{\ 2})(r_2^{\ 2}+\dot{r}_1^{\ 2})$$

and $$(r_1r_2+\dot{r}_1\dot{r}_2)^2+(r_1\dot{r}_2-r_2\dot{r}_1)^2 = (r_1^{\ 2}+\dot{r}_1^{\ 2})(r_2^{\ 2}+\dot{r}_2^{\ 2}).$$

Hence there are two cases of simplication, viz.

(A) when $r_1\dot{r}_1 - r_2\dot{r}_2 = 0$; (B) when $r_1\dot{r}_2 - r_2\dot{r}_1 = 0$.

Case (A) arises when the given curves are so related that

$$r_1^2 - r_2^2 = \text{const.} = a^2.$$

Case (B) arises when $\qquad \dfrac{\dot{r}_1}{r_1} = \dfrac{\dot{r}_2}{r_2}$,

i.e. $\dfrac{r_1}{r_2}$ = constant and the original curves similar and similarly situated with regard to O.

In case (A)

$$(r_1r_2 + \dot{r}_1\dot{r}_2)^2 = (r_2^2 + a^2 + \dot{r}_2^2)(r_1^2 - a^2 + \dot{r}_1^2)$$
$$= (\dot{s}_1^2 - a^2)(\dot{s}_2^2 + a^2)$$

and $\qquad \dot{s}_P^2 = \lambda_1^2\dot{s}_1^2 + \lambda_2^2\dot{s}_2^2 + 2\lambda_1\lambda_2\sqrt{(\dot{s}_1^2 - a^2)(\dot{s}_2^2 + a^2)}$.

If we take $\qquad \lambda_1 = \lambda_2 = \lambda$, say,

$$\dot{s}_P^2 = \lambda^2[\dot{s}_1^2 - a^2 + \dot{s}_2^2 + a^2 + 2\sqrt{\dot{s}_1^2 - a^2}\sqrt{\dot{s}_2^2 + a^2}]$$

and $\qquad \dot{s}_P = \lambda[\sqrt{\dot{s}_2^2 + a^2} + \sqrt{\dot{s}_1^2 - a^2}]$.

If another point Q be taken on the same radius vector such that $\qquad \lambda_1 = -\lambda_2 = \lambda$, say,

then $\qquad \dot{s}_Q = \lambda[\sqrt{\dot{s}_2^2 + a^2} - \sqrt{\dot{s}_1^2 - a^2}]$.

The radicals are placed in this order because

$$\dot{s}_2^2 + a^2 > \dot{s}_1^2 - a^2,$$

as may be seen as follows:

$$\dot{s}_2^2 + a^2 - (\dot{s}_1^2 - a^2) = (r_2^2 + \dot{r}_2^2) - (r_1^2 + \dot{r}_1^2) + 2a^2$$
$$= \dot{r}_2^2 - \dot{r}_1^2 + a^2$$
$$= \frac{r_1^2}{r_2^2}\dot{r}_1^2 - \dot{r}_1^2 + a^2$$
$$= \frac{a^2}{r_2^2}\dot{r}_1^2 + a^2 = a^2\left(1 + \frac{\dot{r}_1^2}{r_2^2}\right),$$

and is positive.

If we take

$$\lambda = \tfrac{1}{2}, \ \textit{i.e.}\ r_P = \frac{r_1 + r_2}{2} \quad \text{and} \quad r_Q = \frac{r_1 - r_2}{2}$$

then the P-curve is the locus of the mid-points of P_1P_2, and the Q-curve is such that $OQ = P_2P = PP_1$ and,

$$r_P \cdot r_Q = \frac{r_1 + r_2}{2} \cdot \frac{r_1 - r_2}{2} = \frac{a^2}{4},$$

so that the P and Q loci are inverse to each other.

For such derived loci we therefore have

$$s_P+s_Q=\int\sqrt{\dot{s}_2{}^2+a^2}\,d\theta,$$

$$s_P-s_Q=\int\sqrt{\dot{s}_1{}^2-a^2}\,d\theta,$$

and when these integrals can be found, s_P and s_Q can be found.

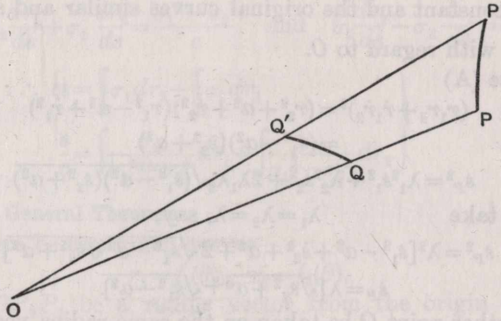

Fig. 164.

Again, the P and Q loci being inverse to each other, the constant of inversion being $\frac{a}{2}$,

$$ds_Q=\left(\frac{a}{2}\right)^2\frac{ds_P}{r_P{}^2}=\frac{r_1{}^2-r_2{}^2}{(r_1+r_2)^2}\,ds_P=\frac{r_1-r_2}{r_1+r_2}\,ds_P;$$

$$\therefore\ ds_P+ds_Q=\frac{2r_1}{r_1+r_2}\,ds_P=\frac{2r_1}{r_1-r_2}\,ds_Q,$$

$$ds_P-ds_Q=\frac{2r_2}{r_1+r_2}\,ds_P=\frac{2r_2}{r_1-r_2}\,ds_Q;$$

$$\therefore\ s_P=\tfrac{1}{2}\int\left(1+\frac{r_2}{r_1}\right)\sqrt{\dot{s}_2{}^2+a^2}\,d\theta=\tfrac{1}{2}\int\left(\frac{r_1}{r_2}+1\right)\sqrt{\dot{s}_1{}^2-a^2}\,d\theta,$$

$$s_Q=\tfrac{1}{2}\int\left(1-\frac{r_2}{r_1}\right)\sqrt{\dot{s}_2{}^2+a^2}\,d\theta=\tfrac{1}{2}\int\left(\frac{r_1}{r_2}-1\right)\sqrt{\dot{s}_1{}^2-a^2}\,d\theta.$$

618. In Case (B), $\quad r_1r_2+\dot{r}_1\dot{r}_2=\dot{s}_1\dot{s}_2;$

whence $\quad \dot{s}_P=\lambda_1\dot{s}_1+\lambda_2\dot{s}_2$

and $\quad s_P=\lambda_1 s_1+\lambda_2 s_2;$

but as the curves are then similar this is an obvious fact, and this part of the investigation does not render any new information.

619. A Useful Case.

In Case (A), it may happen that the derived curves are different branches of the same curve locus,

$$r^2-bF(\theta)r+\frac{a^2}{4}=0, \text{ say,}$$

whose roots are $\quad r_P, r_Q \quad$ and $\quad r_Pr_Q=\frac{a^2}{4},$

and therefore $\quad r_1{}^2-r_2{}^2=4r_Pr_Q=a^2.$

In this case the two branches of the curve are

$$r=\frac{bF(\theta)\pm\sqrt{b^2\{F(\theta)\}^2-a^2}}{2},$$

which are inverse to each other with regard to the pole, the constant of inversion being $\frac{a}{2}$.

And the "given" curves from which this curve is derived are

$$r_1=bF(\theta),$$
$$r_2=\sqrt{b^2\{F(\theta)\}^2-a^2}.$$

And if $\dot{s}_1$ and $\dot{s}_2$ be the differential coefficients of the arcs of these curves, the arcs of the derived P and Q curves are given by

$$2s_P=\int\sqrt{\dot{s}_2{}^2+a^2}\,d\theta+\int\sqrt{\dot{s}_1{}^2-a^2}\,d\theta,$$

$$2s_Q=\int\sqrt{\dot{s}_2{}^2+a^2}\,d\theta-\int\sqrt{\dot{s}_1{}^2-a^2}\,d\theta.$$

620. Ex. 1. Consider the rectification of the curve

$$4(x^2+y^2)(x-a)+a^2x=0.$$

Putting this into Polars,

$$r^2-ar\sec\theta+\frac{a^2}{4}=0,$$

$$r=\frac{a\sec\theta\pm a\tan\theta}{2}.$$

The original curves from which this is derived are obviously

$$r_1=a\sec\theta$$

and $\quad r_2=a\tan\theta,$

the first being a straight line and incidentally an asymptote of the curve we wish to rectify.

The P and Q curves are branches of the same curve and inverse to each other. If N be the node on this curve (see Fig. 165) and A the point where the asymptote $x=2a$ cuts the x-axis, the several arcs are $AP_1=s_1$; $OP_2=s_2$, $NP=s_P$, $NQ=s_Q$.

Now $\dot{s}_1 = a\sec^2\theta, \quad \dot{s}_2{}^2 = a^2(\tan^2\theta + \sec^4\theta),$

$$\dot{s}_1{}^2 - a^2 = a^2\tan^2\theta(\sec^2\theta + 1),$$

$$\dot{s}_2{}^2 + a^2 = a^2\sec^2\theta(\sec^2\theta + 1);$$

$$\therefore \sqrt{\dot{s}_1{}^2 - a^2} = a\frac{\sin\theta}{\cos^2\theta}\sqrt{1+\cos^2\theta},$$

$$\sqrt{\dot{s}_2{}^2 + a^2} = a\sec^2\theta\sqrt{1+\cos^2\theta}.$$

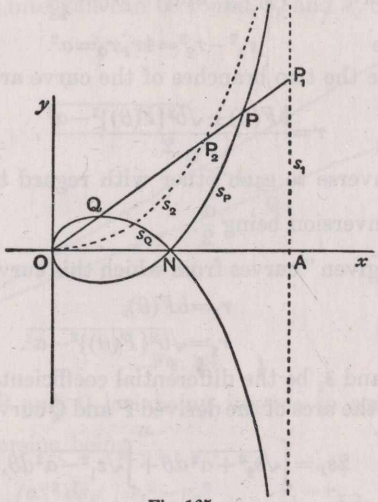

Fig. 165.

Now $\displaystyle\int \frac{\sin\theta}{\cos^2\theta}\sqrt{1+\cos^2\theta}\,d\theta$

$$= \sec\theta\sqrt{1+\cos^2\theta} + \int \frac{\sin\theta}{\sqrt{1+\cos^2\theta}}\,d\theta$$

$$= \sec\theta\sqrt{1+\cos^2\theta} - \sinh^{-1}(\cos\theta)$$

$$= \sqrt{\sec^2\theta + 1} - \sinh^{-1}(\cos\theta),$$

and $\displaystyle\int \sec^2\theta\sqrt{1+\cos^2\theta}\,d\theta = \tan\theta\sqrt{1+\cos^2\theta} + \int \frac{1-\cos^2\theta}{\sqrt{1+\cos^2\theta}}\,d\theta$

$$= \tan\theta\sqrt{1+\cos^2\theta} + \int\left(\frac{2}{\sqrt{1+\cos^2\theta}} - \sqrt{1+\cos^2\theta}\right)d\theta$$

$$= \tan\theta\sqrt{1+\cos^2\theta} + \sqrt{2}\int\frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} - \sqrt{2}\int\sqrt{1-\tfrac{1}{2}\sin^2\theta}\,d\theta$$

$$= \sin\theta\sqrt{\sec^2\theta+1} + \sqrt{2}\,F\left(\theta, \frac{1}{\sqrt{2}}\right) - \sqrt{2}\,E\left(\theta, \frac{1}{\sqrt{2}}\right).$$

Hence

$$\text{arc } NP + \text{arc } NQ = a\left[\sin\theta\sqrt{\sec^2\theta+1} + \sqrt{2}\,F\left(\theta, \frac{1}{\sqrt{2}}\right) - \sqrt{2}\,E\left(\theta, \frac{1}{\sqrt{2}}\right)\right],$$

$$\text{arc } NP - \text{arc } NQ = a\left[\sqrt{\sec^2\theta+1} - \sinh^{-1}\cos\theta\right].$$

Thus arc NP and arc NQ are found by addition and subtraction. It is to be noted in this case, that although each separate arc NP, NQ requires for its expression the elliptic integrals of the first and second kinds, their difference is free from these functions, and expressible in terms of trigonometric and logarithmic functions.

Ex. 2. As a further example, consider the "derived" curves to be the branches of the Cartesian oval

$$r^2-(A+B\cos\theta)r+\frac{a^2}{4}=0.$$

The roots being r_P and r_Q, we have

$$r_1=r_P+r_Q=A+B\cos\theta,$$

$$r_2=r_P-r_Q=\sqrt{(A+B\cos\theta)^2-a^2},$$

and these are the "original" curves from which the Cartesian ovals are derived, the first being a Limaçon.

$$\dot{s}_1{}^2=r_1{}^2+\dot{r}_1{}^2=(A+B\cos\theta)^2+B^2\sin^2\theta$$

$$=A^2+2AB\cos\theta+B^2,$$

$$s_P-s_Q=\int\sqrt{A^2+B^2-a^2+2AB\cos\theta}\,d\theta. \quad \text{(See Art. 573.)}$$

Hence the difference between corresponding portions of the inner and outer loops of the curve

$$r^2-(A+B\cos\theta)r+\frac{a^2}{4}=0$$

can be expressed as the corresponding arc of a certain ellipse.

[This polar equation to the Cartesian oval is an ordinary conversion to polars, retaining one of the poles as origin, of $lr+mr'=n$, writing $r^2+c^2-2rc\cos\theta$ for r'^2 and performing the rationalization.]

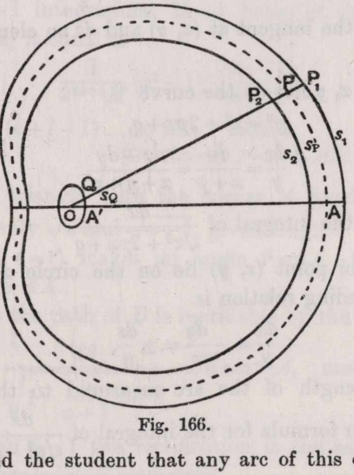

Fig. 166.

We may remind the student that any arc of this curve has already been expressed in terms of three elliptic arcs (Art. 615).

The arcs $s_P=AP$, $s_Q=A'Q$ to which the integration refers are shown in the figure.

We may construct the ovals as follows. Having drawn the limaçon $r_1=A+B\cos\theta$ as explained in Art. 424, *Diff. Calc.*, take any radius vector OP_1, and on OP_1 for diameter construct a circle. Take centre P_1 and radius a and draw a second circle cutting the first at R. Then with centre O and radius OR draw a circle cutting OP_1 at P_2.

Then
$$OP_1=A+B\cos\theta,$$
$$OP_2=\sqrt{(A+B\cos\theta)^2-a^2}.$$

Bisect P_1P_2 at P and make $OQ=PP_1$; then the points P and Q are points on the Cartesian oval.

MISCELLANEOUS PROBLEMS.

1. Prove that the three equations
$$x=c\log\sec\psi,\quad y=c(\tan\psi-\psi),\quad s=c(\sec\psi-1),$$
represent one and the same curve. [I. C. S., 1893.]

2. Find the area of the curve
$$r_1r_2=b^2,$$
considering all cases which may arise.

3. Prove that the value of the integral
$$\int p\,xy\left(\frac{x}{a^3}+\frac{y}{b^3}\right)^2 ds,$$
taken round the ellipse $x^2/a^2+y^2/b^2=1$, is $\frac{\pi}{2}$, p denoting the central perpendicular on the tangent at (x, y) and ds an element of arc.

[I. C. S., 1912.]

4. If the point x, y lies on the curve
$$y^2=x^2+2px+q,$$
prove that
$$\frac{dx}{y}=\frac{dy}{x+p}=\frac{dx+dy}{x+y+p},$$
and hence obtain the integral of $\dfrac{dx}{\sqrt{x^2+2px+q}}$.

If, however, the point (x, y) lie on the circle $x^2+y^2=a^2$, show that the corresponding relation is
$$\frac{dx}{y}=-\frac{dy}{x}=\pm\frac{ds}{a},$$
where s is the length of the arc measured to the point (x, y). Deduce the known formula for the integral of $\dfrac{dx}{\sqrt{a^2-x^2}}$.

[I. C. S., 1908.]

5. Show that if δ stands for $\frac{d}{dx}$,

$$\delta^{-n}(uv)=v\,\delta^{-n}u-n\frac{dv}{dx}\delta^{-n-1}u+\frac{n(n+1)}{1.2}\frac{d^2v}{dx^2}\delta^{-n-2}u-\dots.$$

6. If $\frac{1}{R}\frac{dR}{dx}=\frac{fx+g}{ax^2+bx+c}$,

and if b^2-4ac be positive and the roots of $ax^2+bx+c=0$ be λ and μ, prove that $R=(x-\lambda)^p(x-\mu)^q$, where

$$\frac{p}{q}=\frac{f}{2a}\pm\frac{2ag-bf}{2a^2(\lambda-\mu)}.$$

And if $a=-1$, $b=0$, $c=1$,

$$R=(1-x^2)^{-\frac{f}{2}}\left(\frac{1+x}{1-x}\right)^{\frac{g}{2}}.$$

If b^2-4ac be negative,

$$R=(ax^2+bx+c)^{\frac{f}{2a}}e^{\frac{2ag-bf}{2\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}}.$$

If $b^2-4ac=0$,

$$R=\left(\frac{2ax+b}{2a}\right)^{\frac{f}{a}}e^{-\frac{2ag-bf}{a(2ax+b)}}.$$

[E. J. ROUTH, *Proc. L.M.S.*, vol. xvi., p. 250.]

7. Show that

$$\iiint\dots\int(x^2-1)^{-k-1}\left(\frac{x-1}{x+1}\right)^l dx\,dx\dots dx,$$

there being $2k+1$ integrations, $2k+1$ being an integer, though k may be a fraction, is equal to

$$\frac{1}{2^{2k+1}M}(x^2-1)^k\left(\frac{x-1}{x+1}\right)^l,$$

where $M=(k+l)(k+l-1)\dots$ to $2k+1$ factors.

[Cf. ROUTH, *Proc. L.M.S.*, vol. xvi., p. 249.]

8. ABC is a triangle with the corner A fixed and with sides AC, CB respectively $\sqrt{n}$ and $\sqrt{n+1}$, given lengths.

The side AB $(=r)$ makes an angle $\theta=nA-(n+1)B$ with a fixed straight line AX.

Show (1) that the path of B is rectifiable by the formula

$$s=\sqrt{n}\int_0^A\frac{d\theta}{\sqrt{1-\frac{n}{n+1}\sin^2\theta}}=\sqrt{n}\,\mathrm{am}^{-1}A,\quad \text{mod. }\sqrt{\frac{n}{n+1}}.$$

(2) When $n=1$ the rectification is the same as that of a Bernoulli's Lemniscate.

(3) The inclination of the normal to the radius vector is $A+B$.

(4) The area of the triangle is equal to the area of a sector of the curve starting from the axis AX.

[M. SERRET'S PROBLEM, *Calc. Int.*, p. 269.]

9. C is a point of maximum curvature on the Limaçon

$$r=a\cos\theta+b, \quad b>a;$$

A and A' are the two vertices Prove that the difference between the arcs AC, $A'C$ is $4a$. [ST. JOHN'S, 1891.]

10. If $y=x^3-3a^2x$, prove that

$$\frac{dx}{\sqrt{x^2-4a^2}}=\frac{dy}{3\sqrt{y^2-4a^6}},$$

and by integration express x explicitly in terms of y.

[OXFORD I. P., 1916.]

Apply this method to solve the cubic

$$x^3-3x^2-45x-473=0.$$

11. Prove that

$$\int_0^{\frac{\pi}{2}} e^{-\sin 2x}\cos x\,dx=e^{-1}\left\{1+\frac{1}{3\,.\,1!}+\frac{1}{5\,.\,2!}+\frac{1}{7\,.\,3!}+\dots\right\}.$$

[OXFORD I. P., 1916.]

12. Prove that if n be an odd positive integer greater than 3,

$$n\int_0^{\frac{\pi}{6}}\sin^n x\,dx=(2-\sqrt{3})\frac{n-1}{n-2}\cdot\frac{n-3}{n-4}\dots\frac{4}{3}$$
$$-\frac{\sqrt{3}}{8}\left[\frac{1}{2^{n-3}}+\frac{n-1}{n-2}\,\frac{1}{2^{n-5}}+\dots+\frac{n-1}{n-2}\cdot\frac{n-3}{n-4}\dots\frac{4}{3}\right].$$

[OXFORD I. P., 1916.]

13. The parameters t_1, t_2 of two points A, B of the unicursal curve

$$x/(1-t^2)=y/(t-t^3)=a/(1+t^2)$$

are equal to $\tan\alpha$, $\tan\beta$, where

$$-\tfrac{1}{2}\pi<\alpha<-\tfrac{1}{4}\pi, \quad \tfrac{1}{4}\pi<\beta<\tfrac{1}{2}\pi.$$

Prove that the area of the curvilinear triangle AOB, where O is the double point, is

$$a^2\left[2-\frac{\pi}{2}+\beta-\alpha-\sec\beta\sec\alpha\sin(\beta-\alpha)+\frac{1}{2}\tan\beta\tan\alpha\sin 2(\beta-\alpha)\right].$$

[OXFORD I. P., 1916.]

14. If n be a positive integer, show that

$$\int_0^{\pi} x\sin^2 nx\,\mathrm{cosec}^2 x\,dx=\tfrac{1}{2}n\pi^2.$$ [OXFORD I. P., 1912.]

15. By assuming

$$\int \frac{x^3}{\sqrt{1+x^4}}\,dx \text{ to be of the form } \frac{a+bx+cx^2+dx^3+ex^4}{\sqrt{1+x^4}},$$

obtain the integration by differentiation and equating coefficients, also obtain the result directly by putting $x^4 = z$.

16. Given a rational integral relation between x and y of the form

$$y^n + A_1 y^{n-1} + A_2 y^{n-2} + \ldots + A_n = 0 = F(x, y), \text{ say,}$$

where $A_1, A_2, \ldots A_n$ are rational functions of x, prove that when $\int y\,dx$ can be expressed algebraically in terms of x, then

$$\int y\,dx = B_0 + B_1 y + B_2 y^2 + B_3 y^3 + \ldots + B_{n-1} y^{n-1},$$

where $B_0, B_1, B_2 \ldots$ are rational functions of x. [ABEL.]

17. Assuming X to be a rational function of x, and $y^m = X$, and that $\int y\,dx$ is integrable in algebraic form and expressible as

$$\int y\,dx = P_0 + P_1 y + P_2 y^2 + \ldots + P_{m-1} y^{m-1},$$

where $P_0, P_1, \ldots P_{m-1}$ are rational functions of x, show that

$$P_0 = P_2 = P_3 = \ldots = P_{m-1} = 0,$$

that is that the integration must contain one term only, and that $\frac{1}{y}\int y\,dx$ is a rational algebraic function of x. [LIOUVILLE.]

18. If M and T be two rational polynomials, then, provided $\int \frac{M}{\sqrt[m]{T}}\,dx$ can be integrated in algebraic form at all, the form of the integral is $\frac{\theta}{\sqrt[m]{T}}$, where θ is a function of x.

Show also that

(1) θ is a rational function of x.

(2) That $MT = T\frac{d\theta}{dx} - \frac{1}{m}\theta\frac{dT}{dx}$.

(3) That θ is an integral polynomial expression and not of such form as $\frac{U}{V}$, where U and V are complete polynomials, *i.e.* not such that V contains x.

(4) That the degree of the polynomial θ is greater by unity than the degree of M.

Use these facts to show that $\int \frac{x^2\,dx}{\sqrt{1+x^4}}$ is not expressible algebraically.

[BERTRAND, *C. I.*, p. 94.]

19. P, Q, R being any rational algebraic polynomials, and assuming that when $\int \frac{P}{Q} \frac{dx}{\sqrt{R}}$ is integrable by means of the ordinary elementary functions, the integral must be of the form

$$\int \frac{P}{Q} \frac{dx}{\sqrt{R}} = \eta + \frac{\theta}{\sqrt{R}} + A \log(\alpha_1 + \beta_1 \sqrt{R}) + B \log(\alpha_2 + \beta_2 \sqrt{R}) + \dots,$$

where η, θ, α_1, β_1, α_2, etc., are rational functions of x, a result established by Abel,* show that when the integration can be reduced to one term, the general type of the result is either algebraic of form $\theta/\sqrt{R}$ or may be written as

$$\int \frac{P}{Q} \frac{dx}{\sqrt{R}} = A \tanh^{-1} \frac{\beta\sqrt{R}}{\alpha}.$$

In the latter case show that

$$(1)\ \alpha^2 - \beta^2 R = Q.$$

$$(2)\ \alpha\beta R\left(\frac{\beta'}{\beta} - \frac{\alpha'}{\alpha} + \frac{1}{2}\frac{R'}{R}\right) = \frac{P}{A},$$

$$(3)\ 2\alpha' Q - Q'\alpha = \frac{2P}{A}\beta,$$

where accents denote differentiation with regard to x.

20. Show that

$$\int \frac{x^6(11x^2+37x+28)}{x^{14}-x^3-4x^2-5x-2} \frac{dx}{\sqrt{x+2}} = -2\tanh^{-1} \frac{(x+1)\sqrt{x+2}}{x^7}.$$

21. Prove that

$$\int \frac{x\,dx}{\sqrt{(x^2+2x-5)(x^2+4x-8)}} = \frac{1}{2}\tanh^{-1} \frac{x+4}{x+5}\sqrt{\frac{x^2+4x-8}{x^2+2x-5}}.$$

22. Prove that

$$(1)\ \int \frac{5x^2+3x+1}{2x+1} \frac{dx}{\sqrt{x^4+2x^2+2x+1}} = \tanh^{-1} \frac{x\sqrt{x^4+2x^2+2x+1}}{x^3+x+1}.$$

$$(2)\ \int \frac{3\tan^2\theta+2}{2\tan^4\theta+5\tan^2\theta+4} \sec\theta\, d\theta = \tan^{-1} \frac{\sin\theta}{1+\cos^2\theta}.$$

$$(3)\ \int \frac{\sinh 2x\, dx}{\sqrt{\cosh^2 x+1}\,\sqrt{\cosh^2 x+2}} = 2\tanh^{-1}\sqrt{\frac{\cosh^2 x+1}{\cosh^2 x+2}}.$$

* *Œuvres.* See Bertrand, *Calc Intég*, chap. v.

23. Show that

(i) $$\frac{2n+1}{2}a^2\int\sqrt{\frac{a^{2n-1}+x^{2n-1}}{a^{2n+1}+x^{2n+1}}}\frac{dx}{a^2-x^2}-\frac{2n-1}{2}\int\sqrt{\frac{a^{2n+1}+x^{2n+1}}{a^{2n-1}+x^{2n-1}}}\frac{dx}{a^2-x^2}$$
$$=\tanh^{-1}x\sqrt{\frac{x^{2n-1}+a^{2n-1}}{x^{2n+1}+a^{2n+1}}}.$$

(ii) $$\frac{2n+1}{2}\int\sqrt{\frac{1+\sin^{2n-1}\theta}{1+\sin^{2n+1}\theta}}\sec\theta\,d\theta$$
$$-\frac{2n-1}{2}\int\sqrt{\frac{1+\sin^{2n+1}\theta}{1+\sin^{2n-1}\theta}}\sec\theta\,d\theta=\tanh^{-1}\left(\sin\theta\sqrt{\frac{1+\sin^{2n-1}\theta}{1+\sin^{2n+1}\theta}}\right).$$

24. Integrate the following:

(i) $\displaystyle\int\frac{(2x+1)dx}{\sqrt{x^4+2x^3-3x^2-4x+3}}$. (ii) $\displaystyle\int\frac{x\,dx}{\sqrt{x^4+2x^3-3x^2-ax+a}}$.

[ABEL.]

(iii) $\displaystyle\int\frac{5x^2+15x+12}{5x^2+15x+9}\frac{dx}{\sqrt{(x+1)(x+2)}}$.

(iv) $\displaystyle\int\frac{(1+8x)dx}{\sqrt{1+6x+4x^2}\sqrt{1-2x+4x^2}}$.

(v) $\displaystyle\int\frac{(2x+a)dx}{\sqrt{x^4+2ax^3+3a^2x^2+2a^3x-a^4}}$.

(vi) $\displaystyle\int\frac{3x^4-2x^3+1}{2x^3-x^2+1}\frac{dx}{\sqrt{x^4+1}}$. (vii) $\displaystyle\int\frac{3x+1}{1-x-2x^2-x^3}\frac{dx}{\sqrt{x}}$.

(viii) $\displaystyle\int\frac{3x^4+1}{1-x^2-x^6}\frac{dx}{\sqrt{x^4+1}}$. (ix) $\displaystyle\int\frac{x^2-(a+b)x-ab}{\sqrt{x^2+a^2}\sqrt{x^2+b^2}}dx$.

(x) $\displaystyle\int\frac{1+x-x^2}{1-x+x^2+x^3}\frac{dx}{\sqrt{x^2-1}}$. (xi) $\displaystyle\int\frac{1+x^4}{1+x^2-x^4}\frac{dx}{\sqrt{x^4-1}}$.

(xii) $\displaystyle\int\frac{1+3x^4+2x^5}{1+2x-x^6}\frac{dx}{\sqrt{1+x^4}}$.

25. Show that the whole perimeter and area of a single loop of the curve $r=2a\cos n\theta$ $(n>1)$ are respectively equal to the whole perimeter and area of the ellipse $x^2+n^2y^2=a^2$. [OXF. I. P., 1911.]

26. If an element ds of a curve lie at distance r from the origin, and subtends an angle $d\theta$ there, it is known that unit electric current flowing along ds produces a magnetic force at the origin at right angles to the plane of the curve proportional to $\dfrac{d\theta}{r}$.

Show that if unit current flows through a thin endless wire of given length in the form of an ellipse, the magnetic force due to the current at the centre of the ellipse is inversely proportional to the area of the ellipse. [OXFORD II. P., 1913.

27. A current of electricity is flowing round a fine wire $ABCD \dots KA$ bent into a plane polygon. O is any point within the polygon, and perpendiculars OP, OQ, OR, ... are drawn to the sides KA, AB, BC, etc., respectively, and again perpendiculars whose lengths are α, β, γ, ... from O upon the sides PQ, QR, RS, ... of the inscribed polygon $PQRS\dots$. Show that the magnetic force on unit particle situated at O is

$$\iota \Sigma \frac{\sin A}{\alpha},$$

where ι is the current strength.

28. Show that the perimeter of an ellipse of axes $2a$, $2b$ and small eccentricity e is approximately equal to the perimeter of a circle of diameter $a+b$, with an error which is only about 0·0025 per cent. when e is as great as 0·2. [MATH. TRIP. PART II., 1913.]