CHAPTER XV.

QUADRATURE (IV).

MISCELLANEOUS THEOREMS, CONNEXION OF A LINE-INTEGRAL AND A SURFACE-INTEGRAL, MECHANICAL INTEGRATION, ETC.

466. A THEOREM DUE TO STOKES.

Let u and v be two functions of x and y, finite, single-valued and continuous at every point within and along the boundary of a given region bounded by any given contour line in the plane of x, y having no multiple points, and let the differential coefficients $\frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ *be also functions which are finite, single-valued, and continuous at all points of the region; then the line-integral*

$$\int\left(u\frac{dx}{ds}+v\frac{dy}{ds}\right)ds$$

taken round the perimeter of the contour is equal to the surface-integral

$$\iint\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy$$

taken over the region bounded by the contour. We shall first consider u and v to be *real* functions of x and y.

Let the region referred to be indicated, as shown in the accompanying figure, with an inner boundary and an outer boundary, the inner boundary enclosing a region *within* which the integration is *not* to be performed.

Divide the whole contour into two systems of strips of infinitesimal breadth parallel to the coordinate axes. Two typical strips are shown in the figure, the one parallel to the x-axis being bounded by lines with ordinates y and $y+\delta y$, and that parallel to the y-axis bounded by lines with abscissae x and $x+\delta x$. The first intercepts elementary arcs

$P_1Q_1=\delta s_1,\quad P_2Q_2=\delta s_2,\quad P_3Q_3=\delta s_3$, etc., an even number, and the second intercepts

$P_1'Q_1'=\delta s_1',\quad P_2'Q_2'=\delta s_2',\quad P_3'Q_3'=\delta s_3'$, etc., an even number.

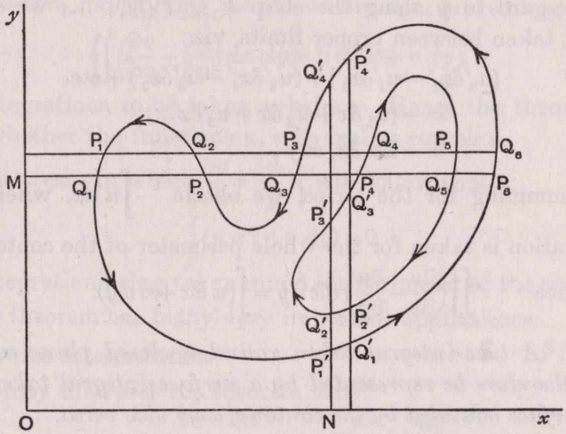

Fig. 88.

The direction of integration is indicated in the figure; the region to be integrated over being on the left hand as a person travels along either boundary, following the direction of increase of s. The signs of δy at the several points P_1, P_2, P_3, P_4, ... are respectively $-\delta y$, $+\delta y$, $-\delta y$, $+\delta y$, ..., and the signs of δx at the points P_1', P_2', P_3', P_4', ... are respectively $+\delta x$, $-\delta x$, $+\delta x$, $-\delta x$, etc.

Let u_r, v_r be the respective values of u, v at P_r, and u_r', v_r' those at P_r'.

And let the abscissae and ordinates of the points P_r, Q_s, P_r', Q_s' be x, y with the corresponding accents and suffixes.

If we integrate $\dfrac{\partial v}{\partial x}\delta y$ with regard to x along the strip $P_1Q_2P_3Q_4$, ..., we have $[v\,\delta y]$, taken between proper limits, viz.

$$\begin{aligned}(v_2\,\delta y_2-v_1\,\delta y_1)+(v_4\,\delta y_4-v_3\,\delta y_3)+\ldots+(v_{2n}\,\delta y_{2n}-v_{2n-1}\,\delta y_{2n-1})\\ =v_1\,\delta y+v_2\,\delta y+v_3\,\delta y+\ldots+v_{2n}\,\delta y\\ =\Sigma v\,\delta y, \text{ say, for the strip.}\end{aligned}$$

If then we sum the result for the whole set of strips parallel to the x-axis by integration, we have $\int v\,dy$, where the integration is taken for the whole perimeter of the contour. Similarly for the strips parallel to the y-axis, if we integrate $\frac{\partial u}{\partial y}\delta x$ with regard to y along the strip $P_1'Q_2'P_3'Q_4'$, ..., we obtain $[u\,\delta x]$, taken between proper limits, viz.

$$\begin{aligned}(u_2'\delta x_2' - u_1'\delta x_1') + (u_4'\delta x_4' - u_3'\delta x_3') + \text{etc.}\\ = -(u_1'\delta x + u_2'\delta x + u_3'\delta x \ldots)\\ = -\Sigma u\,\delta x, \text{ say;}\end{aligned}$$

and, summing for the strips, we obtain $-\int u\,dx$, where the integration is taken for the whole perimeter of the contour.

Hence $$\iint\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy = \int(u\,dx + v\,dy).$$

467. *A line-integral taken round a closed plane contour may therefore be represented by a surface-integral taken over the surface bounded by the contour, and vice versa.*

Or, we may say that if u, v be the components parallel to the axes of x and y of any vector quantity, then $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ may be regarded as another vector quantity at right angles to the plane of xy, and such that the line-integral of u, v round a contour in the plane of x, y is equal to the surface-integral of the vector quantity $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ taken over the surface. This theorem is part of a more general three-dimension theorem due to Professor Stokes.*

468. **Extension to Complex Functions.**

If the functions u and v be not entirely real, let them be separated into their real and imaginary parts, viz.

$$u = u_1 + \iota u_2, \quad v = v_1 + \iota v_2,$$

where u_1, u_2, v_1, v_2 are single-valued finite and continuous functions of x and y for all points within and upon the contour, as also their first differential coefficients.

* Smith's Prize, 1854; Maxwell, *Elect. and Mag.*, vol. i., p. 25.

Then we have

$$\iint\left(\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}\right)dx\,dy=\int(u_1\,dx+v_1\,dy),$$

$$\iint\left(\frac{\partial v_2}{\partial x}-\frac{\partial u_2}{\partial y}\right)dx\,dy=\int(u_2\,dx+v_2\,dy).$$

Therefore, multiplying the second line by ι and adding to the first,

$$\iint\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy=\int(u\,dx+v\,dy),$$

the integrations to be taken as before. Hence the theorem is true whether the functions u, v be real or complex.

In any case in which $\frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}$ it will follow that

$$\int(u\,dx+v\,dy)=0,$$

the integration being taken round the perimeter of the contour.

The theorem has many very important applications.

469. **An Interpretation.**

We may interpret the theorem thus:

Let $$\sigma=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},\quad \rho=u\frac{dx}{ds}+v\frac{dy}{ds}.$$

Then $$\iint\sigma\,dx\,dy=\int\rho\,ds;$$

that is the mass of a plane lamina bounded by any closed contour for surface density $\sigma=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ is equal to the mass of the perimeter with a line density

$$\rho=u\frac{dx}{ds}+v\frac{dx}{ds}.$$

470. **Illustrations.**

Ex. 1. Taking $$u=-y\quad v=x,$$

we have at once $\iint dx\,dy=\frac{1}{2}\int(x\,dy-y\,dx)$, which expressions have been established (Arts. 409 and 452) as measures of the area.

Ex. 2. Let $$u=e^x\sin y-ay,\quad v=e^x\cos y-a.$$

Then $$\int\left[(e^x\sin y-ay)\frac{dx}{ds}+(e^x\cos y-a)\frac{dy}{ds}\right]ds$$

taken round the perimeter of the contour

$$=\iint[e^x\cos y-(e^x\cos y-a)]\,dx\,dy=\iint a\,dx\,dy$$

$$=a\times\text{area of the figure enclosed by the contour.}$$

Ex. 3. Consider the effect of integrating

$$I=\int[(\cos x\cosh y-Ay)\,dx+(\sin x\sinh y-Bx)\,dy]$$

round any closed contour.

Here $\qquad u=\cos x\cosh y-Ay\quad$ and $\quad v=\sin x\sinh y-Bx.$

Therefore $\quad \dfrac{\partial v}{\partial x}=\cos x\sinh y-B\quad$ and $\quad \dfrac{\partial u}{\partial y}=\cos x\sinh y-A.$

Hence

$$I=\iint(A-B)\,dx\,dy=(A-B)\times\text{area enclosed by the contour.}$$

Ex. 4. If U, V be any single-valued conjugate functions of x and y *i.e.* real functions of x and y, such that $U+\iota V=f(x+\iota y)$, and if

$$u=V-Ay,\quad v=U-Bx,$$

then $\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}=\dfrac{\partial U}{\partial x}-\dfrac{\partial V}{\partial y}-B+A=A-B$ [see *Diff. Cal.*, Art. 190],

and $\int[(V-Ay)\,dx+(U-Bx)\,dy]$ round a closed contour

$$=\iint(A-B)\,dx\,dy=(A-B)\times\text{area bounded by the contour.}$$

That many different forms of U and V may lead to the same result is obvious from the consideration that the mass of the area bounded by the contour for a given distribution of surface density may be equal to the mass of the perimeter for many distributions of line density.

471. **Two Resulting Theorems.**

If P, Q, U be any three functions of x and y, finite and continuous throughout and along the boundary of a given contour, as also their first differential coefficients, we have

$$\iint\left\{\frac{\partial}{\partial x}(PU)+\frac{\partial}{\partial y}(QU)\right\}dx\,dy$$

$$=-\int U(Q\,dx-P\,dy)=\int U\left(P\frac{dy}{ds}-Q\frac{dx}{ds}\right)ds,$$

i.e. $$\iint\left(P\frac{\partial U}{\partial x}+Q\frac{\partial U}{\partial y}\right)dx\,dy$$

$$=-\iint U\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dx\,dy+\int U\left(P\frac{dy}{ds}-Q\frac{dx}{ds}\right)ds,$$

the double integrals being understood to be taken over the whole area bounded by the contour, and the single integral being taken round the perimeter in the positive direction, *i.e.* leaving the area bounded to the left in travelling in the direction in which s is measured.

472. If R, S, T, U be any four functions of x, y which, with their first and second differential coefficients, are continuous and finite throughout and along the boundary of a given contour, we have, supposing suffixes to denote partial differential coefficients,

$$\frac{\partial}{\partial x}\left\{\begin{vmatrix} R, & R_x \\ U, & U_x \end{vmatrix}+SU_y\right\}-\frac{\partial}{\partial y}\left\{\begin{vmatrix} U, & U_y \\ T, & T_y \end{vmatrix}+S_xU\right\}$$

$$=(R_xU_x+RU_{xx}-R_{xx}U-R_xU_x+S_xU_y+SU_{xy})$$

$$-(U_yT_y+UT_{yy}-U_{yy}T-U_yT_y+S_{xy}U+S_xU_y)$$

$$=(RU_{xx}+SU_{xy}+TU_{yy})-U(R_{xx}+S_{xy}+T_{yy}).$$

Hence

$$\iint[(RU_{xx}+SU_{xy}+TU_{yy})-U(R_{xx}+S_{xy}+T_{yy})]dx\,dy$$

$$=\iint\left[\frac{\partial}{\partial x}\left\{\begin{vmatrix} R, & R_x \\ U, & U_x \end{vmatrix}+SU_y\right\}-\frac{\partial}{\partial y}\left\{\begin{vmatrix} U, & U_y \\ T, & T_y \end{vmatrix}+S_xU\right\}\right]dx\,dy$$

$$=\int\left[\left\{\begin{vmatrix} U, & U_y \\ T, & T_y \end{vmatrix}+S_xU\right\}\frac{dx}{ds}+\left\{\begin{vmatrix} R, & R_x \\ U, & U_x \end{vmatrix}+SU_y\right\}\frac{dy}{ds}\right]ds,$$

the double integral being taken over the area bounded by the contour and the single integral round the perimeter.

Thus

$$\iint\left(RU_{xx}+SU_{xy}+TU_{yy}\right)dx\,dy=\iint U\left(R_{xx}+S_{xy}+T_{yy}\right)dx\,dy$$

$$+\int\left[\left\{\begin{vmatrix} U, & U_y \\ T, & T_y \end{vmatrix}+S_xU\right\}\frac{dx}{ds}+\left\{\begin{vmatrix} R, & R_x \\ U, & U_x \end{vmatrix}+SU_y\right\}\frac{dy}{ds}\right]ds.$$

These results will be useful later (Chapter XXXIV.).

473. Motion of a Rod in a Plane.

Let O be the origin and Ox, Oy any fixed rectangular axes in the plane.

Let a rod move in any manner in the plane.

Let P_1, P_2, P_3 be points attached to it, their coordinates being

$$(x_1,\ y_1);\quad (x_2,\ y_2);\quad (x_3,\ y_3).$$

Let $\qquad P_2P_3=a_1,\quad P_3P_1=a_2,\quad P_1P_2=a_3,$

so that $\qquad a_1+a_2+a_3=0.$

Let θ be the angle the rod makes at any instant with the x-axis.

Then $x_1=x_2-a_3\cos\theta,\qquad x_3=x_2+a_1\cos\theta,$

$$y_1=y_2-a_3\sin\theta,\qquad y_3=y_2+a_1\sin\theta;$$

$$\therefore\quad dx_1=dx_2+a_3\sin\theta\,d\theta,\quad dx_3=dx_2-a_1\sin\theta\,d\theta,$$

$$dy_1=dy_2-a_3\cos\theta\,d\theta,\quad dy_3=dy_2+a_1\cos\theta\,d\theta;$$

$$\therefore\quad x_1dy_1-y_1dx_1=(x_2-a_3\cos\theta)(dy_2-a_3\cos\theta\,d\theta)$$
$$-(y_2-a_3\sin\theta)(dx_2+a_3\sin\theta\,d\theta)$$
$$=x_2dy_2-y_2dx_2+a_3^2d\theta-a_3(R\cos\theta-S\sin\theta),$$

where $R=dy_2+x_2d\theta$

$$S=dx_2-y_2d\theta,$$

and $x_3dy_3-y_3dx_3=x_2dy_2-y_2dx_2+a_1^2d\theta+a_1(R\cos\theta-S\sin\theta).$

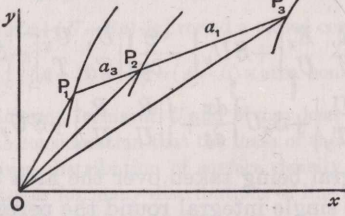

Fig. 89.

Hence, eliminating $R\cos\theta-S\sin\theta$,

$$a_1(x_1dy_1-y_1dx_1)+a_3(x_3dy_3-y_3dx_3)=(a_1+a_3)(x_2dy_2-y_2dx_2)$$
$$+a_1a_3(a_1+a_3)d\theta,$$

i.e. $a_1(x_1dy_1-y_1dx_1)+a_2(x_2dy_2-y_2dx_2)+a_3(x_3dy_3-y_3dx_3)$

$$+a_1a_2a_3d\theta=0.$$

If, then, O be the origin and dA_1, dA_2, dA_3 the elementary sectorial areas described by OP_1, OP_2, OP_3, respectively,

$$a_1dA_1+a_2dA_2+a_3dA_3+\tfrac{1}{2}a_1a_2a_3d\theta=0.$$

Hence, if the points P_1, P_2, P_3 describe closed curves, and A_1, A_2, A_3 be the areas of these curves, and if the rod returns to its original position after making one complete revolution, then

$$a_1A_1+a_2A_2+a_3A_3+\pi a_1a_2a_3=0.$$

474. Various Cases.

If the rod returns to its original position without completing a revolution, rotating in one direction during part of its motion and in the opposite direction during another part, then $\int d\theta = 0$; and

$$a_1A_1 + a_2A_2 + a_3A_3 = 0.$$

475. If then the contours of A_1 and A_3 be such that the rod cannot complete a rotation, but must oscillate as in the case of the connecting rod in a steam engine, we have

$$A_2 = \frac{a_1A_1 + a_3A_3}{a_1 + a_3}.$$

476. If it makes *several complete rotations forwards, say m times, and backwards n times*, whilst the several points P_1, P_2, P_3 describe closed curves once, then $\int d\theta = (m-n)2\pi$; and

$$a_1A_1 + a_2A_2 + a_3A_3 + (m-n)\pi a_1a_2a_3 = 0.$$

477. *If two of the points, say P_1 and P_3, are constrained to move on fixed curves* and the rod rotates *once* round, as, for

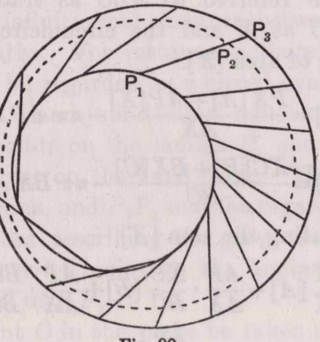

Fig. 90.

instance, if the ends were one on each of a pair of confocal ellipses, or on a pair of circles, as in Fig. 90,

$$A_2 = \frac{a_1A_1 + a_3A_3}{a_1 + a_3} - \pi a_1a_3.$$

478. *If P_1 and P_3 move on the same curve* $A_1 = A_3$, and the theorem reduces to $A_2 = A_1 - \pi a_1a_3$.

This last result is known as Holditch's Theorem.

479. It should be noticed that in the above results, if any of the contours are described in a sense opposite to others, such areas are to be reckoned of opposite sign to the others.

480. Leudesdorf's Theorem.

As an application of this theorem, consider the motion of a lamina on which A, B, C, P are fixed points, the lamina being constrained to move so that A, B, C and P describe closed

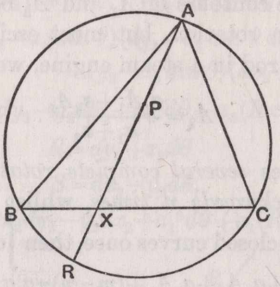

Fig. 91.

curves of areas $[A]$, $[B]$, $[C]$, $[P]$. Let x, y, z be the areal coordinates of P referred to ABC as triangle of reference. Let AP cut BC at X and the circumcircle at R. Let X describe a curve of area $[X]$.

Then $$[P]=\frac{PX[A]+AP[X]}{AX}-n\pi\, AP\,.\,PX,$$

$$[X]=\frac{XC[B]+BX[C]}{BC}-n\pi\, BX\,.\,XC.$$

Hence, eliminating the area $[X]$,

$$[P]=\frac{PX}{AX}[A]+\frac{AP}{AX}\cdot\frac{XC}{BC}[B]+\frac{AP}{AX}\cdot\frac{BX}{BC}[C]$$
$$-n\pi\cdot\frac{AP}{AX}\cdot BX\,.\,XC-n\pi\, AP\,.\,PX.$$

Now $$\frac{PX}{AX}=x,\quad \frac{AP}{AX}\cdot\frac{XC}{BC}=y,\quad \frac{AP}{AX}\cdot\frac{BX}{BC}=z,$$

and $$\frac{AP}{AX}\cdot BX\,.\,XC+AP\,.\,PX=AP\left(\frac{AX\,.\,XR}{AX}+PX\right)$$
$$=AP\,.\,PR$$

= rectangle of segments of any chord of the circumcircle through P;

$\therefore$ $[P]=x[A]+y[B]+z[C]-n\pi\times$ rectangle of segments of chord.

If P lies outside the circle, instead of the rectangle of segments, we may put $-(\text{tangent})^2$, and the theorem may be written

$$[P]=x[A]+y[B]+z[C]+n\pi t^2,$$

t being the tangent from P to the circumcircle.

This theorem is due to Leudesdorf.*

481. Motion of a Plane Lamina sliding in any Manner upon a Fixed Plane. Two Theorems.

When a plane lamina moves in any manner upon a fixed plane, so that in the end it again takes up its original position, it is clear that every point in the lamina will take up its original position, that is that the several points in their motion have travelled along paths back to the same points from which they started, and may therefore be regarded as having travelled along closed curves. This will be supposed to include paths which are retraced, which may be regarded as closed curves of infinitesimal distance between the outgoing and returning paths. For instance, a finite straight line of length $2a$ might be regarded as a closed oval—say an ellipse of semimajor axis a and infinitesimal minor axis.

Suppose two points on the lamina P_1 and P_3 to trace out known closed curves on the fixed plane. This will define the motion of the lamina, and P_1P_3 may be regarded as a straight rod whose ends are describing the given closed curves. Let P be any other carried point on the lamina and PP_2 a perpendicular from P to P_1P_3.

Let a fixed point O in the plane be taken as origin, and let

$$P_2P_3=a_1,\quad P_3P_1=a_2,\quad P_1P_2=a_3 \quad\text{and}\quad PP_2=p,$$

so that

$$a_1+a_2+a_3=0.$$

We shall continue to adopt the convenient notation $[P]$ for the area swept out by the radius vector OP to any moving point P.

Let E be the point of contact of P_1P_3 with its envelope.

Through P draw a parallel PE' to P_3P_1, and let the outward normal to the E locus meet PE' at E'. Then $EE'=p$, and the

* See Williamson, *Int. Calc.*, p. 220; Leudesdorf, *Messenger of Mathematics*, 1878.

E' locus is a parallel to the E locus, the area between them being in the case of n complete revolutions $n\pi p^2+pS$, where S is the perimeter of the envelope of the line P_1P_3 (Art. 435), *i.e.* $[E']-[E]=n\pi p^2+pS$ or πp^2+pS if there be but one revolution of the lamina.

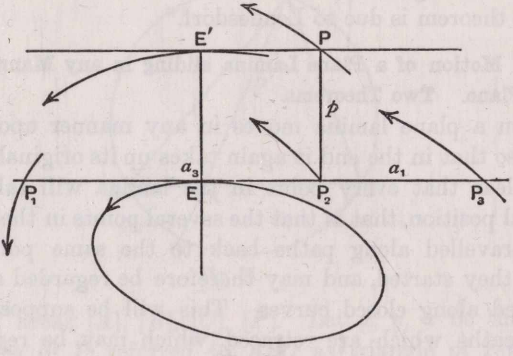

Fig. 92.

Let $E'P=EP_2=r$. Then $P_1E=a_3-r$, $EP_3=a_1+r$, and let P_1P_3 make an angle ψ with any fixed line.

Now $$[P_1]-[E]=\frac{1}{2}\int(a_3-r)^2d\psi,$$

$$[P_2]-[E]=\frac{1}{2}\int r^2d\psi=[P]-[E'],$$

$$[P_3]-[E]=\frac{1}{2}\int(a_1+r)^2d\psi.$$

$\therefore$ multiplying by a_1, a_2, a_3 and adding,

$$a_1[P_1]+a_2[P_2]+a_3[P_3]=-\frac{1}{2}a_1a_2a_3\int d\psi \quad \text{(cf. Art. 473)};$$

and if the lamina reoccupies its original position after n positive revolutions, or if n be the excess of the number of positive revolutions over the number of negative ones, the right-hand side is

$$-\frac{1}{2}a_1a_2a_3\,2n\pi;$$

$$\therefore\ a_1[P_1]+a_2[P_2]+a_3[P_3]+n\pi a_1a_2a_3=0. \quad \text{.........(A)}$$

Also it has been shown that

$$[P]=[P_2]+[E']-[E]=[P_2]+n\pi p^2+pS;$$

$\therefore$ eliminating $[P_2]$,

$$[P]=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}-n\pi a_1a_3+n\pi p^2+pS, \quad \ldots\ldots\text{(B)}$$

which may be written as

$$a_1[P_1]+a_2[P]+a_3[P_3]+n\pi a_1a_2a_3=a_2p(n\pi p+S).$$

482. **Remarks.**

It is assumed that all the areas are described *in the same "sense."* If in any case one of them be described by its tracing point in the clockwise direction, then in this equation the corresponding quantity [ ] is *to be interpreted as the area counted negatively*; and if one of the paths cuts itself so as to form several loops, the interpretation of [ ] is the same as that in Art. 399, viz. *the difference of the odd and even portions.*

The sign of p is positive when in the same sense measured from P_2 as the outward drawn normal of the envelope of P_1P_3.

483. **Deductions.**

Corollary I. When $p=0$ the tracing point P is at P_2, and supposing there to be *one complete revolution* of the lamina we get the case already considered in Art. 477, viz.

$$[P_2]=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}-\pi a_1a_3$$

which is Woolhouse's Extension of Holditch's Theorem.*

484. **Cor. II.** If in addition P_1 and P_3 are *tracing the same curve,* then $[P_1]=[P_3]$ and $[P_2]=[P_1]-\pi a_1a_3$ (Art 478),

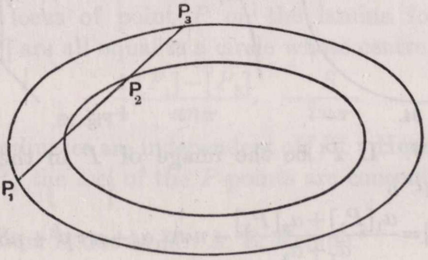

Fig. 93.

and therefore a point upon any chord of constant length inscribed in an oval curve, and which divides the chord into two portions a_1, a_3, traces out another curve whose area is less

* See Williamson's *Integral Calculus*, p. 206.

than that of the original oval by the area of an ellipse whose semiaxes are a_1, a_3. This is **Holditch's** original theorem.*

If a_1, a_3 were interchanged the result would not be affected in this case. If the tracing point be on the chord produced, one of the letters a_1, a_3 is negative and the traced oval is greater than the original oval by the same amount.

485. **Cor. III**. If the line P_1P_3 oscillates back to its original position *without performing* a *complete revolution*, or if *the number of forward revolutions is equal to the number of backward revolutions*, $n=0$, and

$$[P]=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}+pS.$$

This is the case when the contours are two ovals each lying entirely outside the other and the line P_1P_3 cannot revolve completely, but oscillates. It is moreover assumed that the line a_1+a_3 is sufficiently long to allow of the full description of both ovals. If not, the particular oval which is not fully described contributes nothing.

For instance, if P_3 travel along an arc of a circle ACB from A to B via C and back along the same arc, it has described what we may regard as a contour of zero area.

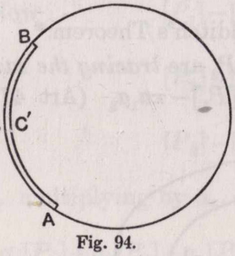

Fig. 94.

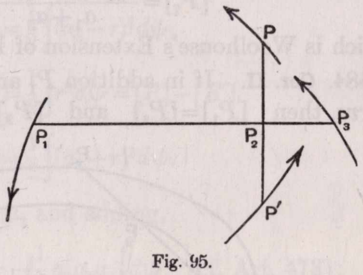

Fig. 95.

486. **Cor. IV**. If P' be the image of P in the line P_1P_3 (*i.e.* $PP_2=P_2P'$),

$$[P]=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}-n\pi a_1a_3+n\pi p^2+pS,$$

$$[P']=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}-n\pi a_1a_3+n\pi p^2-pS,$$

and $[P]-[P']=2pS$, which is independent of the position of P_2.

* See Bertrand, *Calc. Intég.*, p. 365; Williamson, *Integ. Calc.*, p. 206; *Lady's and Gentleman's Diary*, 1858.

487. **Cor. V.** If P_1 and P_3 lie upon the same curve,

$$[P]=[P_1]-n\pi a_1 a_3+n\pi p^2+pS.$$

In case $a_1=0$, we have

$$[P_2]=[P_3] \quad \text{and} \quad [P]=[P_1]+n\pi p^2+pS.$$

488. **Cor. VI.** Let O, the mid-point of P_1P_3, be taken as origin, OP_3 as x-axis, and let $OP_2=x$, $P_2P\equiv p=y$. Let the length of the rod be $2a$.

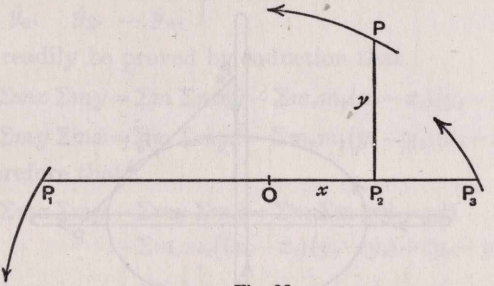

Fig. 96.

Then $$a_1=a-x, \quad a_3=a+x,$$

and $$[P]=\frac{(a-x)[P_1]+(a+x)[P_3]}{2a}-n\pi(a^2-x^2)+n\pi y^2+Sy,$$

i.e. $$x^2+y^2-\frac{[P_1]-[P_3]}{2an\pi}x+\frac{S}{n\pi}y$$

$$+\frac{1}{2n\pi}\{[P_1]+[P_3]-2[P]\}-a^2=0.$$

Hence the locus of point P on the lamina for which the contours $[P]$ are all equal is a circle whose centre is at

$$\frac{1}{4}\,\frac{[P_1]-[P_3]}{an\pi}, \quad -\frac{S}{2n\pi}.$$

These coordinates are independent of $[P]$. Hence, for specific values of $[P]$, the loci of the P-points are concentric circles on the lamina.

This theorem is due to Mr. A. B. Kempe.*

489. We note that if $[P_1]$ and $[P_3]$ be the same contour, the centre of this circle lies on the perpendicular bisector of the line P_1P_3.

* *Messenger of Mathematics*, 1878, cited by Williamson, *Integ. Calc.*, p. 210, where it is deduced from Holditch's form of the theorem geometrically.

490. If the closed "contours" are merely portions of two straight lines $[P_1]=[P_3]=0$, and taking $n=1$,

$$[P]=-\pi a_1 a_3+\pi p^2+pS,$$

or when $p=0$ also, $\quad [P]=-\pi a_1 a_3,$

which is the case of a rod of given length sliding with its ends on the coordinate axes, which are drawn in Fig. 97 as long closed ovals to indicate the direction of rotation.

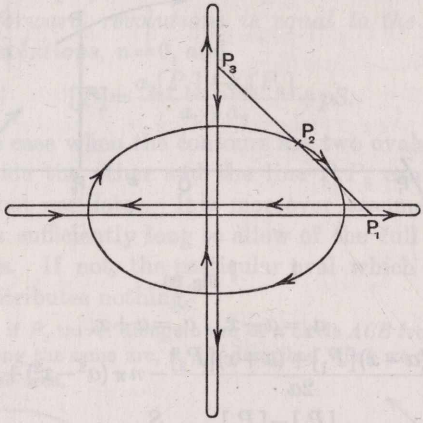

Fig. 97.

Note that in the case shown in Fig. 97 the elliptic area is traced clockwise, the ovals, which are in the limit the axes, are traced one counter-clockwise, one clockwise, and that the areas of the two ovals traced by P_1 and P_3 are both ultimately zero.

It is a well-known theorem that in this case the locus of P_2 is an ellipse of which the product of the semiaxes is the product of the segments of the moving line, whether the axes be rectangular or oblique.

491. **Cor. VII.** If P lie anywhere on the circle on P_1P_3 as diameter, we have $p^2=a_1a_3$, and the theorem reduces to

$$[P]=\frac{a_1[P_1]+a_3[P_3]}{a_1+a_3}+pS,$$

or if $[P_1]$ and $[P_3]$ be the same contour,

$$[P]=[P_1]+pS.$$

492. A General Theorem on the Motion of the Centroid of a System of Moving Particles, connected or otherwise.

If

$$\left.\begin{array}{lllll} m_1, & m_2, & m_3, & \ldots & m_n, \\ x_1, & x_2, & x_3, & \ldots & x_n, \\ y_1, & y_2, & y_3, & \ldots & y_n, \\ \dot{x}_1, & \dot{x}_2, & \dot{x}_3, & \ldots & \dot{x}_n, \\ \dot{y}_1, & \dot{y}_2, & \dot{y}_3, & \ldots & \dot{y}_n, \end{array}\right\} \text{ be five groups of } n \text{ quantities each}$$

it may readily be proved by induction that

$$\Sigma mx\,\Sigma m\dot{y} = \Sigma m\,\Sigma mx\dot{y} - \Sigma m_r m_s(x_r - x_s)(\dot{y}_r - \dot{y}_s)$$

and $$\Sigma my\,\Sigma m\dot{x} = \Sigma m\,\Sigma my\dot{x} - \Sigma m_r m_s(y_r - y_s)(\dot{x}_r - \dot{x}_s),$$

and therefore that

$$\Sigma mx\,\Sigma m\dot{y} - \Sigma my\,\Sigma m\dot{x} = \Sigma m\,\Sigma m(x\dot{y} - y\dot{x})$$
$$-\Sigma m_r m_s[(x_r - x_s)(\dot{y}_r - \dot{y}_s) - (y_r - y_s)(\dot{x}_r - \dot{x}_s)].$$

Let there be n particles of masses in the ratios

$$m_1 : m_2 : m_3 : \ldots : m_n$$

and (x_1, y_1), (x_2, y_2), etc., their coordinates; and let $\dot{x}$, $\dot{y}$ be the differentials of x and y, viz. dx, dy.

The centroid of the system is given by

$$\Sigma m\,.\,\bar{x} = \Sigma mx, \quad \Sigma m\,.\,\bar{y} = \Sigma my;$$

whence $$\Sigma m\,.\,d\bar{x} = \Sigma m\,dx,$$
$$\Sigma m\,.\,d\bar{y} = \Sigma m\,dy.$$

Let each particle describe continuously a closed contour in the plane, m_1 describing a contour of area A_1, m_2 describing a contour of area A_2, and so on, and let $\bar{x}$, $\bar{y}$ in consequence describe a closed contour of area $\bar{A}$. Also let the area of the contour which m_2 describes relatively to m_1 be called S_{12}, and so on for other pairs. Then the above equation may be written

$$[\Sigma m]^2[\bar{x}\,d\bar{y} - \bar{y}\,d\bar{x}] = \Sigma m\,\Sigma m(x\,dy - y\,dx)$$
$$-\Sigma m_r m_s[(x_r - x_s)(dy_r - dy_s) - (y_r - y_s)(dx_r - dx_s)],$$

and therefore integrating round the contours

$$[\Sigma m]^2\bar{A} = \Sigma m\,\Sigma mA - \Sigma m_r m_s S_{rs},$$

an equation which expresses the area of the contour described by the centroid of the system in terms of the areas of the n

contours described by the several particles and of the $\frac{n(n-1)}{2}$ relative contours.

It will be noticed that the particles are in no wise rigidly connected, but are capable of independent motion; also that the result obtained is necessarily homogeneous as regards the masses.

493. If the revolutions of any particles of the system be *not complete*, the various integrals

$$\frac{1}{2}\int(\bar{x}\,d\bar{y}-\bar{y}\,d\bar{x}),\quad \frac{1}{2}\int(x\,dy-y\,dx),$$

$$\frac{1}{2}\int[(x_r-x_s)(dy_r-dy_s)-(y_r-y_s)(dx_r-dx_s)],$$

refer to the sectorial portions of the several contours which have been actually described during the several displacements of the particles, and represent *sectorial areas swept out by the several radii vectores from the origin to the centroid*, or from the origin to x, y in the first two cases, or the relative area by a radius vector from x_r, y_r to x_s, y_s in the third class of integral.

494. *When the several particles are rigidly connected,* the several *relative contours are circles,* with radii the distances between the several pairs, and traced as many times over as the whole system revolves before re-attaining its original position; and in case of no rigid connection, if one or more of the mutual distances returns to its original position without making a complete relative revolution, in such case the corresponding relative area S vanishes.

495. In the case where there are *two particles only*, we have

$$\bar{A}=\frac{m_1A_1+m_2A_2}{m_1+m_2}-\frac{m_1m_2}{(m_1+m_2)^2}S_{12},$$

a result established by Mr. Elliott, and reproduced in Dr. Williamson's *Integral Calculus*, p. 209, with Mr. Elliott's Enunciation of this Theorem.

496. If in this case there be a *rigid connection between the points* A_1 and A_2, say a connecting rod, we may take a_1, a_2 as the distances of A_2, A_1 from the centroid, and $\frac{a_1}{m_1}=\frac{a_2}{m_2}$.

Also the relative contour has area $\pi(a_1+a_2)^2$.

Hence $$\bar{A}=\frac{m_1A_1+m_2A_2}{m_1+m_2}-\frac{m_1m_2}{(m_1+m_2)^2}S_{12}$$

becomes $$\bar{A}=\frac{a_1A_1+a_2A_2}{a_1+a_2}-\frac{a_1a_2}{(a_1+a_2)^2}\pi(a_1+a_2)^2$$
$$=\frac{a_1A_1+a_2A_2}{a_1+a_2}-\pi a_1a_2.$$

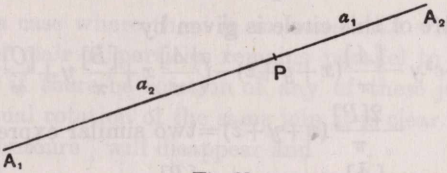

Fig. 98.

Holditch's theorem is therefore deduced as a particular case of the two particle motion, there being a rigid connection.

497. If there be *three particles* the theorem takes the form
$$\bar{A}=\frac{m_1A_1+m_2A_2+m_3A_3}{m_1+m_2+m_3}-\frac{m_2m_3S_{23}+m_3m_1S_{31}+m_1m_2S_{12}}{(m_1+m_2+m_3)^2}.$$

498. Let us apply this result to find the area described by *any* point P *attached to a triangle* ABC which moves in its own plane and after one revolution re-occupies its original position. If x, y, z, be the areal coordinates of P with reference to the triangle ABC, P is the centroid of masses proportional to m_1, m_2, m_3, at A, B, C respectively, where $\frac{x}{m_1}=\frac{y}{m_2}=\frac{z}{m_3}$, and the several "relative areas" are πa^2, πb^2, πc^2;

$$\therefore\ [P]=\frac{m_1[A]+m_2[B]+m_3[C]}{m_1+m_2+m_3}-\frac{m_2m_3\pi a^2+m_3m_1\pi b^2+m_1m_2\pi c^2}{(m_1+m_2+m_3)^2};$$

whence $$[P]=x[A]+y[B]+z[C]-\pi(a^2yz+b^2zx+c^2xy)$$
$$=x[A]+y[B]+z[C]+\pi t^2,$$

where t^2 is the square of the tangent from x, y, z to the circumcircle if the point be without, zero if upon, or $-$the rectangle of the segments of a chord through x, y, z if the point be within the circumcircle; which gives Mr. Leudesdorf's result of Art. 480 already established in a different manner.

499. It is worth observing that the locus of points P which give equal areas $[P]$

is $a^2yz+b^2zx+c^2xy+$linear terms $=0$, *i.e.* a circle,

or making it homogeneous,

$$a^2yz+b^2zx+c^2xy-\left(\frac{[A]}{\pi}x+\frac{[B]}{\pi}y+\frac{[C]}{\pi}z\right)(x+y+z)$$
$$-\frac{[P]}{\pi}(x+y+z)^2=0,$$

and the centre of this circle is given by

$$b^2z+c^2y-\frac{[A]}{\pi}(x+y+z)-\left(\frac{[A]}{\pi}x+\frac{[B]}{\pi}y+\frac{[C]}{\pi}z\right)$$
$$-\frac{2[P]}{\pi}(x+y+z)=\text{two similar expressions},$$

i.e. $$b^2z+c^2y-\frac{[A]}{\pi}=c^2x+a^2z-\frac{[B]}{\pi}=a^2y+b^2x-\frac{[C]}{\pi},$$

which is independent of $[P]$, and therefore indicates that such loci for different values of $[P]$ form a set of concentric circles, which is Mr. Kempe's Theorem of Art. 488 (Cor. VI).

500. It is also worth notice that the area described by *the centroid of the triangle* is given for the case of one complete revolution by

$$[G]=\frac{[A]+[B]+[C]}{3}-\frac{1}{9}\pi(a^2+b^2+c^2);$$

and for *the orthocentre O*,

$$[O]=\frac{[A]\tan A+[B]\tan B+[C]\tan C}{\tan A\tan B\tan C}-8\pi R^2\cos A\cos B\cos C,$$

where R is the radius of the circumcircle.

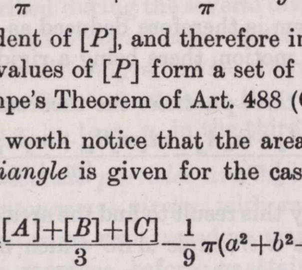

Fig. 99.

501. In the case of *four particles in rigid connection* if a, b, c, d be the sides and e, f the internal diagonals of

the quadrilateral formed, we have, in the one-revolution case,

$$[P]=\frac{m_1[A]+m_2[B]+m_3[C]+m_4[D]}{m_1+m_2+m_3+m_4}$$
$$-\pi\frac{m_1m_2a^2+m_2m_3b^2+m_3m_4c^2+m_4m_1d^2+m_1m_3e^2+m_2m_4f^2}{(m_1+m_2+m_3+m_4)^2},$$

and similarly if there be a greater number of points.

502. In a case where there is *no rotation*, *i.e.* where the line joining each pair of particles remains parallel to its original position, or if there be rotation of any of these joins and an opposite equal rotation of the same join, it is clear that all the "relative contours" will disappear and

$$[P]=\frac{\Sigma m[A]}{\Sigma m}.$$

503. The same result will also hold in the case when the "relative contours," though not individually vanishing, are such as in *the aggregate to destroy each other*, some being positive and others negative, for in such case $\Sigma m_r m_s S_{rs}=0$.

504. If the several particles be *in rigid connection* and the figure describe n revolutions before re-occupying its original position,

$$\Sigma m_r m_s S_{rs}=n\pi\Sigma m_r m_s A_r A_s{}^2=n\pi M\Sigma mGA^2,$$

by Lagrange's "Second Theorem." (Routh, *Anal. Statics*, vol. i., Art. 437); and in that case

$$[G]=\frac{\Sigma m[A]}{M}-n\pi\frac{\Sigma mGA^2}{M}=\frac{\Sigma m[A]}{M}-n\pi\kappa^2,$$

where $M=\Sigma m$ and κ the radius of gyration about the centroid G.

505. Mechanical Integrators or Planimeters.

Consider the case of two rods OP, PQ of lengths a_1 and a_2, freely hinged together at P and the first one OP hinged to a fixed point O in a plane in which both rods can otherwise move freely.

Let x, y be the coordinates of Q relative to a pair of rectangular axes through O, let the rods make angles θ_1, θ_2 respectively with the x-axis, and let $\theta_2-\theta_1=\psi$.

Then

$$x = a_1 \cos\theta_1 + a_2 \cos\theta_2, \qquad y = a_1 \sin\theta_1 + a_2 \sin\theta_2,$$

$$dx = -a_1 \sin\theta_1 d\theta_1 - a_2 \sin\theta_2 d\theta_2, \quad dy = a_1 \cos\theta_1 d\theta_1 + a_2 \cos\theta_2 d\theta_2;$$

$$\therefore\ x\,dy - y\,dx = a_1^2 d\theta_1 + a_2^2 d\theta_2 + a_1 a_2 \cos(\theta_2 - \theta_1)(d\theta_1 + d\theta_2)$$

$$= a_1^2 d\theta_1 + a_2^2 d\theta_2 + a_1 a_2 \cos\psi\, d\psi + 2a_1 a_2 \cos\psi\, d\theta_1.$$

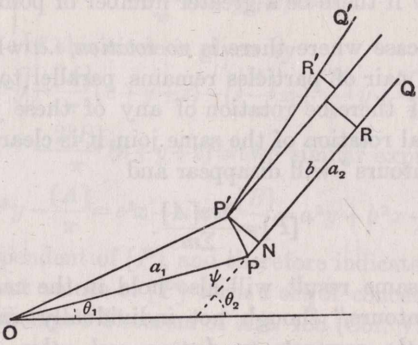

Fig. 100.

Let R be a point on PQ at distance b from P, and let P', Q', R' be the positions taken up by P, Q, R after displacements $d\theta_1$, $d\theta_2$ of the rods.

Then R has advanced perpendicularly to PQ a distance

$$a_1 d\theta_1 \cos\psi + b\,d\theta_2 = ds, \text{ say, to the first order.}$$

Then $x\,dy - y\,dx = a_1^2 d\theta_1 + a_2^2 d\theta_2 + a_1 a_2 \cos\psi\, d\psi + 2a_2(ds - b\,d\theta_2)$.

If Q be made to travel round the contour of any closed curve whose area is to be found, in the positive direction, on completion of the circuit, supposing the point O to be outside the contour and OP and OQ to have *oscillated back to their original positions,*

$$\int d\theta_1 = 0, \quad \int d\theta_2 = 0, \quad \int \cos\psi\, d\psi = [\sin\psi] = 0,$$

and we have

$$\text{Area bounded by the contour} = a_2 S,$$

where S is the total distance travelled over by a point R on the rod PQ, in a direction at right angles to the rod. And it is further to be noticed that this result does not depend upon b, the term involving b disappearing upon integration round the contour. Hence the particular position of the attachment of the point R to the rod is immaterial.

506. But if the point O be within the contour considered, and both rods make a complete revolution before regaining their original position,

$$\int d\theta_1 = 2\pi, \quad \int d\theta_2 = 2\pi, \quad \int \cos\psi\, d\psi = [\sin\psi] = 0;$$

and therefore $\qquad A = \pi(a_1^2 + a_2^2 - 2a_2 b) + a_2 S.$

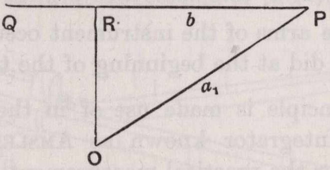

Fig. 101.

Now $a_1^2 + a_2^2 - 2a_2 b$ is the value of OQ^2 when the rods are clamped at the joint P in such a position that OR is perpendicular to PQ. Call this value of OQ^2, r_0^2.

$$\therefore \quad A = \pi r_0^2 + a_2 S.$$

A circle with centre O and radius r_0 is called the zero circle. When the system is clamped in this position the motion of R is at right angles to OR, *i.e.* in the direction of PQ, and R has no motion at all at right angles to the rod PQ on which it lies. Hence when O lies *within* the contour the area of the zero circle, viz. πr_0^2, must be added to $a_2 S$ to give the area of the contour.

Again, if one rod, say OP_1, oscillates back to its original position whilst the other PQ makes a complete turn, then

$$\int d\theta_1 = 0, \quad \int d\theta_2 = 2\pi, \quad \int \cos\psi\, d\psi = 0;$$

and $\qquad A = \pi(a_2^2 - 2a_2 b) + a_2 S.$

Similarly, if PQ oscillates but OP revolves,

$$\int d\theta_1 = 2\pi, \quad \int d\theta_2 = 0, \quad \int \cos\psi\, d\psi = 0,$$

and $\qquad A = \pi a_1^2 + a_2 S.$

507. The general result is therefore that the area traced by the pointer is

$$\begin{array}{ll} & (1)\ a_2S \\ \text{or} & (2)\ \pi(a_1^2+a_2^2-2a_2b)+a_2S \\ \text{or} & (3)\ \pi(a_2^2-2a_2b)+a_2S \\ \text{or} & (4)\ \pi a_1^2+a_2S, \end{array}$$

according as (1) neither a_1 nor a_2 complete a revolution,
(2) both complete a revolution,
(3) a_2 completes a revolution but a_1 does not,
(4) a_1 completes a revolution but a_2 does not,

in each case the arms of the instrument occupying the same position as they did at the beginning of the tracing.

508. This principle is made use of in the construction of a Mechanical Integrator known as AMSLER'S PLANIMETER, which is used for the practical measurement of an area. The rod PQ is provided at R with a small graduated wheel with axis parallel to the rod, which is allowed to rest on the paper and to turn by friction with the paper. It can then only register the amount of travel of R at right angles to the rod, the amount of travel in the direction of the rod being necessarily unregistered as it is due to slide along the surface of the paper and not to the rolling of the wheel. A reading of the wheel gives the value of S. Then

area of contour $=a_2S$ or $a_2S+\pi r_0^2$,

according as the point O is outside or within the contour.

509. Several forms of Mechanical Integrators are in use, but for the most part they are modifications of Professor Amsler's form and based upon the general principle described above.

Description of the Instrument.

The figure shown (Fig. 102) is an illustration of a form of the instrument made by Messrs. John J. Griffin & Sons, Scientific Instrument Makers, Kingsway, London. The lettering corresponds to the preceding general explanation of the principle. O is the fixed point, ABC the contour of the area required, Q the tracing point which is being made to traverse the contour, P is the joint connecting the two beams of the instrument, R the graduated wheel or roller whose axis is parallel to PQ and which rolls upon the paper when there is any motion at right angles to PQ. Its position upon the beam PQ being immaterial, it is placed in this form of the instrument on QP produced. D is a dial whose axis is perpendicular to the axis of the wheel and turned by a worm on the axis of the roller. There is a pointer attached to the beam PQ, serving to mark the amount of rotation of the dial plate. V is a vernier assisting to read small amounts of rotation of the wheel. There is

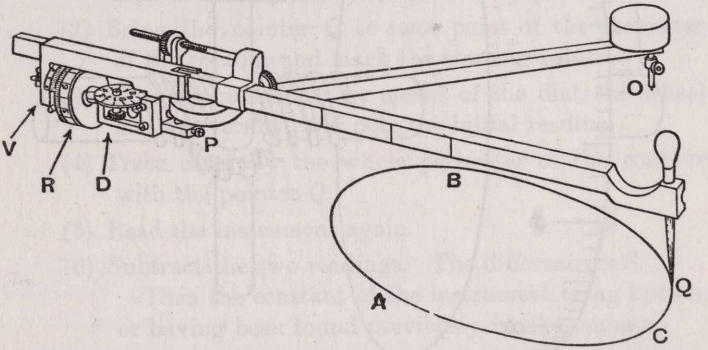

Fig. 102.

a pointer at Q by means of which the contour can be carefully followed.

The graduations on the rim of the wheel are such that the circumference is divided into 10 equal segments indicated by 1, 2, 3, 4, ... 0, and each segment into 10 further subdivisions. The dial D is such as to rotate once for 10 revolutions of the roller, and is itself divided into 10 segments, which are again subdivided, an advance of a segment of the dial indicating one complete revolution of the wheel. The readings of the dial therefore indicate the number of complete revolutions of the wheel. In the vernier a length equal to 9 subdivisions of the wheel is divided into 10 equal portions on the vernier.

If the figures on the dial be taken as units, the figured graduations on the wheel will represent 10ths and the subdivisions

100ths, the difference between the distance of two consecutive divisions of the vernier and two consecutive subdivisions of the wheel, being $(\frac{1}{100}-\frac{1}{10}\times\frac{9}{100})$ of the circumference of the wheel, is $\frac{1}{1000}$ of the circumference of the wheel. Hence, by means of the vernier, readings may be made to three places of decimals. The area to be found has been shown to be

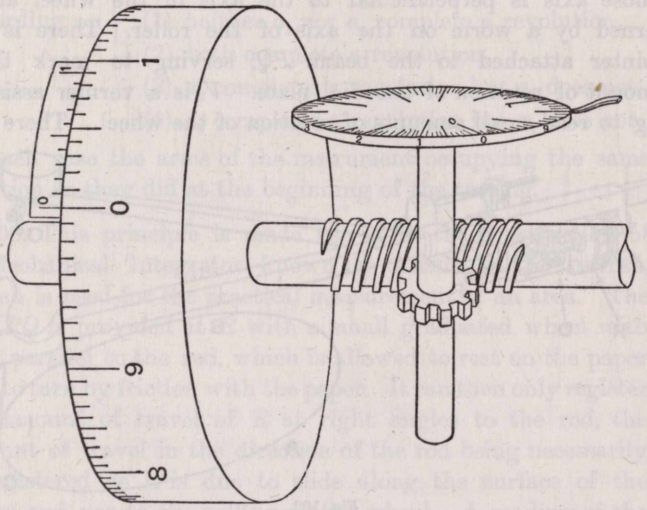

Fig. 103.

proportional to the number registered by the roll of the wheel, the component of motion parallel to the axis, *i.e.* slide, being unregistered. Let S be the number registered by the wheel, then

$$A = CS,$$

where C is some constant called the constant of the instrument. Apply the instrument first to any figure of known area A_0, say a square or a circle, as may be most convenient; let the difference of initial and final readings of the instrument be S_0, then $A_0 = CS_0$, which determines C. If now we apply it to the contour whose quadrature is required and S be the difference of the initial and final readings of the instrument,

$$A = A_0 \frac{S}{S_0}.$$

It has been assumed that the fixed point O has been taken outside the perimeter of the contour. If inside, we have still to add the area of the "zero" circle, and

$$A = \pi r_0^2 + A_0 \frac{S}{S_0}.$$

The area of the zero circle is usually marked on the instrument.

Mode of Procedure.

The procedure is then as follows:

(1) Fix the point O to the drawing board on which the area to be found has been previously pinned.

(2) Bring the pointer Q to some point of the perimeter of the contour and mark the starting point.

(3) Read the instrument by means of the dial, the wheel and the vernier, and note the initial reading.

(4) Trace carefully the whole perimeter of the contour with the pointer Q.

(5) Read the instrument again.

(6) Subtract the two readings. The difference is S.

Then the constant of the instrument being known, or having been found previously in like manner,

$$A = A_0 \frac{S}{S_0} \quad \text{or} \quad A = A_0 \frac{S}{S_0} + \pi r_0^2,$$

according as it has been convenient to take O outside or within the contour.

EXAMPLES.

1. Ox, Oy being perpendicular axes, A, B fixed points on Oy and $AMBA$ any closed region of area S lying in the positive quadrant, show that the integral

$$\int [\{\phi(y)e^x - my\}\,dx + \{\phi'(y)e^x - m\}\,dy],$$

taken round the curve from A to B, is equal to

$$m(S + a - b) + \phi(b) - \phi(a),$$

$\phi(y)$, $\phi'(y)$ being finite and continuous, m a constant and $OA = a$, $OB = b$. [J. Math. Schol. Oxford, 1904.]

2. P_1, P_2 are points on a closed oval of area A, such that P_1, P_2 subtends a right angle at a fixed point O. Show that the area of the curve traced out by the middle point of P_1P_2 is equal to

$$\frac{1}{2}A+\frac{1}{8}\int_0^{2\pi}\left(r_1\frac{dr_2}{d\theta_1}-r_2\frac{dr_1}{d\theta_2}\right)d\theta_1,$$

where $\theta_2=\theta_1+\frac{\pi}{2}$ and $OP_1=r_1$, $OP_2=r_2$.

[COLLEGES β, 1889.]

3. A fixed point O is taken on a central oval which is such that through any point inside it other than the centre one and only one chord can be drawn which is bisected at that point; prove that the locus of the middle point of the chord PQ for a constant sum 2σ of the arcs OP, OQ cuts at right angles the same locus for a constant difference $2\sigma'$ of these arcs; and deduce that the area of the oval is

$$\frac{1}{2}\int_0^l d\sigma\int_0^{\frac{1}{2}l} d\sigma' \sin\theta,$$

where l is the length of the oval, and θ is the angle between the tangents at P and Q. [MATH. TRIPOS, 1889.]

4. A bar AB carries at a point of its length a small wheel having AB for axis and which turns about AB: the end A is constrained to move in a given straight line; show that if the end B is carried round any closed curve without singular points and which does not cut the straight line on which A moves, the area of the curve is measured by the product of AB into the whole length registered by the revolving wheel. [COLLEGES, 1892.]

[This is the principle of construction of Coffin's Planimeter. A full description will be found on p. 159, *Practical Electrical Engineering*, by Briggs and others. It is the case when the rod OP of Fig. 102 is of infinite length, so that P describes a straight line instead of a circle.]

5. A straight line of given length moves with its extremities on the arcs of two closed curves of given areas, and a point is attached to the moving line.

Prove that when the area traced by this attached point has a minimum value for different positions of the point on the line, the difference of the areas of the circles whose radii are the segments into which the point divides the line is equal to the difference of the areas of the given curves. [ST. JOHN'S, 1882.]

6. Show that the path of the mid-point of a rod of constant length $2c$, whose ends lie upon an ellipse, is an oval of area $\pi(ab-c^2)$.

If, instead of both ends being on the ellipse, one end lies on the ellipse and the other on the major axis, or if one end lies on the ellipse and the other on the auxiliary circle, find the areas of the paths described by the centre of the rod in both cases.

7. A rigid cyclic quadrilateral $ABCD$ moves in its plane so as to return to its original position after turning through four right angles. Show that if (A), etc., denote the areas of the curves described by A, etc., and if S_1, S_2, etc., denote the areas of the triangles BCD, CDA, etc., then

$$S_1(A)+S_3(C)=S_2(B)+S_4(D).$$

Find also the equation connecting the areas described by any three vertices with that described by the centre of the circumcircle of the triangle. [I. C. S., 1909.]

8. Two bars OP, RPQ, of lengths $OP=c$, $RPQ=b+a$, respectively turn round a fixed pin at O and a joint at P. dS_1, dS_2 denote the polar elements of area about O of the curves traced by P and Q respectively; prove that

$$dS_2-dS_1=a\,d\zeta+a(\tfrac{1}{2}a+b)\,d\theta-\tfrac{1}{2}a\,dp,$$

where $PQ=a$, $RP=b$, p is the perpendicular from O on RPQ, $d\zeta$ is the displacement of R perpendicular to RPQ and θ is the inclination of RPQ to a fixed line OA. [MATH. TRIP., PT. I., 1914.]