CHAPTER XVI.

RECTIFICATION (I.). ELEMENTARY.

510. In the following five chapters we propose to illustrate further the methods and processes of integration by showing their application to finding the length of a curved line whose equation is given by one of the ordinary modes of description, Cartesian, Polar, Pedal Equation, Tangential Polar, etc.; and further to discuss some subsidiary matters which arise in connection with such problems.

The process of finding the length of an arc of a curve, *i.e.* of finding a straight line whose length is the same as that of a specified arc, is called Rectification. Curves, the lengths of whose arcs can be found, are said to be Rectifiable.

Any formula which may have been established in the Differential Calculus expressing the differential coefficient of the arc "s" with regard to any independent variable, in terms of that variable, gives rise at once by integration to a formula in the Integral Calculus for the finding of s.

In each case the limits of integration to be assigned are the values of the independent variable corresponding to the two points which terminate the arc whose length is sought.

511. THE WORKING FORMULAE.

Below are added a list of the most common of these formulae. The references are to the articles in the author's *Treatise on the Differential Calculus* where they are established.

Formula in the Differential Calculus.	Formula in the Integral Calculus.	Reference.	Observations.
$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$	$s=\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$	Art. 200	For Cartesian Equations of form $y=f(x)$
$\frac{ds}{dy}=\sqrt{1+\left(\frac{dx}{dy}\right)^2}$	$s=\int\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$	Art. 200	For Cartesian Equations of form $x=f(y)$
$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$	$s=\int\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt$	Art. 200	For the case where the curve is defined as $x=f(t)$, $y=F(t)$
$\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$	$s=\int\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}d\theta$	Art. 201	For Polar Equations of form $r=f(\theta)$
$\frac{ds}{dr}=\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$	$s=\int\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}dr$	Art. 201	For Polar Equations of form $\theta=f(r)$
$\frac{ds}{dr}=\sec\phi=\frac{r}{\sqrt{r^2-p^2}}$	$s=\int\frac{r\,dr}{\sqrt{r^2-p^2}}$	Arts. 202 and 203	For Pedal Equations of form $p=f(r)$
$\frac{ds}{d\psi}=p+\frac{d^2p}{d\psi^2}$	$s=\frac{dp}{d\psi}+\int p\,d\psi$	Art. 221	For Tangential-Polars of form $p=f(\psi)$

The formulae $\quad s=\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx \quad$ or $\quad s=\int\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$

are applicable to cases where the Cartesian Equation is given, or can readily be expressed, in the forms $y=f(x)$ or $x=f(y)$ respectively, x being regarded as the independent variable in the first case, y in the second, and the axes being supposed to be rectangular.

As explained in the *Differential Calculus*, Art. 200, these formulae arise from the consideration of the infinitesimal right-angled triangle formed by the increments of abscissa, ordinate, and to the first order, the arc.

512. The amended form of these results for oblique axes would be, with the same description of the figure (Fig. 104) as in the article cited,

$$\delta s^2=(\text{chord } PQ)^2=\delta x^2+\delta y^2+2\delta x\,\delta y\cos\omega,$$

to the second order, and after rejecting infinitesimals of higher order than the second and proceeding to the limit

$$\left(\frac{dx}{ds}\right)^2+2\,\frac{dx}{ds}\,\frac{dy}{ds}\cos\omega+\left(\frac{dy}{ds}\right)^2=1,$$

and accordingly we should write

$$s=\int\sqrt{1+2\,\frac{dy}{dx}\cos\omega+\left(\frac{dy}{dx}\right)^2}\,dx$$

or

$$s=\int\sqrt{1+2\,\frac{dx}{dy}\cos\omega+\left(\frac{dx}{dy}\right)^2}\,dy,$$

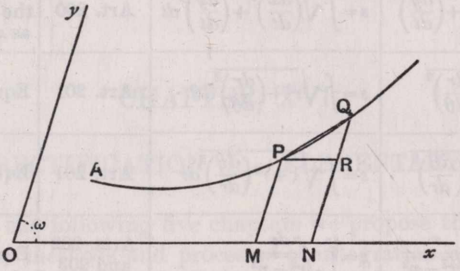

Fig. 104.

according as we take x or y for the independent variable.

513. The formulae may be remembered in a less formal manner as

$$s=\int\sqrt{dx^2+dy^2}$$

or

$$s=\int\sqrt{dx^2+2dx\,dy\cos\omega+dy^2},$$

where the dx or the dy may be brought outside the radical as circumstances demand.

514. Further, when the curve is given by expressing x and y separately in terms of a single variable t, as

$$x=f(t),\quad y=F(t),$$

we have

$$s=\int\sqrt{[\{f'(t)\}^2+F'(t)\}^2]}\,dt$$

or

$$s=\int\sqrt{[\{f'(t)\}^2+2f'(t)\,F'(t)\cos\omega+\{F'(t)\}^2}\,dt,$$

according as the coordinate axes are rectangular or oblique.

The coordinate axes will be always assumed to be rectangular unless the contrary is expressly stated, or to be inferred from the context.

515. The Rectification, therefore, of a curve depends upon the possibility of integration of the radical which occurs in these formulae.

Illustrative Examples.

516. The Earliest Rectification. William Neil's Problem (1637-1670).*

Ex. 1. Rectification of the Semicubical Parabola.

The equation of this curve is $ay^2=x^3$.

Here $$\frac{dy}{dx}=\frac{3}{2}\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}},$$

$$s=\int\sqrt{1+\frac{9}{4}\frac{x}{a}}\,dx=\frac{2}{3}\cdot\frac{4a}{9}\left(1+\frac{9x}{4a}\right)^{\frac{3}{2}}.$$

Taken between $x=0$ (the cusp) and $x=x_1$, for the branch in the first quadrant.

$$s=\frac{1}{27a^{\frac{1}{2}}}[(4a+9x)^{\frac{3}{2}}-(4a)^{\frac{3}{2}}].$$

This is stated by Gregory and Walton to have been the first curve to be rectified. The priority is ascribed to Neil by Wallis, but the rectification of the curve was also independently accomplished by Van Huraet.†

517. The Parabola.

Ex. 2. Consider the arc of the ordinary parabola $y^2=4ax$.

Here $$y=2\sqrt{ax},\quad \frac{dy}{dx}=\sqrt{\frac{a}{x}},$$

$$s=\int\sqrt{1+\frac{a}{x}}\,dx.$$

To effect this integration, let $x=a\tan^2\psi$.

Then $$dx=2a\tan\psi\sec^2\psi\,d\psi,$$

$$s=2a\int\sqrt{1+\cot^2\psi}\tan\psi\sec^2\psi\,d\psi$$

$$=2a\int\sec^3\psi\,d\psi$$

$$=a[\sec\psi\tan\psi+\log(\sec\psi+\tan\psi)]$$

$$=a\left[\sqrt{\frac{x}{a}}\sqrt{1+\frac{x}{a}}+\log\left(\sqrt{1+\frac{x}{a}}+\sqrt{\frac{x}{a}}\right)\right].$$

If taken between any two limits, x_1 and x_2, corresponding to any two points P, Q on the arc, which lie on the same side of the axis,

$$\text{arc } PQ=(\sqrt{x_2}\sqrt{a+x_2}-\sqrt{x_1}\sqrt{a+x_1})+a\log\frac{\sqrt{a+x_2}+\sqrt{x_2}}{\sqrt{a+x_1}+\sqrt{x_1}}.$$

* *Wallisii Opera*, T. 1, 551; Gregory and Walton, p. 420.

† Cajori's *History of Mathematics*, p. 190.

For example, if we require the length from the vertex to the upper end of the latus rectum, $x_1=0,\ x_2=a$,

and
$$\text{arc required}=\sqrt{a}\sqrt{a+a}+a\log\frac{\sqrt{a+a}+\sqrt{a}}{\sqrt{a}}$$
$$=a[\sqrt{2}+\log_e(\sqrt{2}+1)]$$
$$=a(1{\cdot}4142\ldots+{\cdot}8814\ldots)$$
$$=2{\cdot}2956\ldots\times a.$$

Thus the length of an arc of a parabola from one end of the latus rectum to the other is 1·1478... times the latus rectum.

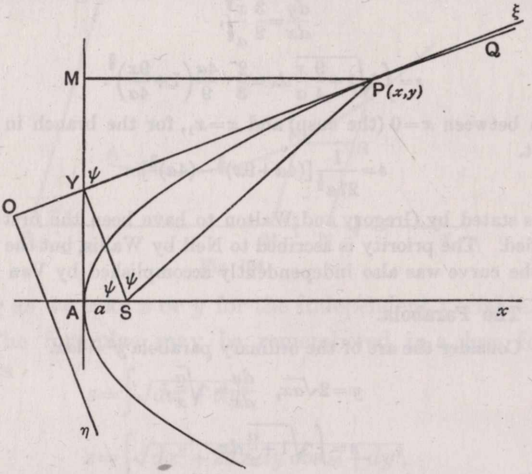

Fig. 105.

It is worth considering the angle ψ which has been used as a subsidiary variable to facilitate integration.

It is the angle which the tangent at the current point P makes with the y-axis, viz. the tangent at the vertex. For if PM be the perpendicular upon the y-axis, PY the tangent, S the focus, SY the perpendicular upon the tangent, and if we call MYP, ψ, we have

$$x=MP=YP\sin\psi=SY\frac{\sin^2\psi}{\cos\psi}=a\tan^2\psi.$$

The intrinsic equation of this curve is therefore

$$s=a\sec\psi\tan\psi+a\log(\sec\psi+\tan\psi)$$

or
$$s=a\sec\psi\tan\psi+a\,\mathrm{gd}^{-1}\psi,$$

the tangent at the vertex being the initial tangent.

Let us call PY, t. Then $t=a\sec\psi\tan\psi$.

Hence
$$s-t=a\log(\sec\psi+\tan\psi).$$

Hence the logarithmic portion of s, viz. $a\log(\sec\psi+\tan\psi)$ denotes the excess of the arcual distance of P from A over the "tail," *i.e.* the portion of the tangent measured from P to the foot of the perpendicular upon the tangent from the focus.

It will be seen later that in many cases this excess "arc − tail" plays an important part.

In the case under consideration—viz. the parabola—let a length $PO=s$ be measured along the tangent. Then $OY=s-t$. The point O is the point on the tangent at which the vertex A would arrive if we regard the tangent as a fixed line, and the parabola to roll upon it without sliding. Consider it in this way. O is then a fixed point. Take the tangent OP as the ξ-axis, and a perpendicular through O as the η-axis. Then, if ξ, η be the coordinates of the focus,

$$\xi=OY=s-t=a\log(\sec\psi+\tan\psi),$$
$$\eta=YS \qquad =a\sec\psi.$$

To find the path of S as the parabola rolls upon its fixed tagent, we have to eliminate ψ.

$$\left.\begin{aligned}&\sec\psi+\tan\psi=e^{\frac{\xi}{a}}.\\ \text{Hence}\qquad&\sec\psi-\tan\psi=e^{-\frac{\xi}{a}}.\end{aligned}\right\}\text{Therefore }\sec\psi=\cosh\frac{\xi}{a}.$$

Therefore the path of the focus of the rolling parabola is

$$\eta=a\cosh\frac{\xi}{a},$$

i.e. the ordinary catenary or chainette.

We also have, putting $\frac{\xi}{a}=u$,

$$\tan\psi=\sinh u,\quad \sin\psi=\tanh u,$$
$$SP=a\sec^2\psi=a\cosh^2 u,$$
$$t=SP\sin\psi=a\sinh u\cosh u=\frac{a}{2}\sinh 2u,$$
$$s=a\sinh u\cosh u+a\log(\sinh u+\cosh u)$$
$$=\frac{a}{2}\sinh 2u+au,$$
$$s-t=au,$$
$$SY=a\sec\psi=a\cosh u,\text{ etc.}$$

Incidentally, we may note that the equation

$$\xi=a\log(\sec\psi+\tan\psi)=a\,\mathrm{gd}^{-1}\psi$$

may be used to indicate the "march" of the function, $\mathrm{gd}^{-1}\psi$ for $a\,\mathrm{gd}^{-1}\psi$ is the abscissa of a point on a catenary curve, and since $\frac{d\eta}{d\xi}=\tan\psi$, ψ is the slope of the tangent to the catenary. Hence a good idea of the graph of $y=a\,\mathrm{gd}^{-1}x$ can be formed by first plotting the catenary itself and then plotting a new curve, taking as abscissae the circular measures of the angles which the tangent to the catenary makes with its directrix, and for ordinates the corresponding abscissae of the catenary.

If PP' be a focal chord of the parabola, the arc AP has been shown to be

$$AP = a\sec\psi\tan\psi + a\log(\sec\psi + \tan\psi),$$

and the arc $P'A$ can be obtained from it by writing $90-\psi$ for ψ,

i.e. $$P'A = a\operatorname{cosec}\psi\cot\psi + a\log(\operatorname{cosec}\psi + \cot\psi).$$

Hence, by addition, the whole arc $P'AP$ cut off by a focal chord which makes an angle 2ψ with the axis is

$$a\left[\frac{\sin^3\psi + \cos^3\psi}{\sin^2\psi\cos^2\psi} + \log(1+\sec\psi)(1+\operatorname{cosec}\psi)\right].$$

The evaluation of the arc might have been conducted by taking y as the independent variable.

Then $$x = \frac{y^2}{4a},\quad \frac{dx}{dy} = \frac{y}{2a},$$

$$s = \frac{1}{2a}\int\sqrt{4a^2+y^2}\,dy$$

$$= \frac{1}{4a}\left[y\sqrt{4a^2+y^2} + 4a^2\log\frac{y+\sqrt{y^2+4a^2}}{2a}\right]_{y_1}^{y_2},$$

which reduces to the same form as already obtained.

518. Sir Christopher Wren's Problem (1632–1723). Rectification of the Cycloid.

Ex. 3. The equations of the curve are

$$\left.\begin{aligned} x &= a\theta + a\sin\theta, \\ y &= a(1-\cos\theta). \end{aligned}\right\} \text{(See } \textit{Diff. Calc.}\text{, pp. 337–339.)}$$

Here $$dx = a(1+\cos\theta)\,d\theta,$$

$$dy = a\sin\theta\,d\theta.$$

Hence $$ds^2 = 2a^2(1+\cos\theta)\,d\theta^2 = 4a^2\cos^2\frac{\theta}{2}\,d\theta^2,$$

$$ds = 2a\cos\frac{\theta}{2}\,d\theta,$$

$$s = 4a\sin\frac{\theta}{2}, \quad\text{......(1)}$$

s being measured from the point at which $\theta = 0$, *i.e.* the vertex.

Again, with the same description of the figure as in *Diff. Calc.*, Art. 394, chord $CQ = 2a\sin\frac{\theta}{2}$.

Therefore $$\text{arc } CP = 2 \text{ chord } CQ. \quad\text{......(2)}$$

Substituting for θ from $y = 2a\sin^2\frac{\theta}{2}$,

$$s = \sqrt{8ay}. \quad\text{......(3)}$$

If the tangent at P is inclined at an angle ψ to the tangent at the vertex,

$$\tan\psi = \frac{dy}{dx} = \frac{\sin\theta}{1+\cos\theta} = \tan\frac{\theta}{2};$$

$$\therefore\ \theta = 2\psi,$$

and $$s = 4a\sin\psi. \qquad (4)$$

This is the intrinsic equation of the curve.

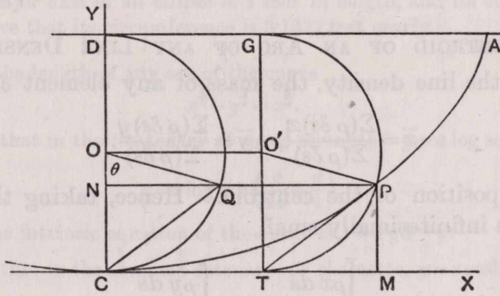

Fig. 106.

The whole length of the curve from cusp to cusp is

$$2\left[4a\sin\psi\right]_0^{\frac{\pi}{2}} = 8a. \qquad (5)$$

The point at which $\psi = 30$ gives $s = 2a$, and therefore bisects the arcual distance from vertex to cusp.

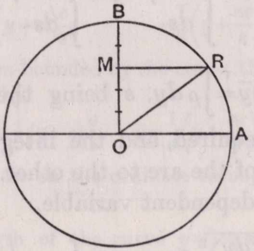

Fig. 107.

If a circle be drawn with any radius, and OA, OB be a pair of radii at right angles, and OB divided into n equal parts so that M being, say, the r^{th} point of division, and MR be then drawn parallel to OA to meet the circle at R, then $\sin AOR = \frac{r}{n}$.

If then in the cycloid a chord CQ of the circle CQD be drawn (Fig 106) so that the angle $XCQ =$ angle AOR, in Fig. 107, the line QP parallel to CX, and cutting the cycloid at P, will cut off an arc $CP = \frac{r}{n}$ of the arc CA, for

$$\text{arc } CP = 4a\sin\psi = 4a\,.\,\frac{r}{n} = \frac{r}{n}\text{arc } CA.$$

Hence an arc of any proposed ratio to the whole arc can be cut off.
Many of the geometers of the seventeenth century devoted considerable attention to the cycloid.* Wren, the architect of St. Paul's Cathedral, discovered the rectification of the curve and determined the centroid; Fermat, the area bounded by an arc; Huygens invented the cycloidal pendulum; Pascal and Wallis also greatly advanced a knowledge of the curve.†

519. Centroid of an Arc of any Line Density.

If ρ be the line density, the mass of any element δs is $\rho\,\delta s$,

and
$$\bar{x}=\frac{\Sigma(\rho\,\delta s)x}{\Sigma(\rho\,\delta s)},\quad \bar{y}=\frac{\Sigma(\rho\,\delta s)y}{\Sigma(\rho\,\delta s)},$$

give the position of the centroid. Hence, taking the limit when δs is infinitesimally small,

$$\bar{x}=\frac{\int\rho x\,ds}{\int\rho\,ds},\quad \bar{y}=\frac{\int\rho y\,ds}{\int\rho\,ds}.$$

If ρ be constant,

$$\bar{x}=\frac{\int x\,ds}{\int ds},\quad \bar{y}=\frac{\int y\,ds}{\int ds};$$

that is, $s\bar{x}=\int x\,ds$, $s\bar{y}=\int\rho\,dy$, s being the length of the arc whose centroid is required, and the integration being taken from one extremity of the arc to the other. (See Art. 446.)

And if x be the independent variable,

$$\bar{x}=\frac{\int\rho x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int\rho\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx},\quad \bar{y}=\frac{\int\rho y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}{\int\rho\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx},$$

with corresponding formulae if it be desirable to express the integral with other independent variables as shown in the table of Art. 511.

*See *Diff. Calc.*, Art. 390.

†Cajori's *Hist. of Math.*, pp. 177, etc.

Examples.

1. Find the length of the arc of the curve $y^2(2a-x)=x^3$, the cissoid of Diocles. [Huygens, 1625–1695.]

2. Find the curve for which the length of the arc measured from the origin varies as the square root of the ordinate.

3. The major axis of an ellipse is 1 foot in length, and its eccentricity is $\frac{1}{10}$. Prove that its circumference is 3·1337 feet nearly. [Trinity, 1883.]

4. Find the length of any arc of the curve

$$x^{\frac{2}{3}}-y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

5. Show that in the "catenary of equal strength," $y=a\log\sec\dfrac{x}{a}$,

$$s=a\log\tan\left(\frac{x}{2a}+\frac{\pi}{4}\right),$$

and that the intrinsic equation of the curve is $s=a\,\mathrm{gd}^{-1}\,\psi$.

6. Show that in the common catenary, or chainette, $y=c\cosh\dfrac{x}{c}$,

$$s=\sqrt{y^2-c^2},\quad s=c\tan\psi,\quad s^2=c(\rho-c),\quad s=c\sinh\frac{x}{c}.$$

The area bounded by the curve, the directrix, the y-axis and an ordinate is $A=cs$.

The centroid of the arc has coordinates

$$\bar{x}=x-c\tan\frac{\psi}{2}\qquad \bar{y}=\tfrac{1}{2}(y+x\cot\psi)$$

$$=x-\left(y-c\right)\frac{c}{s},\qquad =\frac{1}{2}\left(y+\frac{cx}{s}\right).$$

The centroid of the area bounded by the curve, the directrix, the y-axis and an ordinate is given by

$$\bar{x}=x-(y-c)\frac{c}{s},\quad \bar{y}=\frac{1}{4}\left(y+\frac{cx}{s}\right),$$

and that both centroids lie on the ordinate through the intersection of the terminal tangents.

7. Show that the length of the curve $y=\log\coth\dfrac{x}{2}$ from the point (x_1, y_1) to the point (x_2, y_2) is $\log\dfrac{\sinh x_2}{\sinh x_1}$.

8. Show that in the epi- or hypo-cycloid

$$\left.\begin{aligned} x&=(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta,\\ y&=(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta.\end{aligned}\right\}$$

(i) $s=\dfrac{4b}{a}(a+b)\cos\dfrac{a\theta}{2b}$, (ii) $s=\dfrac{4b}{a}(a+b)\cos\dfrac{a}{a+2b}\psi$,

s being measured from the point where $\theta=\dfrac{\pi b}{a}$, *i.e.* a vertex.

9. For the four-cusped hypocycloid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}},$$

show (i) that $s=\dfrac{3a}{4}\cos 2\psi$, s being measured from a vertex;

(ii) the whole length of the curve is $6a$;

(iii) $s^3 \propto x^2$, s being measured from the cusp which lies on the y-axis.

10. In the tractrix

$$x=\sqrt{c^2-y^2}+\frac{c}{2}\log\frac{c-\sqrt{c^2-y^2}}{c+\sqrt{c^2-y^2}},$$

show that $$s=c\log\frac{c}{y}.$$

11. Show that the distance from the vertex of the centroid of a wire in the form of portion of a cycloid, of which the vertex is the middle point, is $\frac{1}{3}$ of the greatest ordinate of the arc.

12. Show that the arc of a parabola of latus rectum $4a$ measured from the vertex, and the radius vector from the focus, are expressible in terms of a parameter t in the respective forms

$$\frac{s}{a}=\frac{t}{1-t^2}+\frac{1}{2}\log\frac{1+t}{1-t},\quad \frac{r}{a}=\frac{1}{1-t^2}.$$

[MATH. TRIP. PT. II., 1915.]

Prove also that $$s=\sqrt{r(r-a)}+a\tanh^{-1}\sqrt{\frac{r-a}{r}}.$$

520. **Polar Formula.**

In the *Differential Calculus* (Art 201) it is shown from consideration of the small infinitesimal right-angled triangle formed by the increments of arc, radius vector and perpendicular on the radius vector from one extremity of the infinitesimal arc, that to the second order

$$\delta s^2=\delta r^2+r^2\delta\theta^2.$$

This gives rise at once, on proceeding to the limit, to the formulae,

$$s=\int\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\,d\theta$$

or $$s=\int\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\,dr,$$

according as we wish to use θ or r as the independent variable, and, as in Art. 513, we may remember it in the less formal manner as

$$s=\int\sqrt{dr^2+r^2\,d\theta^2}.$$

Further, as in the case of Cartesians, if r and θ be given in terms of some third variable t (though this is very unusual) by $r=f(t)$, $\theta=F(t)$, we may say

$$s=\int\sqrt{\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2}\,dt$$

$$=\int\sqrt{[f'(t)]^2+[f(t)]^2\,[F'(t)]^2}\,dt.$$

521. Illustrative Examples.

Ex. 1. In the case of the **Archimedean Spiral** $r=a\theta$,

$$s=a\int\sqrt{\theta^2+1}\,d\theta=\frac{a}{2}\left[\theta\sqrt{\theta^2+1}+\log\left(\theta+\sqrt{\theta^2+1}\right)\right],$$

s being measured from the vertex, where $\theta=0$.

As this may be written

$$s=\frac{1}{2a}\left(r\sqrt{r^2+a^2}+a^2\log\frac{r+\sqrt{r^2+a^2}}{a}\right),$$

we see, on comparison with the result of Art. 517, that this is the same as the arc of the parabola $y^2=2ax$, measured from the vertex of the parabola and expressed in terms of the ordinate.

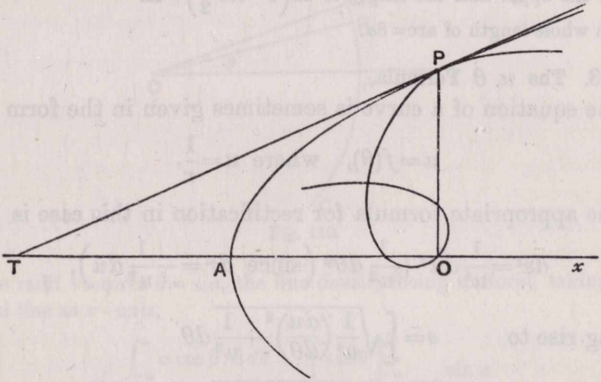

Fig. 108.

Hence it will follow that when an Archimedean spiral $r=a\theta$ rolls without sliding on the concave side of a parabola $y^2=2ax$ so that their vertices come into contact, the roulette of the pole of the spiral is the axis of the parabola. In this case the r of the spiral is the y of the parabola, and the motion of the pole O is always at right angles to the line PO, and arcs AP, OP are equal.

For many examples of this class, see Chapter XIX.

522. Ex. 2. **The Cardioide** $r=a(1-\cos\theta)$. (See Art. 424, *Diff. Calc.*)

The curve is symmetrical about the initial line, and θ varies from 0 to π for the upper half.

$$\frac{dr}{d\theta}=a\sin\theta.$$

Hence $\displaystyle s=\int_0^\theta \sqrt{a^2(1-\cos\theta)^2+a^2\sin^2\theta}\,d\theta$

$$=a\int_0^\theta 2\sin\frac{\theta}{2}\,d\theta=\left[-4a\cos\frac{\theta}{2}\right]_0^\theta=4a\left(1-\cos\frac{\theta}{2}\right)=8a\sin^2\frac{\theta}{4}.$$

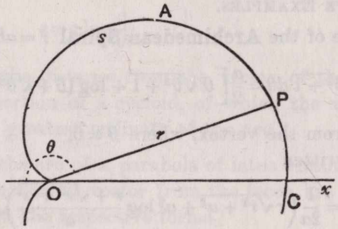

Fig. 109.

This gives the length of any arc OAP.

For the upper half the length is $4a\left(1-\cos\frac{\pi}{2}\right)=4a$.

The whole length of arc $=8a$.

523. **The u, θ Formula.**

The equation of a curve is sometimes given in the form

$$u=f(\theta),\quad \text{where } u=\frac{1}{r}.$$

The appropriate formula for rectification in this case is

$$ds^2=\frac{1}{u^4}du^2+\frac{1}{u^2}d\theta^2\left(\text{since } dr=-\frac{1}{u^2}du\right),$$

giving rise to
$$s=\int\sqrt{\frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2+\frac{1}{u^2}}\,d\theta$$
$$=\int\frac{1}{u^2}\sqrt{\left(\frac{du}{d\theta}\right)^2+u^2}\,d\theta,$$

or
$$s=\int\sqrt{\frac{1}{u^4}+\frac{1}{u^2}\left(\frac{d\theta}{du}\right)^2}\,du$$
$$=\int\frac{1}{u^2}\sqrt{1+u^2\left(\frac{d\theta}{du}\right)^2}\,du,$$

according as θ or u be taken as the independent variable.

524. Centroid of an Arc of any Line Density; Polars.

Again, exactly as in the case of the curve whose equation is given in Cartesian coordinates, if ρ be the line density, the centroid of the arc of a curve is given by

$$\bar{x}=\frac{\int \rho x\,ds}{\int \rho\,ds}=\frac{\int \rho r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}{\int \rho\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta},$$

$$\bar{y}=\frac{\int \rho y\,ds}{\int \rho\,ds}=\frac{\int \rho r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}{\int \rho\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}.$$

525. Centroid of Arc of a Circle.

Ex. In the case of a uniform circular arc of radius a and terminated

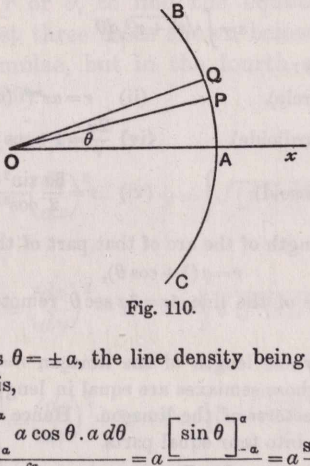

Fig. 110.

by the radii vectores $\theta=\pm\alpha$, the line density being uniform, taking the medial line as x-axis,

$$\bar{x}=\frac{\int_{-\alpha}^{\alpha} a\cos\theta\,.\,a\,d\theta}{\int_{-\alpha}^{\alpha} a\,d\theta}=a\frac{\Big[\sin\theta\Big]_{-\alpha}^{\alpha}}{\Big[\theta\Big]_{-\alpha}^{\alpha}}=a\frac{\sin\alpha}{\alpha}$$

and $\bar{y}=0$ because the x-axis is an axis of symmetry.

526. Moment of Inertia of a Fine Wire.

The moment of inertia of a fine wire of line density ρ about any straight line in the plane of the wire is $\Sigma\rho\,\delta s\times p^2$, where p is the perpendicular from the element δs upon the straight line.

Thus, Moment of inertia about x-axis $=\int \rho y^2\,ds$,

Moment of inertia about y-axis $=\int \rho x^2\,ds$,

and Moment of inertia about a perpendicular to the plane through the pole $=\int \rho r^2\,ds$,

and for ds is to be substituted from the table of Art. 511, the appropriate expression according to the system of coordinates used in any particular case.

The **Product of Inertia** for such a wire with regard to the axes is defined as

$$\int \rho xy\,ds.$$

EXAMPLES.

1. Find the length of any arc of the curve from the formula

$$s=\int \sqrt{r^2+r'^2}\,d\theta$$

for the following cases:

(i) $r=a\cos\theta$ (circle). (ii) $r=ae^{m\theta}$ (equiang. spiral).

(iii) $r=a\sin^2\dfrac{\theta}{2}$ (cardioide). (iv) $\dfrac{2a}{r}=1+\cos\theta$ (parabola).

(v) $r=a\dfrac{\sin^2\theta}{\cos\theta}$ (cissoid). (vi) $r=\dfrac{3a}{2}\dfrac{\sin^2\theta}{\cos^3\theta}$ (semicub. parab.).

2. Show that the length of the arc of that part of the cardioide

$$r=a(1+\cos\theta),$$

which lies on the side of the line $4r=3a\sec\theta$ remote from the pole, is equal to $4a$. [OXFORD.]

3. Show that the whole length of the limaçon $r=a\cos\theta+b$ is equal to that of an ellipse whose semiaxes are equal in length to the maximum and minimum radii vectores of the limaçon. Hence show how to divide the arc of the limaçon into four equal parts. [COLLEGES α, 1888.]

4. Prove that the length of the n^{th} pedal of a loop of the curve

$$r^m=a^m\sin m\theta$$

is

$$a(mn+1)\int_0^{\frac{\pi}{m}} \sin^k m\theta\, d\theta, \quad \text{where } m(k-n+1)=1.$$

5. Show that the length of a loop of the curve

$$3x^2y-y^3=(x^2+y^2)^3$$

$$=2\int_0^1 \frac{d\xi}{\sqrt{1-\xi^6}}.$$ [ST. JOHN'S, 1881.]

6. Show that the rectification of the curve $r^n=a^n \sin n\theta$ is given by the integral

$$s=a\int_0^{\xi} \frac{d\xi}{\sqrt{1-\xi^{2n}}}.$$ [MATH. TRIP., 1896.]

7. Two radii vectores OP, OQ of the curve

$$r=2a\cos^3\left(\frac{\pi}{4}+\frac{\theta}{3}\right)$$

are drawn equally inclined to the initial line; prove that the length of the intercepted arc is $a\alpha$, where α is the circular measure of the angle POQ. [ASPARAGUS, *Educ. Times.*]

8. Show that the centroid of a wire bent into the form of a cardioide $r=a(1+\cos\theta)$, and with a line density $k\sec\frac{\theta}{2}$, k being a constant, is on the axis of the cardioide at distance $\frac{a}{2}$ from the cusp.

527. **The Converse Problem. Given s, find the Curve.**

The converse problem, viz. given s in terms of one of the quantities x, y, r or θ, to find the equation of the curve, leads in the first three cases shown below to an application of the same formulae, but in the fourth case there is more difficulty (Art. 529).

(1) If $s=f(x)$, we have

$$\left(\frac{dy}{dx}\right)^2=\left(\frac{ds}{dx}\right)^2-1, \quad y=\int\sqrt{\{f'(x)\}^2-1}\,dx.$$

(2) If $s=f(y)$,

$$\left(\frac{dx}{dy}\right)^2=\left(\frac{ds}{dy}\right)^2-1, \quad x=\int\sqrt{\{f'(y)\}^2-1}\,dy.$$

(3) If $s=f(r)$,

$$r^2\left(\frac{d\theta}{dr}\right)^2=\left(\frac{ds}{dr}\right)^2-1, \quad \theta=\int\frac{\sqrt{\{f'(r)\}^2-1}}{r}\,dr.$$

528. For example.

1. Find the curve for which $s=\dfrac{x^2}{2a}$.

Here $$\left(\frac{dy}{dx}\right)^2=\frac{x^2}{a^2}-1$$

$$\pm a\,dy=\sqrt{x^2-a^2}\,dx$$

$$\pm 2ay=x\sqrt{x^2-a^2}-a^2\cosh^{-1}\frac{x}{a}+\text{constant}.$$

2. Find the curve in which $s=r\sec\alpha$.

Here $$r^2\left(\frac{d\theta}{dr}\right)^2=\sec^2\alpha-1=\tan^2\alpha,$$

$$\frac{dr}{r}=\pm d\theta\cot\alpha,$$

$$\log r=\pm\theta\cot\alpha+\text{const.},$$

$$r=ae^{\pm\theta\cot\alpha}.\quad\text{(Equiangular spirals.)}$$

3. Find the curve in which $s=\sqrt{8ay}$.

Here $$\frac{ds}{dy}=\sqrt{\frac{2a}{y}},$$

$$\left(\frac{dx}{dy}\right)^2=\frac{2a}{y}-1,\quad dx=\sqrt{\frac{2a-y}{y}}\,dy.$$

Let $$y=a(1-\cos\theta),$$

$$dx=\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\,a\sin\theta\,d\theta=a(1+\cos\theta)\,d\theta,$$

$$\left.\begin{aligned}x+\text{const.}&=a(\theta+\sin\theta),\\ y&=a(1-\cos\theta),\end{aligned}\right\}\text{ a cycloid.}$$

529. (4) But the case when $s=f(\theta)$ leads at once to

$$\left(\frac{dr}{d\theta}\right)^2+r^2=\{f'(\theta)\}^2,$$

and the variables r and θ are not now in general "separable" as in the former cases (see *Integral Calculus for Beginners*, Art. 175); nor does this differential equation fall under any of the standard forms. Nevertheless, in some cases useful information may be derived from its consideration.

For example,

1. Is the circle $r=a$ the only curve for which $s=a\theta$?

Here we have $\left(\frac{dr}{d\theta}\right)^2+r^2=a^2$, which is of course satisfied by $r=a$. But if r is not equal to a, we have

$$\int\frac{dr}{\sqrt{a^2-r^2}}=\pm d\theta,$$

$$\sin^{-1}\frac{r}{a}=\alpha\pm\theta,\quad\text{where }\alpha\text{ is a constant.}$$

Hence $$r=a\sin(\alpha\pm\theta),$$

i.e. a circle of radius $\frac{a}{2}$ and passing through the pole will also give the same result, viz. $s=a\theta$, as is geometrically obvious. But no curve other than $r=a$ or $r=a\sin(\alpha\pm\theta)$ will do so.

2. Is the equiangular spiral $r=ae^{\theta\cot\alpha}$ the only curve for which

$$s=\frac{ae^{\theta\cot\alpha}}{\cos\alpha}?$$

Here $$\left(\frac{dr}{d\theta}\right)^2+r^2=\frac{a^2}{\sin^2\alpha}e^{2\theta\cot\alpha}.$$

Let $r=ave^{\theta\cot\alpha}$, where v is some function of θ to be determined.

Thus $$\left(\frac{dv}{d\theta}+v\cot\alpha\right)^2+v^2=\operatorname{cosec}^2\alpha,$$

which is of course obviously satisfied if $v=1$, which leads back to $r=ae^{\theta\cot\alpha}$.

But we have in addition to this the general solution of

$$\frac{dv}{d\theta}+v\cot\alpha=\pm\sqrt{(\operatorname{cosec}^2\alpha-v^2)},$$

i.e. of $$\int\frac{dv}{v\cot\alpha\mp\sqrt{\operatorname{cosec}^2\alpha-v^2}}=-\theta+\beta,$$

where β is some constant.

To integrate this, let $v=\operatorname{cosec}\alpha\sin\phi$.

Then $$\int\frac{\cos\phi\,d\phi}{\sin(\phi\mp\alpha)}=\frac{\beta-\theta}{\sin\alpha},$$

i.e. $$\int\{\cos\alpha\cot(\phi\mp\alpha)\mp\sin\alpha\}\,d\phi=\frac{\beta-\theta}{\sin\alpha},$$

or $$\left.\begin{aligned}&\cos\alpha\log\sin(\phi\mp\alpha)\mp\phi\sin\alpha=\frac{\beta-\theta}{\sin\alpha},\\ &\text{where}\quad \frac{\sin\phi}{\sin\alpha}=\frac{r}{ae^{\theta\cot\alpha}},\end{aligned}\right\}$$

which upon elimination of ϕ furnishes a set of curves whose arcs are of the same length as the corresponding arcs of the equiangular spiral $r=ae^{\theta\cot\alpha}$.

Examples.

1. Find the curves in which

(i) $s=a\sin^{-1}\dfrac{y}{a}$. (ii) $s=\sqrt{y^2-c^2}$.

(iii) $s=\dfrac{r^2}{2a}$. (iv) $y=ce^{-\frac{s}{c}}$.

(v) $s\propto r$. (vi) $s\propto\sqrt{x}$.

(vii) $s=2\sqrt{2ar}$.

2. Show that the equation

$$nx\frac{d^2s}{d^2x}+\frac{ds}{dx}=0$$

leads to a cycloid or a four-cusped hypocyloid according as $n=2$ or $n=3$.

530. Tangential Polar Equations. Legendre's Formulae.

Formulae $$t=\frac{dp}{d\psi},\quad \frac{ds}{d\psi}=p+\frac{d^2p}{d\psi^2}.$$

These results were proved in Article 221 of the *Differential Calculus*, but are now established in a different manner.

Let PY, $P'Y'$ be the tangents at two contiguous points P, P' of the curve, OY, OY' the perpendiculars upon them from the pole O.

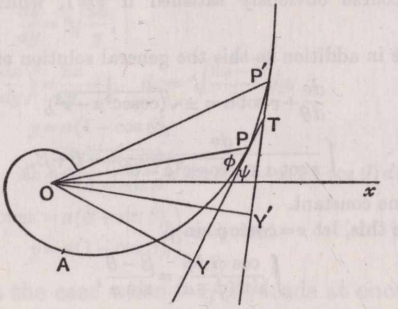

Fig. 111.

Let t be the projection of the radius vector upon the tangent
$$OY=p,\quad OY'=p+\delta p,\quad \text{arc } PP'=\delta s,$$
and $\delta\psi$ the angle YOY'.

Then, projecting the broken line $OYPP'$ upon OY' and upon $Y'P'$,

(1) $p+\delta p=p\cos\delta\psi+t\sin\delta\psi+$second order quantities,

(2) $t+\delta t=\delta s+t\cos\delta\psi-p\sin\delta\psi$,

i.e. $$\left.\begin{aligned}\delta p&=t\,\delta\psi,\\ \delta t&=\delta s-p\,\delta\psi,\end{aligned}\right\}\text{ to the first order.}$$

And ultimately $t=\dfrac{dp}{d\psi}$, $\dfrac{ds}{d\psi}=p+\dfrac{dt}{d\psi}=p+\dfrac{d^2p}{d\psi^2}$.

531. It is to be noted that since $t\equiv\dfrac{dp}{d\psi}=r\cos\phi$, *i.e.* the projection of the radius vector upon the tangent, t is positive or negative according as ϕ is acute or obtuse.

The above figure (Fig. 111) exhibits the standard case. In this case $t\equiv\dfrac{dp}{d\psi}$ is $+PY$, and is in a direction from P opposite to that of the direction of increase of s; p is increasing with ψ and $\dfrac{dp}{d\psi}$ is therefore positive. In cases where p increases or decreases as ψ decreases or increases, $\dfrac{dp}{d\psi}$ (*i.e.* t) is negative, and $=-PY$.

The student should examine the formulae carefully in all four cases:

(1) Curve concave to O, ϕ acute.
(2) Curve convex to O, ϕ acute.
(3) Curve concave to O, ϕ obtuse.
(4) Curve convex to O, ϕ obtuse.

It will be seen that $\dfrac{ds}{d\psi}=p+\dfrac{d^2p}{d\psi^2}$ in all cases and that $t=\pm PY$ according as ϕ is acute or obtuse.

The arc s is measured from a point on the arc on the same side of the radius vector as that on which ϕ is measured; ψ may increase or decrease with the increase of s.

The value of the radius of curvature is, of course, essentially positive; and $\rho=\pm\dfrac{ds}{d\psi}$ according as s and ψ increase together, or the one increases as the other decreases.

Accordingly we have $\rho=\pm\left(p+\dfrac{d^2p}{d\psi^2}\right)$ respectively in these cases. The formulae established are due to LEGENDRE.

532. By integration of

$$\frac{ds}{d\psi}=p+\frac{d^2p}{d\psi^2},$$

we have $$s=\frac{dp}{d\psi}+\int p\,d\psi,$$

i.e. $$s-t=\int p\,d\psi;$$

where t is the "tail" referred to in Art. 517.

In the case of *a closed oval of continuous curvature*, the "tail" t returns to its original value when the integration is conducted round the whole contour.

If the origin be *within the curve* and is only enclosed once by it, the length of the contour is given by

$$\int_0^{2\pi} p\,d\psi.$$

If the *origin is enclosed n times* (Fig. 112), so that the tangent makes n complete revolutions as its point of contact travels continuously round the curve, the length will be

$$\int_0^{2n\pi} p\,d\psi.$$

Further modifications may have to be made, for instance, in integrating round *a loop of a curve* (Fig. 113); it may happen that the initial and final values of $\frac{ds}{d\psi}$ are not the same, and that the tangent does not make a complete

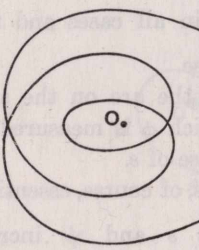

Fig. 112.

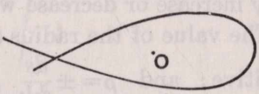

Fig. 113.

revolution, but the student should have no difficulty in such cases in assigning the proper limits.

533. Ex. Show that the perimeter of an ellipse of small eccentricity e exceeds by $\frac{3e^4}{64}$ of its length that of a circle having the same axis.

[γ, 1889.]

Here $$p^2 = a^2\cos^2\psi + b^2\sin^2\psi = a^2(1 - e^2\sin^2\psi),$$

where ψ is the angle p makes with the major axis.

Therefore $$p = a\left(1 - \frac{1}{2}e^2\sin^2\psi - \frac{1}{8}e^4\sin^4\psi - \dots\right).$$

Hence $$s = 4a\left(\frac{\pi}{2} - \frac{1}{2}e^2\frac{1}{2}\frac{\pi}{2} - \frac{1}{8}e^4\frac{3}{4}\frac{1}{2}\frac{\pi}{2} - \dots\right)$$

$$= 2\pi a - \frac{\pi}{2}ae^2 - \frac{3}{32}\pi a e^4 - \dots.$$

The radius r of a circle of the same area is given by

$$r^2 = ab = a^2(1 - e^2)^{\frac{1}{2}},$$

and its circumference is

$$2\pi a\left(1 - \frac{1}{4}e^2 - \frac{3}{32}e^4 - \dots\right);$$

$$\therefore \text{ circumf. ellipse} - \text{circumf. circle} = \left(\frac{3}{16} - \frac{3}{32}\right)\pi a e^4$$
$$= \frac{3}{64} 2\pi a e^4$$
$$= \frac{3e^4}{64}\{\text{circ. of circle}\}$$

as far as terms involving e^4,

$$\textit{i.e.} \quad \frac{\text{circ. ell.} - \text{circ. circle}}{\text{circ. circle}} = \frac{3e^4}{64};$$

$$\therefore \quad \frac{\text{circ. ell.} - \text{circ. circle}}{\text{circ. ellipse}} = \frac{3e^4}{64} \Big/ \left(1 + \frac{3e^4}{64}\right) = \frac{3e^4}{64}$$

to terms involving e^4.

534. Length of the Arc of an Evolute.

It was shown in the *Differential Calculus* (Art. 343) that the difference between the radii of curvature at two points of a curve of continuous curvature is equal to the length of

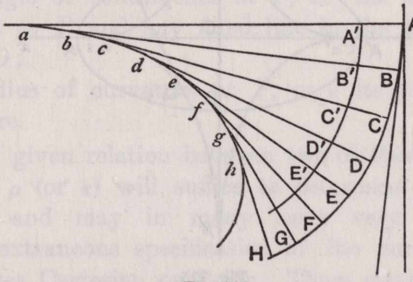

Fig. 114.

the corresponding arc of the evolute; *i.e.* if ah be the arc of the evolute of the portion AH of the original figure, then (Fig. 114)

$$\text{arc } ah = Aa - Hh, \quad \textit{i.e. } \rho \text{ (at } A) - \rho \text{ (at } H).$$

And if the evolute be regarded as a rigid curve, and an inelastic string be unwound from it, being kept tight, then the points of the unwinding string describe a system of parallel curves, each of the parallels being an involute of the curve ha, one of these being the original curve HA itself.

535. Ex. Find the length of the evolute of an ellipse. If α, α', β, β' be the centres of curvature corresponding to the extremities of the axes, viz. A, A', B, B' respectively, the arc $\alpha\beta$ of the evolute corresponds to the arc AB of the ellipse, and we have

$$\text{arc } \alpha\beta = \rho \text{ (at } B) - \rho \text{ (at } A) = \frac{a^2}{b} - \frac{b^2}{a},$$

for the radius of curvature at any point P of the ellipse is $\frac{a^2b^2}{p^3}$ $\left(\text{the pedal equation being } \frac{a^2b^2}{p^2}=a^2+b^2-r^2 \text{ and } \rho=\frac{r\,dr}{dp}\right)$. Thus the length of the entire perimeter of the evolute, which is obviously symmetrical about the axes, is

$$4\left(\frac{a^2}{b}-\frac{b^2}{a}\right).$$

In the application of this rule care is needed, not to pass a point of maximum or minimum curvature on the original curve, for on travelling

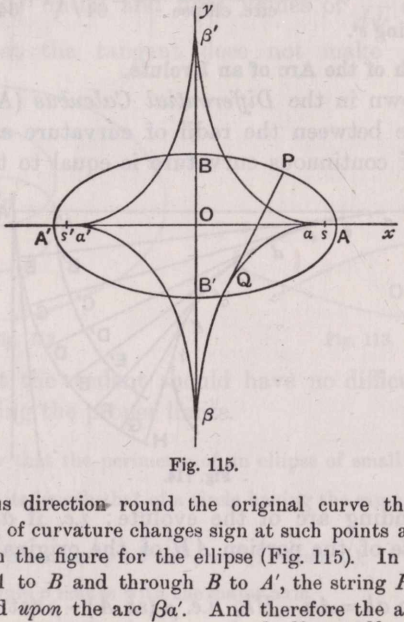

Fig. 115.

in a continuous direction round the original curve the difference of successive radii of curvature changes sign at such points and the evolute has a cusp as in the figure for the ellipse (Fig. 115). In that case, as P travels from A to B and through B to A', the string PQ is wound *off* the arc $\alpha\beta$ and *upon* the arc $\beta\alpha'$. And therefore the arcs $\alpha\beta$ and $\beta\alpha'$ would appear with opposite signs, viz. $\frac{a^2}{b}-\frac{b^2}{a}$ and $\frac{b^2}{a}-\frac{a^2}{b}$, if P travels continuously in one direction. The intervals between the points of maximum and of minimum curvature must therefore be treated separately and the positive results added together.

Examples.

1. Show that in the parabola $y^2=4ax$, the length of the arc of the evolute intercepted within the parabola is

$$4a(3\sqrt{3}-1).$$

2. Find the whole length of the evolute of the cardioide

$$r=a(1+\cos\theta).$$

3. Show that the length of the evolute of the portion of the Folium of Descartes $x^3+y^3=3axy$, which corresponds to the loop, is $\frac{3a}{4}(4-\sqrt{2})$.

536. Intrinsic Equation of a Curve.

Let s be the length of the arc of a curve measured from a fixed point O to the current tracing point P;

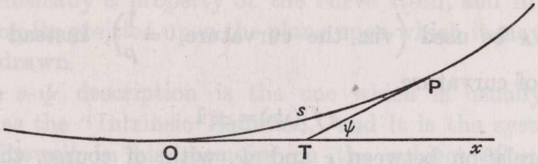

Fig. 116.

ψ the angle of contingence at P, *i.e* the angle between the tangent at P and any fixed line in the plane, say the tangent at O;

ρ the radius of curvature at P, or κ its reciprocal, viz. the curvature.

Then any given relation between two of these three quantities s, ψ, ρ (or κ) will suffice to determine the shape of the curve, and may in many cases very conveniently replace an extraneous specification of the curve by means of coordinates, Cartesian or Polar. These quantities s, ψ, ρ depend upon no external system of coordinates and leave the position of the curve undefined. The nature of the curve itself is specified by the relation existing between two of the three s, ψ, ρ, which has been very aptly styled by Dr. Whewell the Intrinsic Equation of the curve. Some notice has been already taken of Intrinsic Equations in Arts. 346–349 of the *Differential Calculus*. But the subject is more closely allied to Integral Calculus, and it is convenient to develop the matter more fully here, though at the risk of some repetition.

We shall adopt the notation used in the *Differential Calculus* as to the meanings of the letters involved for the following work.

When the relation is between s and ψ,

$$\text{say} \quad s=f(\psi),$$

that between ρ and ψ is

$$\rho = \pm \frac{ds}{d\psi}, \quad \textit{i.e.} \quad \rho = \pm f'(\psi).$$

The sign to be taken $+$ when s is increasing with ψ,

$-$ when s increases or decreases as ψ decreases or increases,

and if κ be used $\left(\text{viz. the curvature, } = \frac{1}{\rho}\right)$, instead of the radius of curvature,

$$\kappa f'(\psi) = \pm 1$$

is the relation between κ and ψ, with, of course, the same rule as to choice of sign.

Conversely, if the connection given be between ρ and ψ,

$$\text{say} \quad \rho = f(\psi),$$

then
$$\frac{ds}{d\psi} = \pm f(\psi),$$

and
$$s = \pm \int f(\psi)\, d\psi + C,$$

C being a constant which may be chosen to correspond to the measurement of s from any arbitrarily chosen point of the curve, and the sign selected as before.

When the relation is given between κ and ψ, it is, of course, the same thing, except that we have

$$\kappa = f(\psi), \quad \text{say},$$

i.e.
$$\frac{d\psi}{ds} = \pm f(\psi),$$

and
$$s = \pm \int \frac{d\psi}{f(\psi)} + \text{const.}$$

Finally, when the relation is between ρ and s,

$$\text{say} \quad \rho = f(s),$$

we have
$$\pm \frac{ds}{d\psi} = f(s),$$

and
$$\psi + C = \pm \int \frac{ds}{f(s)}.$$

Hence, these three systems of description of a curve, by means of a specified relation,

(a) between s and ψ,
(b) between ρ (or κ) and ψ,
(c) between ρ (or κ) and s,

are equivalent, and either forms a mode of specification which is intrinsically a property of the curve itself, and in no way defining its position upon the plane upon which it may happen to be drawn.

The s-ψ description is the one which is usually understood as the "Intrinsic Equation," and it is the system used by Whewell in his memoirs on the subject (*Camb. Phil. Trans.*, viii., p. 659; and ix., p. 150) and discussed in Boole's *Differential Equations*, pages 264–269.

The ρ-ψ specification was used by Euler.

537. To obtain the Intrinsic Equation from the Cartesian Equation.

When the Cartesian Equation is given as $y=f(x)$, then, supposing the initial tangent to be parallel to the x-axis, we have

$$\tan\psi=f'(x), \qquad \text{(1)}$$

and

$$\frac{ds}{dx}=\sqrt{1+[f'(x)]^2},$$

$$s=\int\sqrt{1+[f'(x)]^2}\,dx. \qquad \text{(2)}$$

And if after integration x be eliminated between equations (1) and (2), the required relation between s and ψ,

say $s=F(\psi)$,

will be obtained.

Conversely, if the equation $s=F(\psi)$ be given, and the Cartesian equation be desired, we have

$$\frac{dx}{ds}=\cos\psi, \quad \frac{dy}{ds}=\sin\psi;$$

whence

$$x+A=\int\cos\psi\,F'(\psi)\,d\psi, \qquad \text{(1)}$$

$$y+B=\int\sin\psi\,F'(\psi)\,d\psi, \qquad \text{(2)}$$

A and B being arbitrary constants.

And if after integration ψ be eliminated from equations (1) and (2), the Cartesian Equation of the curve will result.

538. **Illustrative Examples.**

Ex. 1. Intrinsic Equation of a circle.

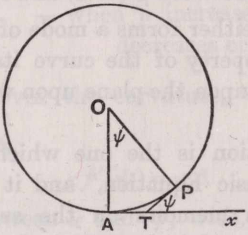

Fig. 117.

If ψ be the angle between the initial tangent at A and the tangent at P, the centre being O and the radius a, we have $P\hat{O}A = P\hat{T}x = \psi$, and therefore $s = a\psi$.

Ex. 2. Intrinsic Equation of a catenary.

In this case the equation of the curve referred to its axis and the tangent at the vertex as coordinate axes is

$$y + c = c\cosh\frac{x}{c}.$$

Hence $$\tan\psi = \frac{dy}{dx} = \sinh\frac{x}{c},$$

and $$\frac{ds}{dx} = \sqrt{1+\sinh^2\frac{x}{c}} = \cosh\frac{x}{c};$$

$$\therefore\ s = c\sinh\frac{x}{c},$$

the constant of integration being zero if we measure s from the vertex where $x=0$; therefore $s = c\tan\psi$ is the intrinsic equation sought.

539. **Case when the Coordinates are expressed in terms of a Parameter.**

If the equations of the curve be given as

$$x = f(t),\quad y = \phi(t),$$

we have $$\tan\psi = \frac{dy}{dx} = \frac{\phi'(t)}{f'(t)}. \quad \text{.......................................(1)}$$

Also $$\frac{ds}{dt} = \sqrt{\{f'(t)\}^2 + \{\phi'(t)\}^2},$$

and $$s = \int\sqrt{\{f'(t)\}^2 + \{\phi'(t)\}^2}\,dt. \quad \text{..................(2)}$$

If s be found in terms of t by integration from equation (2), then between this result and equation (1) we may eliminate t. The required relation between s and ψ will result.

540. Ex. 1. In the cycloid

$$x=a(t+\sin t),$$
$$y=a(1-\cos t).$$

Hence $$\tan\psi=\frac{\sin t}{1+\cos t}=\tan\frac{t}{2};\quad \therefore\ t=2\psi.$$

Also $$\frac{ds}{dt}=a\sqrt{(1+\cos t)^2+\sin^2 t}=2a\cos\frac{t}{2};$$

whence $s=4a\sin\dfrac{t}{2}$, s being measured from the vertex, where $t=0$.

Hence $s=4a\sin\psi$ is the equation required. See *Diff. Calc.*, Arts. 395, 397.

Ex. 2. In the epi- or hypo-cycloid

$$x=(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta,$$

$$y=(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta,$$

$$\frac{dx}{d\theta}=-(a+b)\sin\theta+(a+b)\sin\frac{a+b}{b}\theta,$$

$$\frac{dy}{d\theta}=\quad(a+b)\cos\theta-(a+b)\cos\frac{a+b}{b}\theta,$$

$$\frac{ds}{d\theta}=\pm 2(a+b)\sin\frac{a}{2b}\theta,$$

$$s=\mp\frac{4b(a+b)}{a}\cos\frac{a}{2b}\theta+C.$$

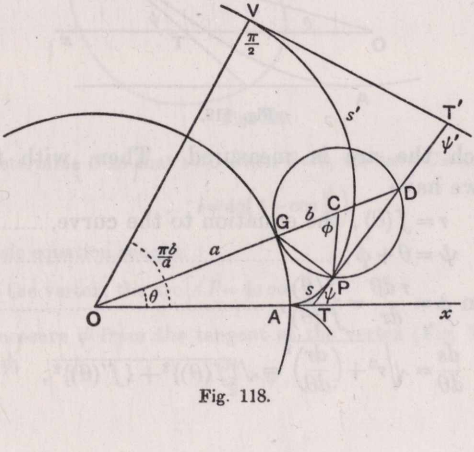

Fig. 118.

Also with the description of figure in Art. 405, *Diff. Calc.*,

$$a\theta=b\phi \quad \text{and} \quad \psi=\theta+\frac{\phi}{2}=\frac{a+2b}{2b}\theta,$$

$$s=\mp\frac{4b(a+b)}{a}\cos\frac{a}{a+2b}\psi+C.$$

If s be measured from the cusp, the tangent at the cusp being the initial line,

$$\text{arc } AP=s=\frac{4b(a+b)}{a}\left(1-\cos\frac{a}{a+2b}\psi\right).$$

If we measure the arc from the vertex V, where $\theta=\frac{\pi b}{a}$,

$$\text{arc } VP=s'=\frac{4b(a+b)}{a}\cos\frac{a}{a+2b}\psi,$$

OA being retained as the initial line for the measurement of ψ. If we measure ψ from the tangent at the vertex, we must write

$$\frac{\pi}{2}+\frac{\pi b}{a}-\psi' \text{ for } \psi, \quad \textit{i.e. } \frac{\pi}{2}-\frac{a}{a+2b}\psi' \text{ for } \frac{a}{a+2b}\psi,$$

and

$$s'=\frac{4b(a+b)}{a}\sin\frac{a}{a+2b}\psi'.$$

Hence the general intrinsic equation of such curves is

$$s=A\sin B\psi \quad \text{or} \quad s=A\cos B\psi.$$

In the case $s=A\sin B\psi$, s is measured from a vertex and ψ is measured from the tangent at that vertex.

In the case $s=A\cos B\psi$, s is measured from a vertex and ψ is measured from the tangent at the next cusp.

541. To obtain the Intrinsic Equation from the Polar.

Suppose the initial line parallel to the tangent at the point A

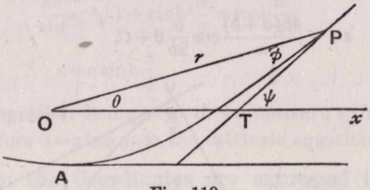

Fig. 119.

from which the arc is measured. Then, with the usual notation, we have

$$r=f(\theta), \text{ the equation to the curve, } \ldots\ldots\ldots\ldots(1)$$

$$\psi=\theta+\phi, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\tan\phi=\frac{r\,d\theta}{dr}=\frac{f(\theta)}{f'(\theta)}, \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}=\sqrt{\{f(\theta)\}^2+\{f'(\theta)\}^2};$$

and therefore

$$s=\int\sqrt{\{f(\theta)\}^2+\{f'(\theta)\}^2}\,d\theta. \quad \text{.......................}(4)$$

If s be found by integration from (4), and θ, ϕ eliminated by means of equations (2) and (3), the required relation between s and ψ will be found. If the initial line of the polar equation be not that from which ψ is measured, equation (2) will need modification accordingly.

542. Ex. 1. Find the intrinsic equation of the cardioide

$$r=a(1-\cos\theta).$$

Here
$$\psi=\theta+\phi,$$

$$\tan\phi=\frac{a(1-\cos\theta)}{a\sin\theta}=\tan\frac{\theta}{2};$$

$$\therefore\ \phi=\frac{\theta}{2}\quad\text{and}\quad\psi=\theta+\frac{\theta}{2}=\frac{3\theta}{2};$$

$$\therefore\ \theta=\frac{2}{3}\psi.$$

Also
$$\frac{ds}{d\theta}=a\sqrt{(1-\cos\theta)^2+\sin^2\theta}=2a\sin\frac{\theta}{2},$$

$$s=-4a\cos\frac{\theta}{2}+C.$$

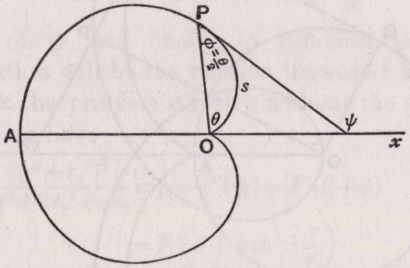

Fig. 120.

If we determine C so that $s=0$ when $\theta=0$, we have $C=4a$;

$$\therefore\ s=4a\left(1-\cos\frac{\psi}{3}\right),$$

the intrinsic equation sought.

If A be the vertex, the arc $AP=4a\cos\frac{\psi}{3}$.

If we measure ψ from the tangent at the vertex (Fig. 121), we must write for ψ,

$$\frac{3\pi}{2}-\psi,$$

and if arc $AP=s'$,
$$s'=4a\cos\left(\frac{\pi}{2}-\frac{\psi'}{3}\right),$$
$$s'=4a\sin\frac{\psi'}{3}.$$

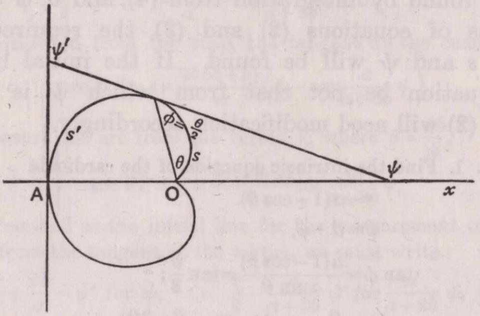

Fig. 121.

Ex. 2. Find the intrinsic equation of the first negative pedal of the Archimedean spiral $r=a\theta$.

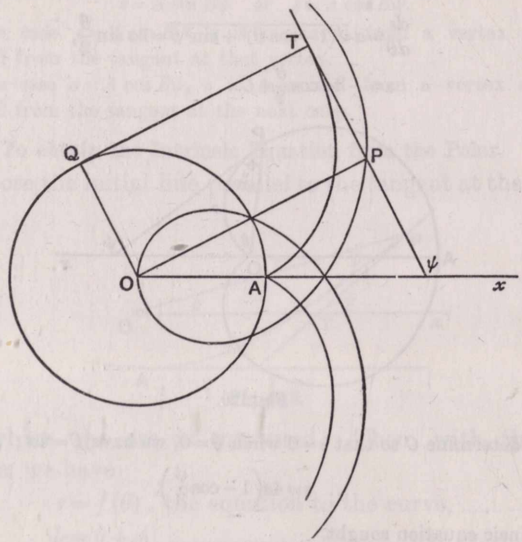

Fig. 122.

If O be the pole, P a point on the spiral, and PT be drawn perpendicular to OP touching the first negative pedal in T, then
$$PT=\frac{dr}{d\theta}=a.$$

Hence the normal TQ to the first negative pedal envelopes a circle with centre O and radius a. It is therefore an involute of the circle. If TQ touches this circle at Q, then $\rho = TQ = \text{arc } AQ$, where A is the cusp of the involute, *i.e.* $\rho = a\psi$, for $\psi = Q\hat{O}A$;

$$\therefore \frac{ds}{d\psi} = a\psi \quad \text{and} \quad s = \frac{a\psi^2}{2}.$$

(See *Diff. Calc.*, Art. 455.)

Otherwise: If $r = a\theta$ be the locus of P, r, θ being the polar coordinates of the foot of the perpendicular from the pole upon a tangent to the first negative pedal, the tangential polar equation of the pedal is $p = a\psi$;

$$\therefore \frac{ds}{d\psi} = p + \frac{d^2p}{d\psi^2} = a\psi; \quad \therefore s = \frac{a\psi^2}{2}.$$

543. To obtain the Polar Equation from the Intrinsic.

When the intrinsic equation $s = F(\psi)$ is given, and it is desired to get the equivalent polar equation, it is usually best to obtain the Cartesian coordinates of a point on the curve first, as above, from

$$x = \int \cos\psi\, F'(\psi)\, d\psi, \quad y = \int \sin\psi\, F'(\psi)\, d\psi,$$

and then, after integration, to form

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{y}{x}$$

as functions of ψ, and finally to eliminate ψ, when the resulting equation will be the relation between r and θ.

If we attack the problem directly without the intervention of Cartesians, we have

$$\frac{(r^2 + r_1{}^2)^{\frac{3}{2}}}{r^2 + 2r_1{}^2 - rr_2} = \rho = F'(\psi) = F'(\theta + \phi)$$
$$= F'\left(\theta + \tan^{-1}\frac{r}{r_1}\right)$$

which is a troublesome second order differential equation; but one which, of course, theoretically furnishes the required relation between r and θ.

544. Illustrative Examples.

Ex. 1. Find the s, ψ relation for the equiangular spiral

$$r = ae^{\theta \cot \alpha}.$$

Here $\phi = \alpha$, $\psi = \theta + \alpha$,

$$s = a \operatorname{cosec} \alpha \int e^{\theta \cot \alpha} d\theta = \frac{a}{\cos \alpha} e^{\theta \cot \alpha}$$

the constant being determined so that s shall be measured from the point at which $\theta=-\infty$, *i.e.* from the pole;

$$\therefore\ s=\frac{a}{\cos\alpha}e^{(\psi-\alpha)\cot\alpha}.$$

Ex. 2. Conversely, find the polar equation corresponding to

$$s=\frac{a}{\cos\alpha}e^{(\psi-\alpha)\cot\alpha}.$$

We have $x\dfrac{\sin\alpha}{a}=\displaystyle\int\cos\psi\, e^{(\psi-\alpha)\cot\alpha}d\psi=\frac{e^{\overline{\psi-\alpha}\cot\alpha}}{\operatorname{cosec}\alpha}\cos(\psi-\alpha),$

$$y\frac{\sin\alpha}{a}=\int\sin\psi\, e^{(\psi-\alpha)\cot\alpha}d\psi=\frac{e^{\overline{\psi-\alpha}\cot\alpha}}{\operatorname{cosec}\alpha}\sin(\psi-\alpha),$$

the constants vanishing if we make $x=a$ and $y=0$, when $\psi=\alpha$;

$$\therefore\ \frac{r}{a}=e^{\overline{\psi-\alpha}\cot\alpha};$$

and $\tan\theta=\tan(\psi-\alpha);\quad\therefore\ \psi=\theta+\alpha;$

$$\therefore\ r=ae^{\theta\cot\alpha}.$$

545. Intrinsic Equation deduced from the Tangential Polar.

When the tangential polar equation of the curve is given,

$$p=F(\psi),\quad\text{say},$$

we have at once

$$\frac{ds}{d\psi}=p+\frac{d^2p}{d\psi^2}=F(\psi)+F''(\psi),$$

and

$$s=F'(\psi)+\int F(\psi)\,d\psi,$$

the intrinsic equation required.

546. Tangential Polar form deduced from the Intrinsic Equation.

To get back to the tangential polar form from the intrinsic equation

$$s=f(\psi),$$

we have, of course, $\dfrac{d^2p}{d\psi^2}+p=f'(\psi).$ ....(1)

To solve this differential equation we may either say at once

$$p=A\sin\psi+B\cos\psi+\frac{1}{D^2+1}f'(\psi),$$

and perform the operation indicated. (See Integral for Beginners, Chap. XVI.), or we may proceed thus:

(*a*) multiply (1) by $\cos\psi$ and then by $\sin\psi$, giving respectively,

$$\frac{d}{d\psi}\left(\frac{dp}{d\psi}\cos\psi+p\sin\psi\right)=f'(\psi)\cos\psi$$

and
$$\frac{d}{d\psi}\left(\frac{dp}{d\psi}\sin\psi-p\cos\psi\right)=f'(\psi)\sin\psi;$$

(*b*) integrating we have

$$\left.\begin{aligned}\frac{dp}{d\psi}\cos\psi+p\sin\psi&=\int_0^\psi f'(\psi)\cos\psi\,d\psi+A,\\ \frac{dp}{d\psi}\sin\psi-p\cos\psi&=\int_0^\psi f'(\psi)\sin\psi\,d\psi-B,\end{aligned}\right\}$$

where A and B are arbitrary constants;

(*c*) eliminating $\dfrac{dp}{d\psi}$,

$$p=\sin\psi\int_0^\psi f'(\psi)\cos\psi\,d\psi-\cos\psi\int_0^\psi f'(\psi)\sin\psi\,d\psi$$
$$+A\sin\psi+B\cos\psi;$$

and the tangential polar result is obtained.

The result may obviously be written as

$$p-A\sin\psi-B\cos\psi=\int_0^\psi f'(\omega)\sin(\psi-\omega)\,d\omega.$$

Moreover, if we choose our origin of measurement of p to be such that A and B both vanish, and suppose s to have been measured from a point where $\psi=0$, so that $f(0)=0$, we may integrate by parts and further reduce this equation to

$$p=\int_0^\psi f(\omega)\cos(\psi-\omega)\,d\omega.$$

547. Intrinsic Equation deduced from the Pedal Equation.

When the pedal equation (p, r) is given, say $p=f(r)$,

$$\frac{dr}{ds}=\cos\phi=\sqrt{1-\frac{p^2}{r^2}}.$$

Then s can be found in terms of r by integrating

$$s=\int\frac{r\,dr}{\sqrt{r^2-[f(r)]^2}}.\quad\text{..........................(1)}$$

Again,
$$\frac{ds}{d\psi}=\rho=\frac{r\,dr}{dp}=\frac{r}{f'(r)}.\quad\text{..........................(2)}$$

If r be eliminated between equations (1) and (2) we get a differential equation between s and ψ, whose solution furnishes the intrinsic equation sought.

548. Ex. Consider $p=r\sin\alpha$ (equiangular spiral).

$$s=\int\frac{r\,dr}{\sqrt{r^2-p^2}}=\int\frac{r\,dr}{r\cos\alpha}=\frac{r}{\cos\alpha},$$

$$\frac{ds}{d\psi}=\frac{r\,dr}{dp}=\frac{r}{\sin\alpha}=s\cot\alpha,$$

$$\frac{ds}{s}=\cot\alpha\,d\psi,$$

$$\log s=\psi\cot\alpha+\text{constant},$$

$$s=Ce^{\psi\cot\alpha}.$$

549. **Pedal Equation from the Intrinsic.**

Conversely, if it be required to derive the pedal equation from the intrinsic equation $s=f(\psi)$, we have

$$r\frac{dr}{dp}=\rho=\frac{ds}{d\psi}=f'(\psi),\quad\ldots\ldots(1)$$

and $$p=\sin\psi\int^{\psi}f'(\psi)\cos\psi\,d\psi-\cos\psi\int^{\psi}f'(\psi)\sin\psi\,d\psi.\quad\ldots(2)$$

Upon elimination of ψ we have a differential equation between $\dfrac{dr}{dp}$, r and p, which when solved gives the required p-r equation.

550. Ex. Starting with $s=Ce^{\psi\cot\alpha}$,

$$r\frac{dr}{dp}=\frac{ds}{d\psi}=C\cot\alpha\,e^{\psi\cot\alpha},$$

and $$p=\sin\psi\int C\cot\alpha\,e^{\psi\cot\alpha}\cos\psi\,d\psi-\cos\psi\int C\cot\alpha\,e^{\psi\cot\alpha}\sin\psi\,d\psi$$

$$=C\cot\alpha\,e^{\psi\cot\alpha}\frac{\sin\psi\cos(\psi-\alpha)-\cos\psi\sin(\psi-\alpha)}{\operatorname{cosec}\alpha},$$

i.e. $$p=C\cos\alpha\sin\alpha\,e^{\psi\cot\alpha};$$

$\therefore$ dividing, $\dfrac{r}{p}\dfrac{dr}{dp}=\dfrac{1}{\sin^2\alpha}$,

i.e. $r^2=\dfrac{p^2}{\sin^2\alpha}$, if p and r are taken to vanish together,

i.e. $$p=r\sin\alpha.$$

551. Variations on these modes of procedure may of course be adopted to suit special cases.

552. WELL-KNOWN INTRINSIC EQUATIONS.

The following are the most common intrinsic equations of the "well-known" curves:

(1) For the circle, $s=a\psi$, *Diff. Calc.*, p. 273.

(2) For the catenary, $s=c\tan\psi$, „ p. 273.

(3) For the cycloid, $s=4a\sin\psi$, „ p. 340.

(4) For the epi- or hypo-cycloid, $\left\{\begin{array}{l} s=\dfrac{4b}{a}(a+b)\cos\dfrac{a}{a+2b}\psi; \\ \text{or, generally,} \\ \qquad s=A\sin B\psi \\ \text{or} \quad s=A\cos B\psi, \end{array}\right\}$ „ p. 345.

(5) Involute of a circle, $\left\{s=\dfrac{a\psi^2}{2},\right.$ „ p. 275.

(6) Parabola $y^2=4ax$, $\left\{\begin{array}{l}\rho=2a\sec^3\psi, \\ s=a[\sec\psi\tan\psi+\log(\sec\psi+\tan\psi)],\end{array}\right\}$ *Int. Calc.*, Art. 517.

(7) Evolute of a parabola $27ay^2=4(x-2a)^3$, $\left\{s=2a(\sec^3\psi-1),\right.$ *Diff. Calc.*, p. 275.

Semicubical parabola $3ay^2=2x^3$, $\left\{9s=4a(\sec^3\psi-1)\right\}$ *Int. Calc. for Beginners*, p. 151.

(8) Equiangular spiral, $\left\{s=Ae^{m\psi}.\right.$

(9) Tractory, $s=c\log\operatorname{cosec}\psi$, *Diff. Calc.*, p. 358.

(10) Cardioide $r=a(1+\cos\theta)$, $r=a(1-\cos\theta)$, $\left\{\begin{array}{l} s=4a\sin\left(\dfrac{\psi}{3}-\dfrac{\pi}{6}\right), \\ \text{included as a case of} \\ \text{the epi-cycloid,} \\ s=4a\left(1-\cos\dfrac{\psi}{3}\right), \end{array}\right\}$ *Int. Calc.*, Art. 542.

(11) Catenary of equal strength, $y=a\log\sec\dfrac{x}{a}$, $\left\{\begin{array}{l} s=a\,\mathrm{gd}^{-1}\psi, \\ \text{i.e. } s=a\log\tan\left(\dfrac{\pi}{4}+\dfrac{\psi}{2}\right), \end{array}\right\}$ *Int. Calc.*, Ex. 8, Art. 517.

553. Intrinsic Equation of the Evolute.

Let $s=f(\psi)$ be the equation of the given curve. Let s' be the length of the arc of the evolute measured from some fixed point A to any other point Q on the evolute. Let O and P be the points on the original curve corresponding to the points A, Q on the evolute; ρ_0, ρ the radii of curvature at O and P; ψ the angle the tangent QP makes with OA produced, or, which is the same thing, the angle the tangent PT makes with the tangent at O.

Then
$$s'=\rho-\rho_0=\frac{ds}{d\psi}-\rho_0$$
$$s'=f'(\psi)-\rho_0.$$

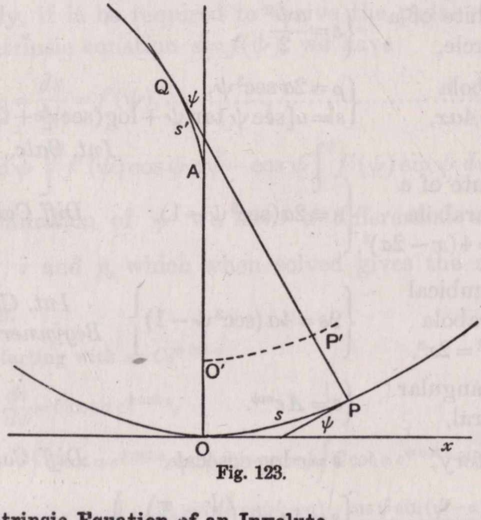

Fig. 123.

554. Intrinsic Equation of an Involute.

With the same figure, if the curve AQ be the original curve given by the equation $s'=f(\psi)$, we have

$$\rho=s'+\rho_0, \quad \rho=\frac{ds}{d\psi};$$
$$\frac{ds}{d\psi}=f(\psi)+\rho_0,$$
$$s=\int[f(\psi)+\rho_0]\,d\psi.$$

ρ_0 is now an arbitrary constant, and

$$s = \int f(\psi)\, d\psi + \rho_0 \psi + \text{const.}$$

The intrinsic equation of an involute

$$s = \rho_0 \psi + \int_0^\psi f(\psi)\, d\psi$$

is the particular involute whose radius of curvature at O is ρ_0. For any of the other involutes, the whole set of which form a family of parallel curves, replace ρ_0 by a where a is the radius of curvatnre of the parallel, corresponding to ρ_0 for the particular involute considered.

Then $$s = a\psi + \int_0^\psi f(\psi)\, d\psi.$$

The difference of the arcs OP, $O'P'$ of these parallel curves is therefore $(\rho_0 - a)\psi$; and if the involutes form closed ovals, the tangent making one complete revolution, the difference of their perimeters is $2\pi(\rho_0 - a)$.

555. In the case of an **involute of a circle**, already discussed in

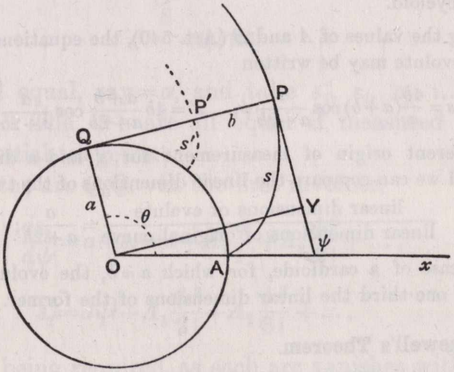

Fig. 124.

Art. 542, if a be the radius, O the centre, A the cusp of the involute, and Q the point of contact of the radius of curvature at P,

$$PQ = a\theta = a\psi \quad \text{and} \quad s = \int a\psi\, d\psi = \frac{a\psi^2}{2}.$$

For a parallel traced by a point at distance b from P

$$s' = \int (a\psi - b)\, d\psi$$

$$= \frac{a\psi^2}{2} - b\psi + C$$

$$= \frac{a}{2}\left(\psi - \frac{b}{a}\right)^2, \text{ if we measure } s' \text{ so that}$$

$$s' = 0 \text{ when } \psi = \frac{b}{a},$$

i.e. another involute of the circle.

556. In the case of the **epi- and hypo-cycloids**

$$s = A \sin B\psi,$$

the evolute is $$s' = AB \cos B\psi' - \rho_0,$$

or, dropping the accent, and writing $\rho_0 + s' = s$, *i.e.* changing the origin of measurement of s suitably,

$$s = AB \cos B\psi,$$

s being measured from the point where $\psi = \frac{\pi}{2B}$, or $s = AB \sin B\psi$ if we choose a suitable initial tangent, viz. that at the point from which s is measured.

Hence the evolute of an epi- or hypo-cycloid is a similar epi- or hypo-cycloid.

Putting $B = 1$ we have a case which shows that the evolute of a cycloid is an equal cycloid.

Supplying the values of A and B (Art. 540), the equations of the curve and of the evolute may be written

$$s = \frac{4b}{a}(a+b) \cos \frac{a}{a+2b}\psi, \quad s' = 4b\frac{a+b}{a+2b} \cos \frac{a}{a+2b}\psi';$$

with a different origin of measurement for s' and a different initial tangent, and we can compare the linear dimensions of the two curves, viz.

$$\frac{\text{linear dimensions of evolute}}{\text{linear dimensions of original curve}} = \frac{a}{a+2b};$$

e.g. in the case of a cardioide, for which $a = b$, the evolute is another cardioide of one-third the linear dimensions of the former.

557. **Whewell's Theorem.**

An interesting theorem is quoted by Boole from Whewell's *Memoir* (above referred to) with regard to the ultimate form to which the successive involutes of a given curve tend, the involutes being such as have equal "tails."

Whewell takes as his original curve $s = F(\psi)$, which he supposes capable of expansion in powers of ψ, and s vanishing with ψ, so that

$$s = A_1\psi + A_2\psi^2 + A_3\psi^3 + \dots,$$

and he further supposes the successive involutes to be defined as having the same "rectilinear tail" at starting.

Let P_0P be the original curve, and Q_0Q, R_0R, $S_0S, \dots$ the successive involutes, and the several "tails" Q_0P_0, R_0Q_0,

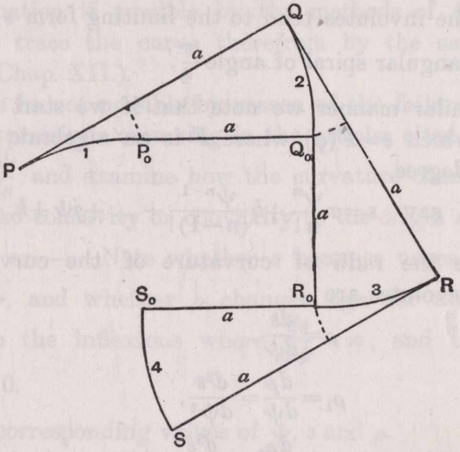

Fig. 125.

$S_0R_0, \dots$ all equal, say $= a$, and take $s_1, s_2, s_3, \dots$ the successive arcs. The ψ's are all equal if measured from the respective initial tangents.

Then for the arc Q_0Q, viz. the first involute,

$$\frac{ds_2}{d\psi} = a + s_1 = a + A_1\frac{\psi}{1!} + A_2\frac{\psi^2}{2!} + \dots,$$

$$s_2 = a\psi + A_1\frac{\psi^2}{2!} + A_2\frac{\psi^3}{3!} + \dots,$$

no constant being required, as each arc vanishes with ψ.

Similarly, $$s_3 = a\psi + a\frac{\psi^2}{2!} + A_1\frac{\psi^3}{3!} + A_2\frac{\psi^4}{4!} + \dots,$$

$$s_4 = a\psi + a\frac{\psi^2}{2!} + a\frac{\psi^3}{3!} + A_1\frac{\psi^4}{4!} + \dots.$$

Proceeding thus,

$$s_n = a\left(\psi + \frac{\psi^2}{2!} + \frac{\psi^3}{3!} + \dots + \frac{\psi^{n-1}}{(n-1)!}\right) + \left(A_1\frac{\psi^n}{n!} + A_2\frac{\psi^{n+1}}{(n+1)!} + \dots\right).$$

And when n is very large the terms in the first bracket (which are unaffected by the form of the original curve) approximate to $e^\psi - 1$.

And those in the second bracket have coefficients which are ultimately infinitesimally small.

Hence the involutes tend to the limiting form $s = a(e^\psi - 1)$, *i.e.* an equiangular spiral of angle $\frac{\pi}{4}$.

In a similar manner we note that if we start off with a curve in which $s = F(\phi)$, where F is an algebraic expression of the n^{th} degree,

$$\text{say} \quad s = a\frac{\psi^n}{n!} + b\frac{\psi^{n-1}}{(n-1)!} + \dots + j\psi + k,$$

then, since the radii of curvature of the curve and its successive evolutes are

$$\rho = \frac{ds}{d\psi},$$

$$\rho_1 = \frac{d\rho}{d\psi} = \frac{d^2s}{d\psi^2},$$

$$\rho_2 = \frac{d\rho_1}{d\psi} = \frac{d^3s}{d\psi^3}, \text{ etc.,}$$

it follows that $\rho_{n-1} = a$.

Hence the $(n-1)^{\text{th}}$ evolute is a circle.

$$\text{Therefore} \quad s = a\frac{\psi^n}{n!} + b\frac{\psi^{n-1}}{(n-1)!} + \dots + j\psi + k$$

is one of the $(n-1)^{\text{th}}$ involutes of a circle of radius a, or parallels to such involutes, the "tails" being the successive coefficients k, j, etc.

558. **Involute of a Catenary.**

Ex. The intrinsic equation of the catenary is $s = c\tan\psi$.

Hence the intrinsic equation of its evolute is

$$s = c\sec^2\psi - \rho_0,$$

and ρ_0 is the radius of curvature at the vertex $= c$

$$\left[\text{for } \rho = \frac{ds}{d\psi} = c\sec^2\psi = c, \text{ when } \psi = 0\right].$$

Hence the evolute is $s = c(\sec^2\psi - 1) = c\tan^2\psi$.

The intrinsic equation of an involute is

$$s=\int (c\tan\psi+A)\,d\psi$$
$$=c\log\sec\psi+A\psi+\text{constant},$$

and if s be so measured that $s=0$ when $\psi=0$, we have

$$s=c\log\sec\psi+A\psi.$$

559. **Tracing of a Curve from the Intrinsic Equation $s=f(\psi)$.**

(1) Generally it is best to obtain the Cartesian or polar form of equation if possible by the methods of Arts. 537, 543, and to trace the curve therefrom by the usual rules (*Diff. Calc.*, Chap. XII.).

(2) If this be not possible by reason of the failure to integrate the expressions occurring in the articles cited, find the curvature $\dfrac{d\psi}{ds}$, and examine how the curvature changes with ψ. Note also concavity or convexity to the origin according as $\dfrac{ds}{d\psi}$ is + or −. Note whether s becomes unreal for any values of ψ, and whether ρ changes sign for any values of ψ. Also the inflexions where $\dfrac{ds}{d\psi}=\infty$, and the cusps where $\dfrac{ds}{d\psi}=0$.

Tabulate corresponding values of ψ, s and ρ.

Observe whether a change of sign in ψ would alter the value of s. If not there is symmetry about the initial line from which ψ is measured.

Examine whether

$$\left.\begin{aligned} x&=\int_0^{\psi}\cos\chi\, f'(\chi)\,d\chi,\\ y&=\int_0^{\psi}\sin\chi\, f'(\chi)\,d\chi,\end{aligned}\right\}$$

even though not (as in the case considered) integrable in general terms, can be evaluated as definite integrals for any particular values of ψ. Approximate values of these integrals may lead to important information as to the position of some points through which the curve passes. For accurate plotting the tabulated values of these integrals for various values of ψ in general becomes necessary. For a general idea of the shape of the curve when close accuracy of plotting is not necessary, an examination of the integrals and the behaviour of the integrand may furnish sufficient information.

560. Ex. Trace the curve $ks^2=\psi$, (k +^{ve}). **Cornu's Spiral.***

Here $$\rho=\frac{ds}{d\psi}=\frac{1}{2ks}.$$

The curvature continuously increases with s. Hence, as s increases, the osculating circle at any point will contain the whole of the remainder of the curve; and ρ diminishes more and more slowly as s increases.

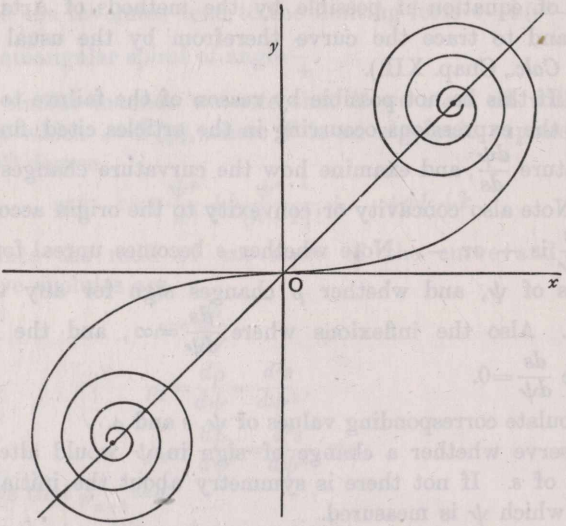

Fig. 126.

Negative values of ψ would give unreal values of s. Each value of ψ gives two values of s, one positive, one negative. It is to be inferred that the origin of measurement of s is a point of symmetry.

We have $$x=\int\cos\psi\,ds=\int_0^s\cos ks^2\,ds,$$

$$y=\int\sin\psi\,ds=\int_0^s\sin ks^2\,ds.$$

These integrals are not integrable in general terms.

But $\int_0^\infty\cos ks^2\,ds=\frac{\sqrt{\pi}}{2\sqrt{2k}}$ is a known result (Art. 1163, Ch. XXVIII.), and $\int_0^\infty\sin ks^2\,ds$ has the same value. These are known as Fresnel's integrals.

* *Journal de Physique*, t. iii., 1874, M. A. Cornu.

Hence, when s becomes very large the curve dwindles down to a point on the line $y=x$ after an infinite number of convolutions about the point. And the point is at a distance from the origin $=\frac{\sqrt{\pi}}{2\sqrt{k}}$. The value of ρ is infinite and changes sign when $s=0$. There is therefore a point of inflexion there.

Also $$\frac{dx}{ds}=\cos ks^2, \quad \frac{dy}{ds}=\sin ks^2,$$

which show that the tangent is parallel to the initial line

$$\text{when } ks^2=0, \quad \pi, \quad 2\pi \ldots,$$

and perpendicular to it

$$\text{when } ks^2=\frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{5\pi}{2} \ldots,$$

which, indeed, is obvious from the equation $ks^2=\psi$

Taking k as unity for convenience,

	$\psi=0,$	$1,$	$2,$	$3,$	$4,$	$5,$	$6,$	$\infty,$
give	$s=0,$	$\pm 1,$	$\pm 1{\cdot}414,$	$\pm 1{\cdot}732,$	$\pm 2,$	$\pm 2{\cdot}236,$	$\pm 2{\cdot}449 \ldots$	$\pm \infty,$
	$\rho=\infty,$	$0{\cdot}500$	$0{\cdot}354,$	$0{\cdot}289,$	$0{\cdot}250,$	$0{\cdot}224,$	$0{\cdot}204 \ldots$	$0.$

We are now in a position to form an idea of the curve which is shown in the figure.

This spiral is of considerable importance in the theory of light, the length and direction of the radius vector at any point giving a graphical representation of the amplitude of the resultant of a system of superposed vibrations.*

The values of Fresnel's Integrals

$$C=\int_0^v \cos\frac{\pi v^2}{2}\,dv, \quad S=\int_0^v \sin\frac{\pi v^2}{2}\,dv,$$

have been calculated for values of v from 0 to ∞ by Gilbert.† The tabulated values are necessary for accurate plotting.

The general methods of evaluating these integrals are discussed by Verdet (*Œuvres*, vol. v.), Fresnel (*Œuvres*, tom. i.), Knockenhauer (*Die undul. des Lichts*), Cauchy (*Comptes Rendus*, t. xv.) and others. See Preston, *Theory of Light*, page 220 onwards.

Incidentally the spiral exhibits graphically the march of these integrals, the abscissa and the ordinate representing the integrals and s being the independent variable, showing their oscillatory character.

Thus $x=\int_0^s \cos\frac{\pi s^2}{2}\,ds$ increases from $s=0$ to $s=\sqrt{1}$, decreases from $s=\sqrt{1}$ to $s=\sqrt{2}$, increases from $s=\sqrt{2}$ to $s=\sqrt{3}$, and so on. And similarly for y.

These integrals will be discussed more fully later.

* Preston, *Theory of Light*, Art. 141, onwards.

† *Mém. couronnés de l' Acad. de Bruxelles*, t. xxxi., 1863.

561. Length of Arc of First Positive Pedal Curve.

Let p be the perpendicular from the origin upon the tangent to any curve, and χ the angle this perpendicular makes with the initial line. We may then regard p, χ as the polar coordinates of a current point on the pedal curve.

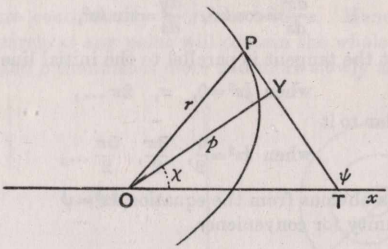

Fig. 127.

Hence the length of an arc s' of the pedal curve may be calculated by the formula

$$s' = \int \sqrt{p^2 + \left(\frac{dp}{d\chi}\right)^2}\, d\chi. \qquad (1)$$

562. Ex. Apply the above method to find the length of any arc of the pedal of a circle with regard to a point on the circumference (*i.e.* a cardioide).

Here, if $2a$ be the diameter, we have from the figure,

$$p = OP\cos\frac{\chi}{2} = 2a\cos^2\frac{\chi}{2}.$$

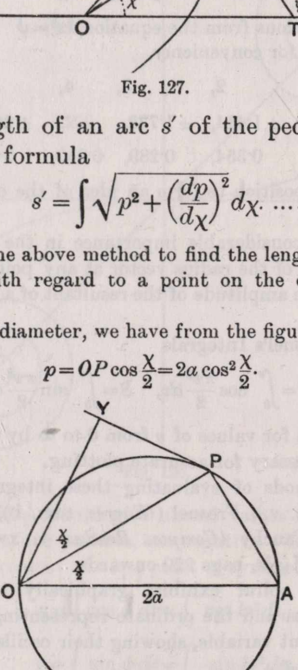

Fig. 128.

Hence, $$\text{arc of pedal} = \int 2\sqrt{a^2\cos^4\frac{\chi}{2} + a^2\sin^2\frac{\chi}{2}\cos^2\frac{\chi}{2}}\, d\chi$$

$$= \int 2a\cos\frac{\chi}{2}\, d\chi = 4a\sin\frac{\chi}{2} + C.$$

The limits for the upper half of this curve are $\chi=0$ and $\chi=\pi$. Hence the whole perimeter of the pedal

$$=2\left[4a\sin\frac{\chi}{2}\right]_0^\pi$$
$$=8a.$$

563. Arc of the Pedal Curve.

Again, the tangent to the locus of Y, the foot of the perpendicular, makes with OY the same angle that the radius vector OP makes with the tangent at the corresponding point P of the original curve.

Thus $$\frac{dp}{ds'}=\frac{dr}{ds};$$

$$\therefore\quad s'=\int\frac{dp}{dr}\,ds=\int\frac{r}{\rho}\,ds, \quad\text{.......................(2)}$$

which again expresses the arc of the pedal in terms of elements of the original curve.

The result may be presented in various forms.

Thus $$s'=\int\frac{r}{\rho}\,ds=\int r\,d\psi, \quad\text{..........................(3)}$$

which is equivalent to (1) for

$$\psi=\frac{\pi}{2}+\chi \quad\text{and}\quad r^2=p^2+\left(\frac{dp}{d\psi}\right)^2.$$

Also $$s'=\int\frac{\sqrt{x^2+y^2}\,\dfrac{d^2y}{dx^2}}{1+\left(\dfrac{dy}{dx}\right)^2}\,dx, \quad\text{..............for Cartesians (4)}$$

or $$=\int\frac{r\left\{r^2+2\left(\dfrac{dr}{d\theta}\right)^2-r\dfrac{d^2r}{d\theta^2}\right\}}{r^2+\left(\dfrac{dr}{d\theta}\right)^2}\,d\theta, \quad\text{.......for polars (5)}$$

or $$=\int\frac{r\dfrac{dp}{dr}}{\sqrt{r^2-p^2}}\,dr, \quad\text{............for pedal equations (6)}$$

(from equation 2).

564. Arc of a First Negative Pedal.

If the original curve be $r=f(\theta)$, then r, θ are the polar coordinates of the foot of the perpendicular from the pole upon the tangent to the first negative pedal, whose tangential polar equation may therefore be written $p=f(\chi)$, χ being the angle the perpendicular to the tangent makes with the initial line

$$\left(\text{viz. } \chi=\psi-\frac{\pi}{2}\right).$$

Also
$$\frac{ds}{d\chi}=p+\frac{d^2p}{d\chi^2};$$

$$\therefore\ s=\frac{dp}{d\chi}+\int f(\chi)\,d\chi+\text{constant}.$$

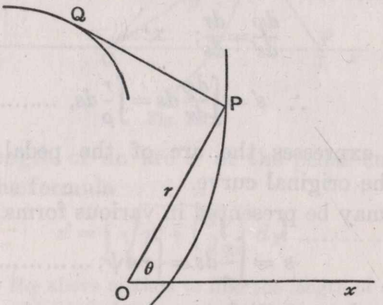

Fig. 129.

565. Ex. Find the intrinsic equation of the first negative pedal of an ellipse $\frac{l}{r}=1+e\cos\theta$, with regard to the pole.

Here
$$s=\frac{d}{d\chi}\frac{l}{1+e\cos\chi}+\int\frac{l}{1+e\cos\chi}\,d\chi;$$

$$\therefore\ s=\frac{el\sin\chi}{(1+e\cos\chi)^2}+\frac{l}{\sqrt{1-e^2}}\cos^{-1}\frac{e+\cos\chi}{1+e\cos\chi}+\text{const.}$$

If we choose to measure s from the point where $\chi=0$,

$$\text{the constant}=-\frac{l}{\sqrt{1-e^2}}\cos^{-1}1=0.$$

PROBLEMS.

1. Show that the whole length of the curve
$$y^2(a^2-y^2)=8a^2x^2 \quad \text{is} \quad \pi a\sqrt{2}.$$ [Oxford I. P., 1890.]

2. Find the whole length of the loop of the curve
$$3ay^2=x(x-a)^2.$$ [Oxford I. P., 1889.]

3. Show that the arcs of an equiangular spiral, measured from the pole to the different points of its intersection with another equiangular spiral having the same pole but a different angle, will form a series in geometrical progression. [TRINITY, 1884.]

4. Show that the length of an arc of the curve $y^n = x^{m+n}$ can be found in finite terms in the cases when $\frac{n}{2m}$ or $\frac{n}{2m}+\frac{1}{2}$ is an integer.

5. Evaluate the expressions,

$$\text{(i)} \int y\frac{dx}{ds}\,ds, \qquad \text{(ii)} \int x\frac{dy}{ds}ds, \qquad \text{(iii)} \int\left(\frac{x}{r^2}\frac{dy}{ds}-\frac{y}{r^2}\frac{dx}{ds}\right)ds,$$

wherein the line-integrals are taken round the perimeter of a closed curve. [ST. JOHN'S, 1890.]

6. If s be the length of the curve

$$r = a\tanh\frac{\theta}{2}$$

between the origin and $\theta = 2\pi$, and A be the area between the same points, show that

$$A = a(s-a\pi). \qquad \text{[OXFORD I., 1888.]}$$

7. Show that if the arc of the curve

$$r = a\tanh^n\frac{\theta}{2n} \quad (n \text{ being integral}),$$

measured from the origin, be called s, and if A be the corresponding area swept out by the radius vector from the origin,

$$A = \frac{a^2\theta}{2} - na^2\left[\left(\frac{r}{a}\right)^{\frac{1}{n}} + \frac{1}{3}\left(\frac{r}{a}\right)^{\frac{3}{n}} + \dots + \frac{1}{2n-1}\left(\frac{r}{a}\right)^{\frac{2n-1}{n}}\right],$$

$$\left\{\begin{array}{l} s+r = a\theta - 2na\left[\left(\frac{r}{a}\right)^{\frac{1}{n}} + \frac{1}{3}\left(\frac{r}{a}\right)^{\frac{3}{n}} + \dots + \frac{1}{n-2}\left(\frac{r}{a}\right)^{\frac{n-2}{n}}\right], \\ \hfill \text{if } n \text{ be odd and } > 2, \\ s+r = 2na\left[\log\cosh\frac{\theta}{2n} - \left\{\frac{1}{2}\left(\frac{r}{a}\right)^{\frac{2}{n}} + \frac{1}{4}\left(\frac{r}{a}\right)^{\frac{4}{n}} + \dots + \frac{1}{n-2}\left(\frac{r}{a}\right)^{\frac{n-2}{n}}\right\}\right], \\ \hfill \text{if } n \text{ be even,} \end{array}\right.$$

the results for $r = a\tanh\frac{\theta}{2}$ giving $2A = a(s-r) = a(2s-a\theta)$.

8. Show that the length of an arc of the Cissoid of Diocles

$$r = a\frac{\sin^2\theta}{\cos\theta}$$

is $a\sqrt{3}(z-\tanh^{-1}z)$ taken between limits θ_1 and θ_2 where

$$3z^2\cos\theta = 1 + 3\cos^2\theta.$$

9. Show that the intrinsic equation of the semicubical parabola

$$3ay^2 = 2x^3 \quad \text{is} \quad 9s = 4a(\sec^3\psi - 1).$$

10. In a certain curve

$$x = e^{\theta}\sin\theta, \quad y = e^{\theta}\cos\theta.$$

Show that $s = e^{\theta}\sqrt{2} + C$.

Also that for the curve

$$x = e^{a\theta}\sin b\theta, \quad y = e^{a\theta}\cos b\theta, \quad s = e^{a\theta}\sqrt{a^2+b^2} + C.$$

Name these curves.

11. Show that the length of an arc of the curve

$$x\sin\theta + y\cos\theta = f'(\theta),$$
$$x\cos\theta - y\sin\theta = f''(\theta),$$

is given by $s = f(\theta) + f''(\theta) + C$.

12. Trace the curve $y^2 = \dfrac{x}{3a}(a-x)^2$, and find the length of that part of the evolute which corresponds to the loop.

[St. John's, 1881 and 1891.]

13. Show that the curve whose pedal equation is $p^2 = r^2 - a^2$ has for its intrinsic equation $s = a\dfrac{\psi^2}{2}$.

What curve is this?

14. The coordinates of a point on a plane curve are given by the relations

$$x = a[(1-\theta^2)\cos\theta + 2\theta\sin\theta - 1],$$
$$y = a[(1-\theta^2)\sin\theta - 2\theta\cos\theta];$$

prove that

$$3sa^{\frac{1}{2}} = (2a+\rho)(\rho-a)^{\frac{1}{2}},$$

ρ being the radius of curvature at the point and s the arcual distance from the origin.

[Oxford II. P., 1888.]

15. The evolute of a parabola whose vertex is A meets the axis in C, and the parabola in Q. Find the perimeter of the figure bounded by AC, the parabolic arc AQ, and the arc of the evolute CQ.

[Oxford I. P., 1889.]

16. Prove that the length of the first negative pedal, taken with respect to the origin, of the loop of the folium of Descartes

$$x^3 + y^3 - 3axy = 0$$

is equal to

$$6a - a\{\pi - \sqrt{2}\log(\sqrt{2}+1)\}.$$

17. Find the length of the arc between two consecutive cusps of the curve

$$(c^2 - a^2)p^2 = c^2(r^2 - a^2).$$

[Oxford I. P., 1889.]

18. Show that the length of the arc of the hyperbola $xy=a^2$ between the limits $x=b$ and $x=c$ is equal to the arc of the curve $p^2(a^4+r^4)=a^4r^2$, between the limits $r=b$, $r=c$. [OXFORD I. P., 1888.]

19. By means of the formula $s=\int \frac{r\,dr}{\sqrt{r^2-p^2}}$, find the length of the curve $r=a\sin^2\frac{\theta}{2}$. [COLLEGES α, 1887.]

20. If s be the arc of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ measured from the end of the major axis to a point whose eccentric angle is ϕ, prove that

$$s+ae^2\cos\phi\sin\theta=\int_0^\theta \sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta,$$

where $$\theta=\tan^{-1}\left(\frac{a}{b}\tan\phi\right).$$ [COLLEGES α, 1883.]

21. Show that the circumference of an ellipse can be expressed either as

$$4a\int_0^{\frac{\pi}{2}}(1-e^2\sin^2\theta)^{\frac{1}{2}}\,d\theta$$

or as $$4a(1-e^2)\int_0^{\frac{\pi}{2}}(1-e^2\sin^2\theta)^{-\frac{3}{2}}d\theta,$$

where a is the semi-major axis, e the eccentricity. [TRINITY, 1887.]

22. Show that the three-cusped hypocycloid has equations of the forms

(i) $p=b\cos 3\psi$,

(ii) $r^4+8br^3\cos 3\theta+18b^2r^2=27b^4$.

Show that the length of an arc of the inverse of this curve with respect to the centre is proportional to

$$\tan^{-1}(2\sqrt{2}\sin 3\psi).$$ [ST. JOHN'S, 1887.]

23. Prove that the intrinsic equation of the curve

$$r^{\frac{1}{5}}=a^{\frac{1}{5}}\cos\frac{1}{5}\theta \quad \text{is} \quad \frac{s}{a}=\frac{5}{16}\psi+\frac{5}{4}\sin\frac{\psi}{3}+\frac{5}{32}\sin\frac{2\psi}{3},$$

where s and ψ are measured from the point a, 0, and the tangent at that point. [ST. JOHN'S, 1889.]

24. A circle of perimeter S and area A rolls externally in its plane entirely round an oval curve of perimeter S and area B. Prove that its centre describes an oval of perimeter $2S$ and area $3A+B$. [OXFORD I. P., 1918.]

25. Find the centroid of a sector bounded by two radii vectores, and an arc of the curve whose polar equation is

$$r^2 = a^2(1 - \sin 2\theta)(1 + \sin 2\theta)^{-1},$$

and show that an arc of this curve is expressible as

$$\frac{5a}{2}\int \frac{\cos^2\chi\, d\chi}{(1+\sqrt{5}\sin\chi)\sqrt{1-5\sin^2\chi}}.$$

[MATZ, *Educ. Times.*]

26. A rod moves always to pass through a fixed point and have one extremity on a straight line distant h from the point. Show that the arc of the curve traced out by its centre of instantaneous rotation, as the rod moves from the perpendicular position to one inclined at 45° to the line, is

$$\tfrac{1}{4}\{\log(\sqrt{5}+2) + 2\sqrt{5}\}\, h.$$ [MATH. TRIPOS, 1883.]

27. On the tangent at any point P of a curve, PT is taken equal to the radius vector of P; show how to find the length of any arc of the locus of T. For example, take the equiangular spiral and verify the result geometrically. [ST. JOHN'S, 1884.]

28. Find the arc of the curve enveloped by the line

$$x\cos\phi + y\sin\phi = (a\cos^2\phi + b\sin^2\phi)^{-3}$$

between the points corresponding to $\phi = 0$, $\phi = \frac{\pi}{2}$. [ST. JOHN'S, 1891.]

29. Find the whole area of the curve

$$x = a\sin\theta - b\sin 2\theta,$$
$$y = a\cos\theta - b\cos 2\theta,$$

and show that the whole length of its perimeter is equal to that of an ellipse whose semiaxes are $a+2b$, $a-2b$. [COLLEGES a, 1885.]

30. Prove that if s be the arc of the curve

$$\left.\begin{aligned} r &= a\sec\alpha, \\ \theta &= \tan\alpha - \alpha, \end{aligned}\right\}$$

where α is a variable parameter, measured from the initial line to a point P on the curve, and if A be the area bounded by the curve, the initial line, and the radius vector to P, then

$$9A^2 = 2as^3.$$

Find the area swept out in any portion of its progress by the intercept of the tangent to the curve between the curve and the first positive pedal with regard to the origin. [TRINITY, 1890.]

31. If a curve be given by $r^{2n}=\sin^2\phi+m^2\cos^2\phi$, where r is the radius vector and ϕ the angle it makes with the tangent, show that

$$m\tan\frac{1}{m}(\phi-n\theta)=\tan\phi,$$

θ being the angle the radius vector makes with the initial line (which is to be appropriately chosen).

Obtain also a formula for the rectification of the curve. (The result is not obtainable in *finite terms.*) [I. C. S., 1898.]

32. Consider the nature of the curves

$$\text{(i) } s\psi^2=a, \qquad \text{(ii) } \frac{\psi}{2\pi}=\sin\frac{s}{a}, \qquad \text{(iii) } s=l\sin m\psi,$$

when $m<1$ and when $m>1$.

33. Given a closed oval of continuous curvature without any singularities: a series of parallel curves is drawn. Prove that if A denote the area of any one of them and l its perimeter, then

$$4\pi A-l^2$$

is the same for all. [I. C. S., 1895.]

34. In the equation of the curve $r=a+\epsilon u$, a and ϵ are constants, the latter being small; and u is a function of θ finite for all values of θ and periodic, with a period 2π. Show that if A denote the area of the curve, then its length is $2\sqrt{\pi A}$ accurately as far as small quantities of the first order inclusive. [I. C. S., 1896.]

35. The area of an ellipse differs from that of its auxiliary circle by 10 per cent. of the area of the latter. Show that the perimeter of the ellipse differs from that of the auxiliary circle by 4·93 per cent. approximately of the perimeter of the latter. [I. C. S., 1910.]

36. Assuming that for the catenary formed by a hanging elastic wire

$$\frac{x}{c}=u+k\,\mathrm{sh}\,u, \quad \frac{y}{c}=\mathrm{ch}\,u+\tfrac{1}{2}k\,\mathrm{ch}^2u,$$

prove that
$$\frac{s}{c}=\tfrac{1}{2}ku+\mathrm{sh}\,u+\tfrac{1}{4}k\,\mathrm{sh}\,2u,$$

reducing to the common catenary when $k=0$ and approximating to a parabola when k is large. [B.A. Hon. Lond., 1899.]

37. In the cycloid $y=\dfrac{s^2}{8a}$. Show that the only curve for which *both* x and y are finite integral functions of s is a straight line. [Oxf. I. P., 1913.]

38. Find the Cartesian equation (choosing convenient axes of coordinates) of the curve in which

$$\rho^2/a^2=(d\rho/ds)^2+1. \qquad \text{[Oxf. I. P., 1917.]}$$

39. Find the intrinsic equation of the curve $27ay^2 = 4x^3$. Prove that the involutes of the curve $27ay^2 = 4x^3$ are given by the equations

$$x = a\tan^2\psi + c\cos\psi - 2a,$$
$$y = -2a\tan\psi + c\sin\psi,$$

c being an arbitrary constant.

What happens when $c = 0$? [OXF. I. P., 1915.]

40. Show that the length of a quadrant of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is equal to $\dfrac{3a}{2}$, and find the length of one quadrant of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$

[MATH. TRIPOS, PART I., 1910.]

41. Trace the form of the curve

$$x(1+t^2) = 1 - t^2, \quad y(1+t^2) = 2t,$$

as t increases from $-\infty$ to ∞, and show that its area is π.

Find also the length of any arc of the curve in terms of t.

[MATH. TRIP., PART I., 1910.]

42. Show that the length of the arc of the parabola $y^2 = 4ax$ which is intercepted between the point of intersection of the parabola and $3y = 8x$ is

$$a(\log 2 + \tfrac{15}{16}).$$ [MATH. TRIP. I., 1908.]

43. Prove that the perimeter of an ellipse of small eccentricity e and semiaxes a, b is equal to

$$\tfrac{1}{2}\pi\{3(a+b) - 2\sqrt{ab}\},$$

neglecting e^7 and higher powers. [MATH. TRIP. I., 1917.]

44. Prove that the length of an ellipse may be expressed by $\displaystyle\iint\frac{dS}{\rho}$ taken over the area, where dS is an element of the area of the ellipse and ρ the radius of curvature of the similar, similarly situated and concentric ellipse passing through the element dS.

[COLLEGES, 1892.]

45. Find the intrinsic equations of a circle, a catenary, and a cycloid, and trace the curves $s = a\phi^3$, $s\phi = a$ and $s^2 = a^2\phi$.

At any point P of a cycloid the tangent is produced to a length PT equal to the arc measured from the vertex, and at T a perpendicular is drawn equal to the radius of curvature at P. Prove that the locus of the extremity of this perpendicular is the same cycloid moved parallel to its axis through a distance equal to twice the diameter of the generating circle. [ST. JOHN'S COLLEGE, 1882.]