CHAPTER XIII.

QUADRATURE (II).

TANGENTIAL POLARS, PEDAL EQUATIONS AND PEDAL CURVES, INTRINSIC EQUATIONS, ETC.

416. Other Expressions for an Area

Many other expressions may be deduced for the area of a plane curve, or proved independently, specially adapted to the cases when the curve is defined by systems of coordinates other than Cartesians or Polars, or for regions bounded in a particular manner.

To avoid continual redefinition of the symbols used we may state that in the subsequent work the letters

$$x,\ y,\ r,\ \theta,\ s,\ p,\ \psi,\ \phi,\ \rho$$

have the meanings assigned to them throughout the treatment of Curvature in the author's *Differential Calculus.*

417. The (p, s) formula.

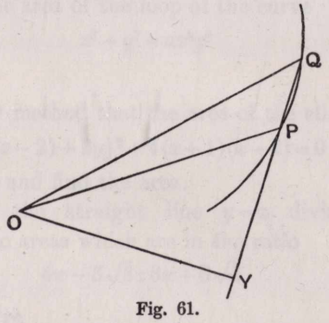

Fig. 61.

Let PQ be an element δs of a plane curve and OY the perpendicular from the pole upon the chord PQ. Then

$$\triangle OPQ = \tfrac{1}{2} OY \,.\, PQ,$$

and any sectorial area

$$=Lt\,\Sigma\triangle OPQ=\tfrac{1}{2}Lt\,\Sigma OY.PQ,$$

the summation being conducted along the whole bounding arc. In the notation of the Integral Calculus this is $\dfrac{1}{2}\displaystyle\int p\,ds.$

This might be deduced from the polar formula at once.

For $A=\dfrac{1}{2}\displaystyle\int r^2 d\theta=\dfrac{1}{2}\int r^2\dfrac{d\theta}{ds}\,ds=\dfrac{1}{2}\int r\sin\phi\,ds=\dfrac{1}{2}\int p\,ds,$

where ϕ is the angle between the tangent and the radius vector.

418. **Tangential-Polar Form** $(p,\ \psi)$.

Again, since $\rho=\dfrac{ds}{d\psi}=p+\dfrac{d^2p}{d\psi^2},$

we have Area $=\dfrac{1}{2}\displaystyle\int p\,ds=\dfrac{1}{2}\int p\rho\,d\psi=\dfrac{1}{2}\int p\left(p+\dfrac{d^2p}{d\psi^2}\right)d\psi,$

a form suitable for use when the Tangential-Polar (*i.e.* p,ψ) form of the equation to the curve is given.

This gives the sectorial area bounded by the curve and the initial and final radii vectores.

419. **Caution.**

In using the formula

$$A=\frac{1}{2}\int p\left(p+\frac{d^2p}{d\psi^2}\right)d\psi,$$

care should be taken not to integrate over a point, between the proposed limits, at which the integrand changes sign. If such points exist the whole integration is to be conducted in sections along each of which the sign of the integrand is permanent. The results for the several sections are then to be taken positively and added together. When a point of inflexion is passed $p+\dfrac{d^2p}{d\psi^2}$ passes through an infinite value and changes sign.

420. **The Case of a Closed Curve.**

When the curve is closed the formula admits of some simplification.

For integrating by parts

$$\int p\frac{d^2p}{d\psi^2}d\psi=\left[p\frac{dp}{d\psi}\right]-\int\left(\frac{dp}{d\psi}\right)^2d\psi.$$

Hence $$\text{Area}=\frac{1}{2}\left[p\frac{dp}{d\psi}\right]+\frac{1}{2}\int\left\{p^2-\left(\frac{dp}{d\psi}\right)^2\right\}d\psi.$$

In integrating round the whole perimeter the term between square brackets, viz. $\frac{1}{2}\left[p\frac{dp}{d\psi}\right]$ disappears, for it resumes the same value as it originally had when we return to the starting-point after integrating round the contour of the curve. Hence, for a closed curve,

$$\text{Area}=\frac{1}{2}\int\left\{p^2-\left(\frac{dp}{d\psi}\right)^2\right\}d\psi.$$

421. Ex. 1. Let A_1CA_2 be one foil of the epicycloid $p=A\sin B\psi$ and OA_1 the initial line. Then p vanishes if $B\psi=0, \pi, 2\pi, \ldots$.

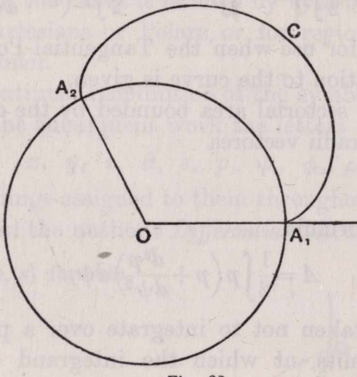

Fig. 62.

Therefore, for the area bounded by OA_1, OA_2 and a foil of the epicycloid, viz. the kite-shaped figure OA_1CA_2O in Fig. 62,

$$\text{Area}=\frac{1}{2}\int_0^{\frac{\pi}{B}}p\left(p+\frac{d^2p}{d\psi^2}\right)d\psi=\frac{1}{2}\int_0^{\frac{\pi}{B}}A\sin B\psi\{A\sin B\psi-AB^2\sin B\psi\}d\psi$$

$$=\frac{A^2(1-B^2)}{2}\int_0^{\frac{\pi}{B}}\sin^2 B\psi\,d\psi$$

$$=\frac{A^2(1-B^2)}{2}\cdot\frac{1}{B}\int_0^{\pi}\sin^2\phi\,d\phi,\quad\text{if }\phi=B\psi,$$

$$=\frac{\pi}{4}\frac{A^2}{B}(1-B^2).$$

Thus, for the whole cardioide, which is a one-cusped epicycloid formed as the path of a point attached to the circumference of a circle of radius a rolling upon an equal circle whose centre is at the origin O,

$$p=3a\sin\frac{\psi}{3}. \qquad \text{(See } \textit{Diff. Calc.}\text{, p. 345.)}$$

And the area is

$$\frac{\pi}{4}(3a)^2\times 3\left(1-\frac{1}{9}\right)=6\pi a^2.$$

Ex. 2. Otherwise, the cardioide $p=3a\sin\dfrac{\psi}{3}$ is a "closed" curve.

Let us apply the second formula

$$\frac{1}{2}\int\left(p^2-\overline{\frac{dp}{d\psi}}\Big|^2\right)d\psi \text{ in this case.}$$

The whole area $=\dfrac{1}{2}\displaystyle\int\left(9a^2\sin^2\frac{\psi}{3}-a^2\cos^2\frac{\psi}{3}\right)d\psi$ taken between limits

$$\psi=0 \quad\text{and}\quad \psi=3\pi.$$

Putting $\psi=3\theta$, this becomes

$$\frac{3a^2}{2}\int_0^{\pi}(9\sin^2\theta-\cos^2\theta)\,d\theta$$

$$=3a^2\left(9\,\frac{1}{2}\,\frac{\pi}{2}-\frac{1}{2}\,\frac{\pi}{2}\right)=6\pi a^2, \text{ as before.}$$

422. Pedal Curves.

If $p=f(\psi)$ be the tangential-polar equation of a given curve, $\delta\psi$ is the angle between the perpendiculars from the pole

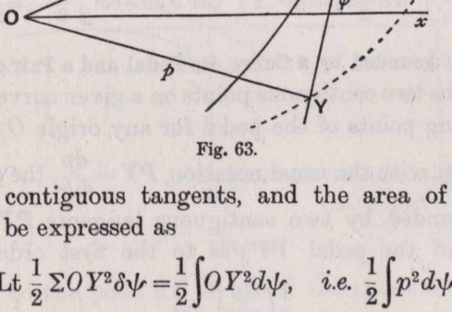

Fig. 63.

upon two contiguous tangents, and the area of the pedal curve may be expressed as

$$\text{Lt}\,\frac{1}{2}\Sigma OY^2\delta\psi=\frac{1}{2}\int OY^2 d\psi, \quad \textit{i.e.}\ \frac{1}{2}\int p^2 d\psi,$$

p, ψ being the polar coordinates of Y.

423. Ex. Find the area of the pedal of a circle with regard to a point on the circumference (*i.e.* the cardioide).

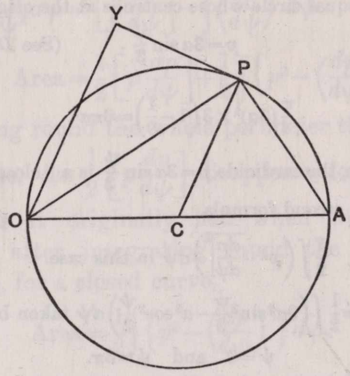

Fig. 64.

Here, if OY is the perpendicular on the tangent at P, and OA the diameter $=2c$, it is geometrically obvious that OP bisects the angle AOY. Hence calling $A\hat{O}Y$, ψ, we have for the tangential polar equation of the circle

$$p=OY=OP\cos\frac{\psi}{2}=OA\cos^2\frac{\psi}{2},$$

i.e.

$$p=2c\cos^2\frac{\psi}{2}.$$

Hence Area$=\frac{1}{2}\int 4c^2\cos^4\frac{\psi}{2}\,d\psi$, where the limits are to be taken as 0 and π, and the result is to be doubled so as to include the lower portion of the pedal.

Then

$$A=4c^2\int_0^{\pi}\cos^4\frac{\psi}{2}\,d\psi=4c^2\,.\,2\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta=8c^2\,\frac{3}{4}\,\frac{1}{2}\,.\,\frac{\pi}{2}=\frac{3}{2}\pi c^2.$$

424. **Area bounded by a Curve, its Pedal and a Pair of Tangents.** Let P, Q be two contiguous points on a given curve; Y, Y' the corresponding points of the pedal for any origin O (Fig. 65).

Then since, with the usual notation, $PY=\frac{dp}{d\psi}$, the elementary triangle bounded by two contiguous tangents PY, QY', and the chord of the pedal YY', is to the first order of small quantities

$$\frac{1}{2}\left(\frac{dp}{d\psi}\right)^2\delta\psi.$$

Hence the area of any portion bounded by the two curves and a pair of tangents to the original curve may be expressed as

$$\frac{1}{2}\int\left(\frac{dp}{d\psi}\right)^2 d\psi,$$

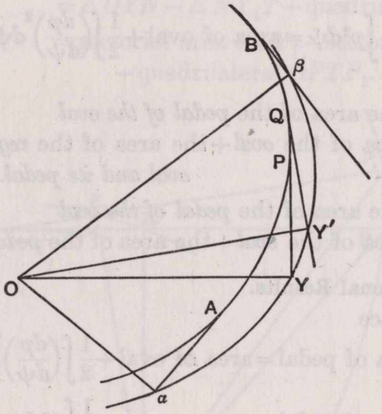

Fig. 65.

and is the same as the *corresponding portion of the pedal of the evolute*, for $PY=$ the perpendicular from O upon the normal at P (Fig. 66).

425. Pedal of Evolute of a Closed Curve.

In the case of a closed curve, then, the equation

$$\text{Area}=\frac{1}{2}\int\left\{p^2-\left(\frac{dp}{d\psi}\right)^2\right\}d\psi$$

admits of two interpretations.

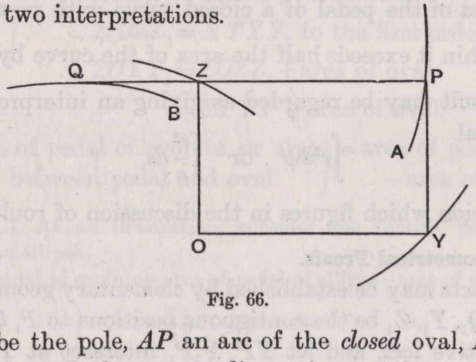

Fig. 66.

Let O be the pole, AP an arc of the *closed* oval, BQ an arc of the evolute, P, Q corresponding points on the curve and the evolute, OY, OZ, perpendiculars from O on the tangent and normal at P.

Then the Y locus is the pedal to the curve, the Z locus is the pedal to the evolute. Hence the equation

$$\frac{1}{2}\int p^2 d\psi = \text{area of oval} + \frac{1}{2}\int\left(\frac{dp}{d\psi}\right)^2 d\psi$$

expresses

(A) That the area of the *pedal of the oval*
= area of the *oval* + the area of the *region between the oval and its pedal.*

(B) That the area of the *pedal of the oval*
= area of the *oval* + the area of the *pedal of the evolute.*

426. Additional Results.

Further, since

$$\text{area of pedal} = \text{area of oval} + \frac{1}{2}\int\left(\frac{dp}{d\psi}\right)^2 d\psi$$

and

$$\text{area of pedal} = \frac{1}{2}\int p^2 d\psi,$$

we have upon addition

$$2 \times \text{area of pedal} = \text{area of oval} + \frac{1}{2}\int\left\{p^2 + \left(\frac{dp}{d\psi}\right)^2\right\} d\psi$$

$$= \text{area of oval} + \frac{1}{2}\int r^2 d\psi$$

or

$$= \text{area of oval} + \frac{1}{2}\int \frac{r^2}{\rho}\, ds,$$

i.e. the area of the pedal of a closed curve with regard to any origin within it exceeds half the area of the curve by $\frac{1}{4}\int \frac{r^2}{\rho}\, ds$.

This result may be regarded as giving an interpretation for the integral

$$\int r^2 d\psi \quad \text{or} \quad \int \frac{r^2}{\rho}\, ds,$$

an expression which figures in the discussion of roulettes.

427. Geometrical Proofs.

These facts may be established by elementary geometry thus.

Let P_1, Q_1, Y_1, Z_1 be the contiguous positions to P, Q, Y, Z on the respective loci, and let YP, Y_1P_1 intersect at T and YP, OY_1 at N.

Then

$$\begin{aligned}\triangle OYP - \triangle OY_1P_1 &= (\triangle OYN + \triangle ONP) \\ &\quad - (\triangle ONP + \triangle NY_1T + \text{quadrilateral } OPTP_1) \\ &= \triangle OYN - \triangle NY_1T - \text{quadrilateral } OPTP_1 \\ &= \text{sectorial area } OYY_1 - \text{sectorial area } TYY_1 \\ &\quad - \text{quadrilateral } OPTP_1.\end{aligned}$$

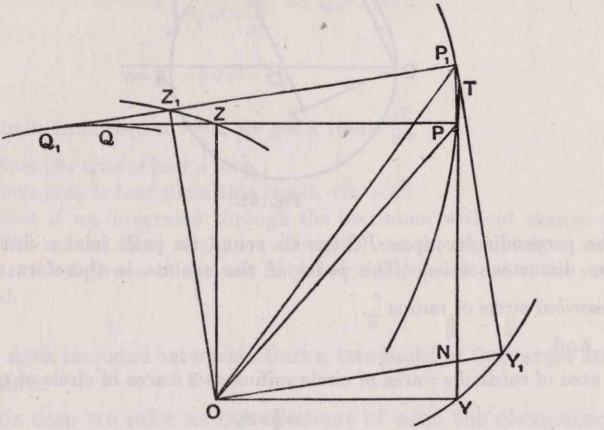

Fig. 67.

And summing for a closed oval,

$$\Sigma(\triangle OYP - \triangle OY_1P_1) = 0;$$

$$\therefore\ \Sigma OYY_1 = \Sigma TYY_1 + \Sigma OPTP_1,$$

and $\qquad \triangle OZZ_1 = \triangle TYY_1$ to the first order;

$$\therefore\ \Sigma OYY_1 = \Sigma OZZ_1 + \text{area of oval}$$

or $\qquad = \Sigma TYY' + \text{area of oval},$

i.e. $\left.\begin{array}{l}\text{area of pedal of evolute, or area}\\ \text{between pedal and oval}\end{array}\right\} = \begin{array}{l}\text{area of pedal of oval}\\ \quad - \text{area of oval.}\end{array}$

428. Ex. 1. As an illustration, consider the central pedal of the evolute of an ellipse.

Area of pedal of evolute = area of pedal of ellipse − area of ellipse

$$= \frac{\pi}{2}(a^2+b^2) - \pi ab$$

$$= \frac{\pi}{2}(a-b)^2.$$

Ex. 2. The pedal of a circle of radius c and centre C with regard to a point O on the circumference is $r=c(1+\cos\theta)$, a cardioide. The evolute of the circle is a point, viz. the centre. As the current point P travels round the circumference of the circle once, the path of Z, the foot of

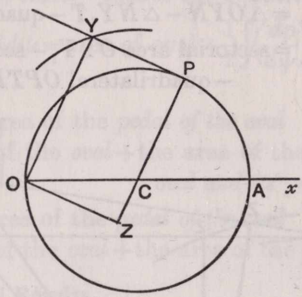

Fig. 68.

the perpendicular upon PC travels round its path (viz. a circle on OC for diameter) *twice*. The pedal of the evolute is therefore the *twice described* circle of radius $\frac{c}{2}$.

And

$$\text{area of cardioide} = \text{area of circle radius } c + 2\times\text{area of circle of radius } \frac{c}{2}$$
$$=\tfrac{3}{2}\pi c^2.$$

429. **Pedal Equation** (p, r).

When the relation between p, r is given, *i.e.* the pedal equation, we have

$$\text{Area} = \frac{1}{2}\int p\,ds = \frac{1}{2}\int p\frac{ds}{dr}\,dr = \frac{1}{2}\int p\sec\phi\,dr$$
$$=\frac{1}{2}\int\frac{rp}{\sqrt{r^2-p^2}}\,dr.$$

This again gives the sectorial area between the curve and a definite pair of radii vectores.

Again care is required in the use of the formula to avoid integration through a value of r for which $\sec\phi$ changes sign, *i.e.* when ϕ changes from acute to obtuse, as it will do at points where r has a maximum or minimum value. If such points occur, the integration must be conducted separately for each of the portions into which these points divide the perimeter and the results taken positively added together.

430. Ex. 1. In the equiangular spiral $p=r\sin\alpha$, and any sectorial area

$$=\frac{1}{2}\int_{r_1}^{r_2}\frac{r^2\sin\alpha}{r\cos\alpha}dr=\frac{1}{4}(r_2^2-r_1^2)\tan\alpha.$$

Ex. 2. Find the area of the lemniscate $p=\frac{r^3}{a^2}$.

Here $$A=\frac{1}{2}\int\frac{r\frac{r^3}{a^2}dr}{\sqrt{r^2-\frac{r^6}{a^4}}}=\frac{1}{2}\int\frac{r^3}{\sqrt{a^4-r^4}}dr$$

$$=\tfrac{1}{2}\left[-\tfrac{1}{2}\sqrt{a^4-r^4}\right].$$

Taking limits from $r=0$ to $r=a$, we get a result $\frac{a^2}{4}$.

This gives the area of half a loop.

The whole area is four times this result, viz. $=a^2$.

Note, that if we integrated through the maximum without change of sign of the radical from $r=0$ to $r=0$ again, we should obtain a zero result—*i.e.* the *difference* of the two halves of the loop instead of the sum as desired.

431. Area included between a Curve, two Radii of Curvature and the Evolute.

In this case we take as our element of area the elementary

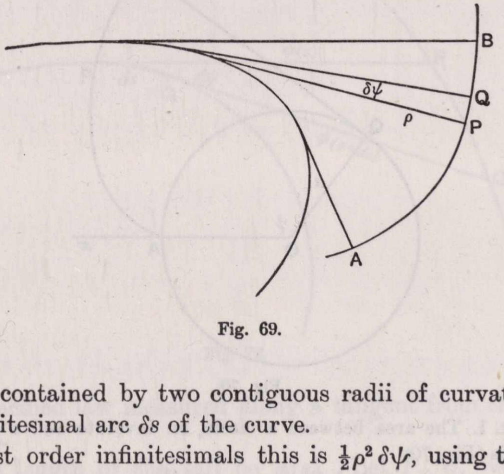

Fig. 69.

triangle contained by two contiguous radii of curvature and the infinitesimal arc δs of the curve.

To first order infinitesimals this is $\frac{1}{2}\rho^2\,\delta\psi$, using the same notation as before.

And area required

$$=Lt\,\Sigma\,\tfrac{1}{2}\rho^2\,\delta\psi,$$

i.e. $$=\frac{1}{2}\int\rho^2\,d\psi, \quad\text{or}\quad =\frac{1}{2}\int\rho\,ds,$$

or $$=\frac{1}{2}\int\left(p+\frac{d^2p}{d\psi^2}\right)ds, \quad\text{or}\quad \frac{1}{2}\int\left(p+\frac{d^2p}{d\psi^2}\right)^2 d\psi,$$

or other forms adapted to the particular species of coordinates in use.

For instance, for Cartesians

$$=\frac{1}{2}\int\frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}\sqrt{1+y_1^2}\,dx$$

$$=\frac{1}{2}\int\frac{(1+y_1^2)^2}{y_2}\,dx, \quad\text{where } y_1=\frac{dy}{dx},\; y_2=\frac{d^2y}{dx^2};$$

or for Polars

$$=\frac{1}{2}\int\frac{(r^2+r_1^2)^{\frac{3}{2}}}{r^2+2r_1^2-rr_2}\sqrt{r^2+r_1^2}\,d\theta$$

$$=\frac{1}{2}\int\frac{(r^2+r_1^2)^2}{r^2+2r_1^2-rr_2}\,d\theta, \quad\text{where } r_1=\frac{dr}{d\theta},\; r_2=\frac{d^2r}{d\theta^2},$$

etc.

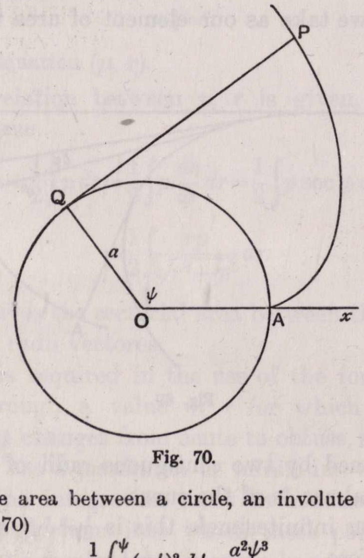

Fig. 70.

432. Ex. 1. The area between a circle, an involute and a tangent to the circle is (Fig. 70)

$$\frac{1}{2}\int_0^{\psi}(a\psi)^2\,d\psi=\frac{a^2\psi^3}{6}.$$

Ex. 2. The area between the tractrix and its asymptote is found in similar manner. The tractrix is described in *Diff. Calc.*, Art. 444. The portion of the tangent between the point of contact and the x-axis is of constant length c.

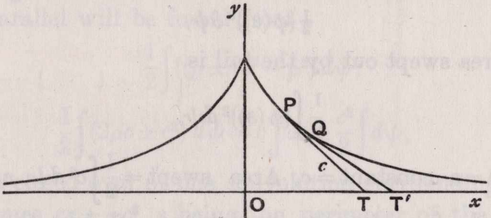

Fig. 71.

Taking two adjacent tangents and the axis of x as forming an elementary triangle (Fig. 71),

$$\text{Area} = 2 \cdot \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} c^2 d\psi$$

$$= \frac{\pi c^2}{2}.$$

433. Area swept by a "Tail."

In exactly the same way as in the last example we may find the area swept out by a "tail" of length varying according

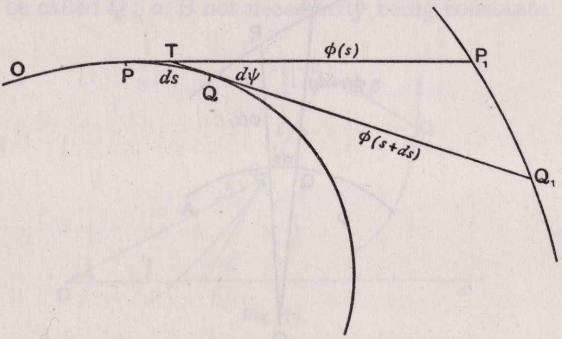

Fig. 72.

to any specified law measured along a tangent from the point of contact.

Let the length of the tail be $\phi(s)$. Let P_1, Q_1 be at the distances $\phi(s)$, $\phi(s+\delta s)$ measured along the tangents at contiguous points P and Q respectively from the points of contact. Then the area of the triangular element bounded by the two contiguous tails and the arc P_1Q_1 is to the first order

$$\tfrac{1}{2}\{\phi(s)\}^2\,\delta\psi,$$

and the area swept out by the tail is

$$\frac{1}{2}\int\{\phi(s)\}^2\,d\psi.$$

If $\phi(s) = $ a constant $= c$, Area swept $= \frac{1}{2}\int c^2\,d\psi$, and for a closed oval of continuous curvature $= \pi c^2$, viz. the area of a circle of radius c.

If the tail be of length equal to the corresponding radius of curvature, the area swept out $= \frac{1}{2}\int\rho^2\,d\psi = \frac{1}{2}\int\rho\,ds.$

434. **If lengths be taken along the normal drawn outwards,** and specified in the same way, viz. $\phi(s)$, the area between the original curve and the locus traced is

$$\frac{1}{2}\int[\{\rho+\phi(s)\}^2-\rho^2]\,d\psi = \int\phi(s)\,ds + \frac{1}{2}\int\{\phi(s)\}^2\,d\psi,$$

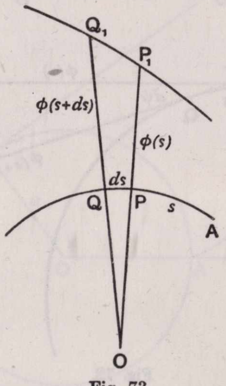

Fig. 73.

or if the distance $\phi(s)$ be on the inward drawn normal

$$\int\phi(s)\,ds - \frac{1}{2}\int\{\phi(s)\}^2\,d\psi.$$

435. **Parallel Curves.**

If, in this case (Art. 434), $\phi(s)$ be constant $=c$, a 'parallel' to the original curve is traced, and the area between a curve and its parallel will be found from

$$\frac{1}{2}\int\{(\rho \pm c)^2 \sim \rho^2\}\, d\psi,$$

i.e.
$$\frac{1}{2}\int(2\rho c \pm c^2)\, d\psi = c\int ds \pm \frac{c^2}{2}\int d\psi,$$

and for a closed oval of one convolution surrounding the pole this becomes $cs \pm \pi c^2$, s being the perimeter of the oval, the positive sign being taken for exterior parallels, the negative sign for interior ones. If the normal makes n revolutions before returning to its original position, the area swept over by PP_1 will be numerically

$$cs \pm n\pi c^2.$$

436. **General Case.**

More generally, let us construct a new curve from a given one by measuring a distance α along the tangent from the point of contact, in the direction of measurement of the arc, and a distance β through the extremity of α, parallel to the outward drawn normal at P, and let the point at which we arrive be called Q; α, β not necessarily being constants.

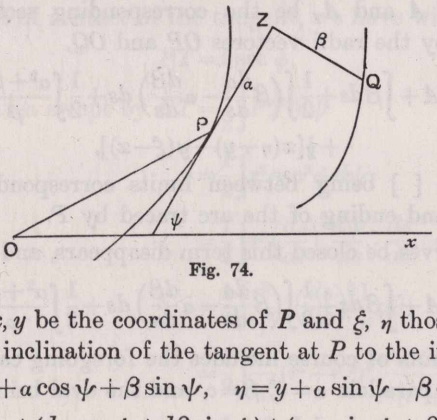

Fig. 74.

Then if x, y be the coordinates of P and ξ, η those of Q, and if ψ be the inclination of the tangent at P to the initial line,

$$\xi = x + \alpha\cos\psi + \beta\sin\psi, \quad \eta = y + \alpha\sin\psi - \beta\cos\psi.$$

Then $d\xi = dx + (d\alpha\cos\psi + d\beta\sin\psi) + (-\alpha\sin\psi + \beta\cos\psi)\,d\psi$,

$$d\eta = dy + (d\alpha\sin\psi - d\beta\cos\psi) + (\alpha\cos\psi + \beta\sin\psi)\,d\psi;$$

$$\begin{aligned}\therefore\quad \xi\,d\eta-\eta\,d\xi &= (x+\alpha\cos\psi+\beta\sin\psi)\{dy+(d\alpha\sin\psi-d\beta\cos\psi)\\ &\qquad +(\alpha\cos\psi+\beta\sin\psi)\,d\psi\}\\ &\quad -(y+\alpha\sin\psi-\beta\cos\psi)\{dx+(d\alpha\cos\psi+d\beta\sin\psi)\\ &\qquad +(-\alpha\sin\psi+\beta\cos\psi)\,d\psi\}\\ &= x\,dy-y\,dx+\{(\alpha\cos\psi+\beta\sin\psi)\sin\psi\\ &\qquad -(\alpha\sin\psi-\beta\cos\psi)\cos\psi\}\,ds\\ &\quad +x(d\alpha\sin\psi-d\beta\cos\psi)-y(d\alpha\cos\psi+d\beta\sin\psi)\\ &\qquad +x(\alpha\cos\psi+\beta\sin\psi)\,d\psi\\ &\qquad -y(-\alpha\sin\psi+\beta\cos\psi)\,d\psi\\ &\quad +(\alpha\cos\psi+\beta\sin\psi)(d\alpha\sin\psi-d\beta\cos\psi)\\ &\qquad -(\alpha\sin\psi-\beta\cos\psi)(d\alpha\cos\psi+d\beta\sin\psi)\\ &\qquad +(\alpha^2+\beta^2)\,d\psi\\ &\qquad\qquad (\text{for } dx=ds\cos\psi,\ dy=ds\sin\psi)\\ &= (x\,dy-y\,dx)+2\beta\,ds+(\beta\,d\alpha-\alpha\,d\beta)+(\alpha^2+\beta^2)\,d\psi\\ &\qquad +d\{x(\alpha\sin\psi-\beta\cos\psi)\}\\ &\qquad -d\{y(\alpha\cos\psi+\beta\sin\psi)\},\end{aligned}$$

a term $\{-dx(\alpha\sin\psi-\beta\cos\psi)+dy(\alpha\cos\psi+\beta\sin\psi)\}$,

i.e. $\left\{-dx\left(\alpha\dfrac{dy}{ds}-\beta\dfrac{dx}{ds}\right)+dy\left(\alpha\dfrac{dx}{ds}+\beta\dfrac{dy}{ds}\right)\right\}$,

that is $\beta\,ds$ having been added and subtracted in the arrangement.

Hence, if A and A_1 be the corresponding sectorial areas swept out by the radii vectores OP and OQ,

$$A_1=A+\int\beta\,ds+\frac{1}{2}\int\left(\beta\frac{d\alpha}{ds}-\alpha\frac{d\beta}{ds}\right)ds+\frac{1}{2}\int\frac{\alpha^2+\beta^2}{\rho}\,ds$$
$$+\tfrac{1}{2}[x(\eta-y)-y(\xi-x)],$$

the portion [ ] being between limits corresponding to the beginning and ending of the arc traced by P.

If the curves be closed this term disappears, and

$$A_1=A+\int\beta\,ds+\frac{1}{2}\int\left(\beta\frac{d\alpha}{ds}-\alpha\frac{d\beta}{ds}\right)ds+\frac{1}{2}\int\frac{\alpha^2+\beta^2}{\rho}\,ds.$$

This formula of course includes the foregoing cases.

Thus, for parallels $\alpha=0$, $\beta=c$, and the oval being closed,

$$A_1=A+cs+\frac{1}{2}\int c^2\,d\psi=A+cs+\pi c^2 \text{ as before.}$$

437. Polar Subtangent.

The area bounded by any portion of a given curve, two tangents, and the corresponding portion of the locus of the extremity of the polar subtangent is given by

$$A=\frac{1}{2}\int\frac{r^2}{r_1^{\,2}}(r^2+2r_1^{\,2}-rr_2)\,d\theta,$$

where $$r_1=\frac{dr}{d\theta},\quad r_2=\frac{d^2r}{d\theta^2}.$$

For if OT be the polar subtangent corresponding to a point

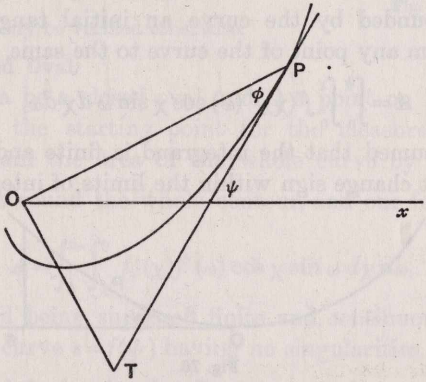

Fig. 75.

P, the point of contact of the tangent, we have with the usual notation

$$PT=r\sec\phi,$$

and
$$\text{Area swept by } PT=\frac{1}{2}\int PT^2\,d\psi$$
$$=\frac{1}{2}\int r^2\sec^2\phi\,d\psi$$
$$=\frac{1}{2}\int r^2\left(\frac{ds}{dr}\right)^2\frac{d\psi}{ds}\cdot\frac{ds}{d\theta}\cdot d\theta$$
$$=\frac{1}{2}\int\frac{r^2}{r_1^{\,2}}\cdot\left(\frac{ds}{d\theta}\right)^3\frac{1}{\rho}\,d\theta$$
$$=\frac{1}{2}\int\frac{r^2}{r_1^{\,2}}(r^2+r_1^{\,2})^{\frac{3}{2}}\frac{r^2+2r_1^{\,2}-rr_2}{(r^2+r_1^{\,2})^{\frac{3}{2}}}\,d\theta$$
$$=\frac{1}{2}\int\frac{r^2}{r_1^{\,2}}(r^2+2r_1^{\,2}-rr_2)\,d\theta,$$

the limits being the initial and final values of θ for the arc specified.

For a closed curve this area therefore exceeds twice the area of the original curve by

$$\frac{1}{2}\int \frac{r^3}{r_1^{\,2}}(r-r_2)\,d\theta.$$

438. **Intrinsic Equation.**

When the intrinsic equation is given, viz.

$$s=f(\psi),$$

the area bounded by the curve, an initial tangent, and an ordinate from any point of the curve to the same, is given by

$$A=\int_0^\psi\int_0^\chi f'(\chi)f'(\omega)\cos\chi\sin\omega\,d\chi\,d\omega,$$

it being assumed that the integrand is finite and continuous and does not change sign within the limits of integration.

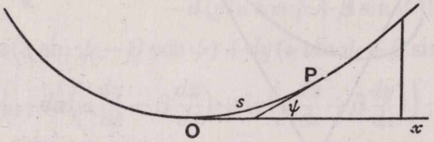

Fig. 76.

This is merely a transformation of

$$A=\int y\,dx.$$

For $\dfrac{dy}{ds}=\sin\psi$ and $y=\displaystyle\int_0^s \sin\psi\,ds=\int_0^\psi f'(\psi)\sin\psi\,d\psi$

$$=\int_0^\psi f'(\omega)\sin\omega\,d\omega.$$

Also $dx=\cos\psi\,ds=f'(\psi)\cos\psi\,d\psi.$

Hence $A=\displaystyle\int_0^\psi f'(\psi)\cos\psi\left\{\int_0^\psi f'(\omega)\sin\omega\,d\omega\right\}d\psi.$

$$=\int_0^\psi f'(\chi)\cos\chi\left\{\int_0^\chi f'(\omega)\sin\omega\,d\omega\right\}d\chi.$$

This may clearly be written

$$A=\int_0^\psi\int_0^\chi f'(\chi)\cos\chi f'(\omega)\sin\omega\,d\chi\,d\omega.$$

439. Ex. Taking as a test the case of the circle $s=a\psi$,

$$A=a^2\int_0^\psi\int_0^\chi \cos\chi\sin\omega\,d\chi\,d\omega$$

$$=a^2\int_0^\psi \cos\chi\Big[-\cos\omega\Big]_0^\chi d\chi$$

$$=a^2\int_0^\psi \cos\chi(1-\cos\chi)\,d\chi$$

$$=a^2\int_0^\psi\left(\cos\chi-\frac{1+\cos 2\chi}{2}\right)d\chi$$

$$=a^2\left[\sin\psi-\frac{\psi}{2}-\frac{\sin 2\psi}{4}\right],$$

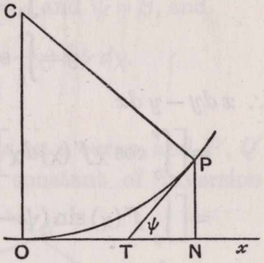

Fig. 77.

which may readily be verified otherwise.

440. Closed Oval.

If the area be a closed oval and O a point on the circumference, viz. the starting point for the measurement of s, we may obtain the area of the whole curve by integrating $-\int y\cos\psi\,ds$ round the whole contour, and our formula may be written

$$A=\int_0^{2\pi}\int_\chi^0 f'(\chi)f'(\omega)\cos\chi\sin\omega\,d\chi\,d\omega,$$

the integrand being supposed finite and continuous throughout, and the curve $s=f(\psi)$ having no singularities.

441. Closed Oval. Another Form

Another form may be given for the area of a closed curve whose intrinsic equation is $s=f(\psi)$.

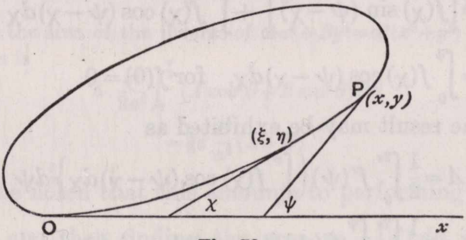

Fig. 78.

Measuring s from the point at which $\psi=0$, we have at any point ξ, η, where the inclination of the tangent to the initial tangent is χ, and the element of arc ds_1,

$$\frac{d\xi}{ds_1}=\cos\chi,\quad \frac{d\eta}{ds_1}=\sin\chi\,;$$

$$\therefore\ x=\int_0^s \cos\chi\, ds_1=\int_0^\psi \cos\chi f'(\chi)d\chi,$$

$$y=\int_0^s \sin\chi\, ds_1=\int_0^\psi \sin\chi f'(\chi)d\chi;$$

$\therefore\ x\,dy-y\,dx$

$$=\left[\int_0^\psi \cos\chi f'(\chi)d\chi\right]\sin\psi\, ds-\left[\int_0^\psi \sin\chi f'(\chi)\,d\chi\right]\cos\psi\, ds$$

$$=\left[\int_0^\psi f'(\chi)\sin(\psi-\chi)d\chi\right]ds;$$

$\therefore$ area of curve

$$=\frac{1}{2}\int(x\,dy-y\,dx),\text{ taken round the perimeter,}$$

$$=\frac{1}{2}\int_0^{2\pi} f'(\psi)\left[\int_0^\psi f'(\chi)\sin(\psi-\chi)\,d\chi\right]d\psi,$$

or, as we may write it,

$$A=\frac{1}{2}\int_0^{2\pi}\int_0^\psi f'(\psi)f'(\chi)\sin(\psi-\chi)\,d\psi\,d\chi,$$

it being understood that the first integration is with regard to χ, considering ψ a constant, from 0 to ψ, and then the result from 0 to 2π with regard to ψ.

Also
$$\int_0^\psi f'(\chi)\sin(\psi-\chi)d\chi$$
may be integrated by parts, and becomes

$$=\Big[f(\chi)\sin(\psi-\chi)\Big]_0^\psi+\int_0^\psi f(\chi)\cos(\psi-\chi)d\chi$$

$$=\int_0^\psi f(\chi)\cos(\psi-\chi)d\chi,\quad \text{for } f(0)=0.$$

Hence the result may be exhibited as

$$A=\frac{1}{2}\int_0^{2\pi} f'(\psi)\left\{\int_0^\psi f(\chi)\cos(\psi-\chi)d\chi\right\}d\psi$$

or
$$=\frac{1}{2}\int_0^{2\pi}\int_0^\psi f'(\psi)f(\chi)\cos(\psi-\chi)d\psi\,d\chi,$$

it being understood as before that the first integration is with regard to χ from 0 to ψ.

442. If the curve be not closed, and the limits for ψ are from $\psi=\alpha$ to $\psi=\beta$, we find by these formulae, a sectorial

area bounded by the arc and two specified radii vectores, viz. from the origin to the points where $\psi=\alpha$ and $\psi=\beta$, and

$$A=\frac{1}{2}\int_\alpha^\beta\int_0^\psi f'(\psi)f(\chi)\cos(\psi-\chi)\,d\psi\,d\chi.$$

443. Inverse Curve.

If the points P, Q be contiguous points on a curve, and P', Q' their respective inverses, k being the constant of inversion

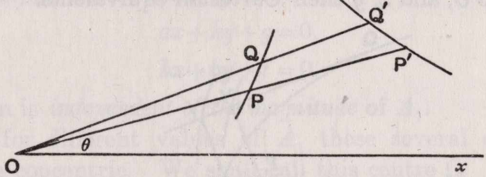

Fig. 79.

and O the pole, we have for any sectorial element $OP'Q'$ of the new curve,

$$\text{area } OP'Q'=\tfrac{1}{2}OP'.OQ'\sin\delta\theta=\tfrac{1}{2}k^4\frac{1}{OP.OQ}\delta\theta \text{ to the first order}$$

$$=\frac{k^4}{2}\frac{1}{r^2}\delta\theta \text{ to the first order,}$$

and the area of any sectorial portion of the inverse is

$$\frac{k^4}{2}\int\frac{1}{r^2}d\theta,$$

r being the radius vector of the original curve.

Ex. Thus the area of the inverse of $Ax^4+By^4=a^2(x^2+y^2)$ with regard to the origin is

$$4\cdot\frac{k^4}{2a^2}\int_0^{\frac{\pi}{2}}(A\cos^4\theta+B\sin^4\theta)\,d\theta$$
$$=\tfrac{3}{8}\pi\frac{k^4}{a^2}(A+B).$$

It will be noted that this amounts to performing the inversion first, and then finding the area as $\frac{1}{2}\int r'^2\,d\theta$, so that our formula $\frac{k^4}{2}\int\frac{1}{r^2}d\theta$ is of but little additional convenience.

444. Locus of Origins of Pedals of given Area.

Let O be a fixed point. Let p, ψ be the polar coordinates of the foot of the perpendicular OY upon any tangent to a given curve. Let P be any other fixed point, $PY_1\,(=p_1)$, the perpendicular from P upon the same tangent. Then the areas of the pedals, with O and P respectively as origins, are

$$\frac{1}{2}\int p^2\,d\psi \quad \text{and} \quad \frac{1}{2}\int p_1{}^2\,d\psi,$$

taken between the same definite limits. Call these A and A_1 respectively. Let r, θ be the polar coordinates of P with regard to O, and x, y their Cartesian equivalents.

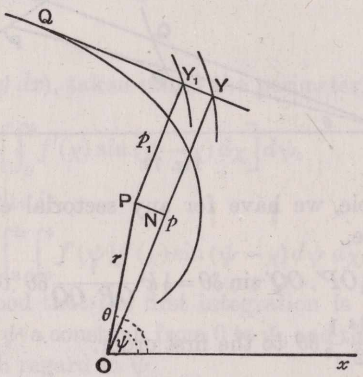

Fig. 80.

Then $\quad p_1 = p - r\cos(\theta-\psi) = p - x\cos\psi - y\sin\psi,$

and p is a known function of ψ.

Hence

$$2A_1 = \int p_1{}^2\,d\psi = \int (p - x\cos\psi - y\sin\psi)^2\,d\psi$$

$$= \int p^2\,d\psi - 2x\int p\cos\psi\,d\psi - 2y\int p\sin\psi\,d\psi$$

$$+x^2\int\cos^2\psi\,d\psi + 2xy\int\cos\psi\sin\psi\,d\psi + y^2\int\sin^2\psi\,d\psi$$

Now $\quad 2\int p\cos\psi\,d\psi, \quad 2\int p\sin\psi\,d\psi, \quad \int\cos^2\psi\,d\psi,$ etc.,

taken between such limits that the whole pedal is described, will be definite constants. Call them respectively

$$-2g,\ -2f,\ a,\ 2h,\ b,$$

and we thus obtain

$$2A_1 = 2A + 2gx + 2fy + ax^2 + 2hxy + by^2.$$

If then P move in such a manner that A_1 is constant, its locus must be a conic section.

By Article 342,

$$\int_0^\psi \cos^2\psi\, d\psi \times \int_0^\psi \sin^2\psi\, d\psi > \left\{\int_0^\psi \cos\psi \sin\psi\, d\psi\right\}^2,$$

i.e. $$ab > h^2.$$

Hence this conic section is **in general an ellipse.**

Moreover, its centre being given by

$$ax+hy+g=0,$$
$$hx+by+f=0,$$

its position is *independent of the magnitude* of A_1.

Hence for different values of A_1 these several conic-loci will all be concentric. We shall call this centre Ω.

445. Closed Oval.

Next suppose that the original curve is a closed oval curve, and that the point P is within it. Then the limits of integration are 0 and 2π.

Thus $$a=\int_0^{2\pi} \cos^2\psi\, d\psi = \pi = \int_0^{2\pi} \sin^2\psi\, d\psi = b$$

and $$h=\int_0^{2\pi} \cos\psi \sin\psi\, d\psi = 0.$$

Hence the conic becomes

$$\pi(x^2+y^2)+2gx+2fy+2(A-A_1)=0,$$

that is a circle whose centre is at the point

$$\frac{1}{\pi}\int_0^{2\pi} p\cos\psi\, d\psi,\ \frac{1}{\pi}\int_0^{2\pi} p\sin\psi\, d\psi.$$

Now, if x, y be the point of contact of the tangent, viz. Q,

$$QY=\frac{dp}{d\psi},$$

and $$\left.\begin{aligned} x&=p\cos\psi-\frac{dp}{d\psi}\sin\psi,\\ y&=p\sin\psi+\frac{dp}{d\psi}\cos\psi,\end{aligned}\right\}$$ by projecting p, $\frac{dp}{d\psi}$ upon the coordinate axes;

$$\therefore \int x\, d\psi = \int p\cos\psi\, d\psi - [p\sin\psi] + \int p\cos\psi\, d\psi = 2\int p\cos\psi\, d\psi,$$

and

$$\int y\,d\psi=\int p\sin\psi\,d\psi+[p\cos\psi]+\int p\sin\psi\,d\psi=2\int p\sin\psi\,d\psi,$$

for the portions in square brackets disappear in integrating round the whole curve.

Hence the **coordinates of the centre** of the circle may be written

$$\left.\begin{aligned} x_1&=\frac{1}{2\pi}\int x\,d\psi, \text{ or } \frac{\int x\,d\psi}{\int d\psi}, \text{ or } \frac{1}{2\pi}\int\frac{x}{\rho}\,ds,\\ y_1&=\frac{1}{2\pi}\int y\,d\psi, \text{ or } \frac{\int y\,d\psi}{\int d\psi}, \text{ or } \frac{1}{2\pi}\int\frac{y}{\rho}\,ds,\end{aligned}\right\}\text{ where } \frac{1}{\rho} \text{ is the curvature at the element } ds.$$

446. Another Determination of the Centre.

If the original curve be regarded as a material curve of uniform section ω and with a density proportional to the curvature at each point, $=\frac{k}{\rho}$, say, the mass of each element δs is $\frac{k}{\rho}\omega\,\delta s$, and the formulae

$$\bar{x}=\frac{\Sigma mx}{\Sigma m},\quad \bar{y}=\frac{\Sigma my}{\Sigma m}$$

of Statics show that the centroid of any arc of this curve is given by

$$\bar{x}=\frac{\int\frac{k}{\rho}\omega x\,ds}{\int\frac{k}{\rho}\omega\,ds}=\frac{\int\frac{x}{\rho}\,ds}{\int\frac{1}{\rho}\,ds}, \text{ or } \frac{\int x\,d\psi}{\int d\psi},$$

$$\bar{y}=\frac{\int\frac{k}{\rho}\omega y\,ds}{\int\frac{k}{\rho}\omega\,ds}=\frac{\int\frac{y}{\rho}\,ds}{\int\frac{1}{\rho}\,ds}, \text{ or } \frac{\int y\,d\psi}{\int d\psi}.$$

Hence the point Ω, which is the centre of these loci, is identical with the centroid of a material wire of fine uniform section, bent into the form of the original curve, and having a density proportional to the curvature at each point; or, which comes to the same thing, having uniform density and cross-section infinitesimally small but proportional at each point to the curvature.

447. Connection of Areas.

The point Ω having been found, let us transfer our origin from O to Ω. The linear terms of the conic will thereby be removed. Thus Ω is a point such that the integrals

$$\int p\cos\psi\, d\psi \quad \text{and} \quad \int p\sin\psi\, d\psi,$$

where p is now measured from Ω, both vanish, and if Π be the area of the pedal whose pole is Ω, we have for any other,

$$2A_1 = 2\Pi + ax^2 + 2hxy + by^2 \text{ in the general case,}$$

and $\quad 2A_1 = 2\Pi + \pi(x^2+y^2)$ in the particular case, when the oval is closed.

The area of the conic is $\dfrac{2\pi(A_1-\Pi)}{\sqrt{ab-h^2}}$. (Smith, *Conic Sections*, Art. 171.)

Thus, in the general case,

$$A_1 = \Pi + \frac{\sqrt{ab-h^2}}{2\pi} \times \text{area of conic.}$$

And in the particular case of the closed oval,

$$A_1 = \Pi + \tfrac{1}{2}\pi r^2,$$

where r is the radius of the circle on which P lies for constant values of A_1, *i.e.* the distance of P from Ω.

448. Position of the Point Ω for a Centric Closed Oval.

In any oval which has a centre the point Ω is plainly at that centre. For when the centre is taken as origin, the integrals

$$\int p\cos\psi\, d\psi \quad \text{and} \quad \int p\sin\psi\, d\psi, \quad i.e. \ \frac{1}{2}\int x\, d\psi \ \text{and} \ \frac{1}{2}\int y\, d\psi,$$

both vanish when the integration is performed for the complete oval, opposite elements of the integration cancelling ; or, which is the same thing, the centroid of a material centric oval curve for a law of density, which varies as the curvature at each point, is obviously at the centre of the oval.

449. Origin for Pedal of Minimum Area.

When Ω is taken as origin, it appears that

$$2A_1 = 2\Pi + \int (x\cos\psi + y\sin\psi)^2\, d\psi.$$

Hence, as the term $\int (x\cos\psi + y\sin\psi)^2\, d\psi$ is necessarily positive, it is clear that A_1 can never be less than Π.

Ω is therefore the origin for which the corresponding pedal curve has a minimum area.

450. A Statical View of the Case.

Let O be the origin, QRS the closed oval, OY the perpendicular from O upon a tangent to the curve. Let P be any other point, and Ω the centre of gravity of the curve, QRS having a density at each point proportional to the curvature.

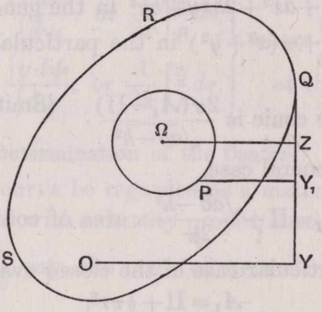

Fig. 81.

A theorem by Lagrange (Routh, *Statics*, vol. i Art. 436) states that if $m_1, m_2, m_3, \ldots$ be the masses of a system of heavy particles at $Q_1, Q_2, Q_3, \ldots$, and Ω their centre of gravity, and if P be any other point, then

$$m_1PQ_1^2+m_2PQ_2^2+m_3PQ_3^2+\ldots=m_1\Omega Q_1^2+m_2\Omega Q_2^2+m_3\Omega Q_3^2 +\ldots+(m_1+m_2+m_3+\ldots)\Omega P^2.$$

Applying this theorem to our curve of density $\frac{k}{\rho}$, uniform small section ω, and total mass $\lambda k\omega$, say,

$$\int\frac{PQ^2}{\rho}ds=\int\frac{\Omega Q^2}{\rho}ds+\lambda . P\Omega^2.$$

Now it has been proved in Art. 426 that the area of the pedal of a closed oval exceeds $\frac{1}{2}$ the area of the oval by $\frac{1}{4}\int\frac{r^2}{\rho}ds$.

$$\therefore \text{ pedal with regard to } P=\frac{1}{2}\text{ oval }+\frac{1}{4}\int\frac{PQ^2}{\rho}ds;$$

and $$\text{pedal with regard to } \Omega=\frac{1}{2}\text{ oval }+\frac{1}{4}\int\frac{\Omega Q^2}{\rho}ds;$$

$\therefore$ pedal with regard to P = pedal with regard to $\Omega + \frac{\lambda}{4} P\Omega^2$

and $\quad \lambda k\omega = \text{mass of curve} = \int \frac{k}{\rho}\omega\, ds = k\omega \int d\psi = 2\pi k\omega$;

$\therefore\ \lambda = 2\pi$.

$\therefore$ pedal with regard to P = pedal with regard to $\Omega + \frac{\pi}{2} P\Omega^2$.

Hence we are led by statical considerations to the same result as already obtained, viz. that the loci of the origins P, of which the pedal curves of a closed oval are of constant area, are concentric circles, their centre being the origin of the pedal of minimum area and the centroid of a fine wire bent into the form of the original oval, and having uniform cross-section and a density varying as the curvature.

Illustrative Examples.

Ex. 1. Find the area of the pedal of a circle with regard to any point within the circle at a distance c from the centre *i.e.* a limaçon.

Here $$A_1 = \Pi + \frac{\pi c^2}{2}$$

and $$\Pi = \pi a^2.$$

Hence $$A_1 = \pi a^2 + \tfrac{1}{2}\pi c^2.$$

Ex. 2. Find the area of the pedal of an ellipse with regard to any point at a distance c from the centre.

In this case, Π is the area of the pedal with regard to the centre

$$= 2\int_0^{\frac{\pi}{2}} (a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta = (a^2+b^2)\frac{\pi}{2}.$$

Hence $\quad A_1 = \frac{\pi}{2}(a^2+b^2+c^2)$.

Ex. 3. The area of the pedal of the cardioide $r = a(1-\cos\theta)$ taken with respect to an internal point on the axis at a distance c from the pole is

$$\frac{3\pi}{8}(5a^2 - 2ac + 2c^2).$$

[Math. Tripos, 1876.]

Let O be the pole, P the given internal point; p and p_1 the two perpendiculars OY_2 and PY_1 upon any tangent from O and P respectively; ϕ the angle $Y_2\hat{O}P$ and $OP = c$; then $p_1 = p - c\cos\phi$, and

$$2A_1 = 2A - 2c\int p\cos\phi\, d\phi + \int c^2\cos^2\phi\, d\phi.$$

Now, in order that p may sweep out the whole pedal, we must integrate between limits $\phi=0$ and $\phi=\frac{3\pi}{2}$ and double. Now in the cardioide (Fig. 82).

$$p=OQ\sin Y_2QO=OQ\sin\tfrac{1}{2}xOQ.$$

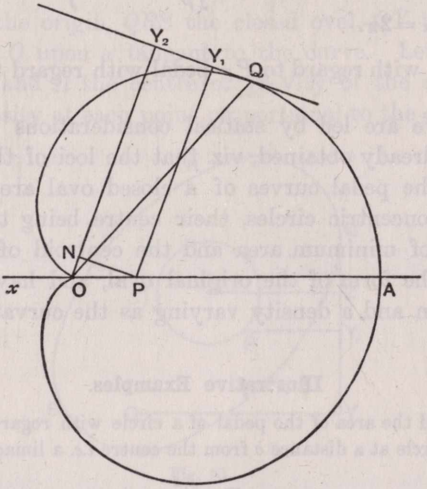

Fig. 82.

For $$Y_2QO=\tfrac{1}{2}xOQ=\frac{\theta}{2}.$$

Hence $$\frac{\pi}{2}-\left\{\phi-(\pi-\theta)\right\}=\frac{\theta}{2},$$

or $$\frac{3\pi}{2}-\phi=\frac{3\theta}{2}\quad\text{and}\quad\frac{\theta}{2}=\frac{\pi}{2}-\frac{\phi}{3}.$$

So $$p=r\sin\frac{\theta}{2}=2a\sin^3\frac{\theta}{2}=2a\cos^3\frac{\phi}{3}.$$

Hence

$$\int p\cos\phi\,d\phi=2\int_0^{\frac{3\pi}{2}}2a\cos^3\frac{\phi}{3}\cos\phi\,d\phi=4a\times 3\int_0^{\frac{\pi}{2}}\cos^3 z\cos 3z\,dz$$

$$=12a\int_0^{\frac{\pi}{2}}\left[4\cos^6 z-3\cos^4 z\right]dz$$

$$=12a\left[4\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}-3\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right]=\frac{3\pi a}{4}.$$

Also $\int c^2\cos^2\phi\,d\phi=3\cdot 2c^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3\pi c^2}{2}.$

Finally, $$2A=2\int_0^{\frac{3\pi}{2}}4a^2\cos^6\frac{\phi}{3}\,d\phi=24a^2\int_0^{\frac{\pi}{2}}\cos^6 z\,dz;$$

$$\therefore\ A=12a^2\,\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{15\pi a^2}{8}.$$

Thus, $$A_1 = \frac{15\pi a^2}{8} - \frac{3\pi ac}{4} + \frac{3\pi c^2}{4}$$
$$= \frac{3\pi}{8}(5a^2 - 2ac + 2c^2).$$

Ex. 4. Let A, B, C be any three points and P a fourth point whose areal coordinates are x, y, z when the triangle ABC is regarded as the triangle of reference. To find the relation of the areas of the pedals of any closed curve with respect to A, B, C and P.

Let $[A]$, $[B]$, $[C]$, $[P]$ represent the areas of the pedals. Let X, Y, Z be the areal coordinates of Ω, the centre for the pedal of minimum area.

Then
$$[A] = [\Omega] + \tfrac{1}{2}\pi A\Omega^2,$$
$$[B] = [\Omega] + \tfrac{1}{2}\pi B\Omega^2,$$
$$[C] = [\Omega] + \tfrac{1}{2}\pi C\Omega^2,$$
$$[P] = [\Omega] + \tfrac{1}{2}\pi P\Omega^2;$$

$$\therefore\ [P] - [A]x - [B]y - [C]z = \frac{\pi}{2}(P\Omega^2 - xA\Omega^2 - yB\Omega^2 - zC\Omega^2).$$

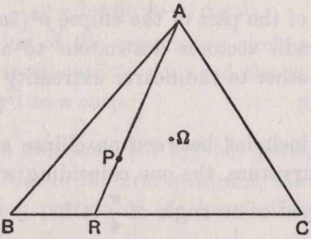

Fig. 83.

Now (Ferrers' *Trilinears*, p. 6) the distance from x, y, z to X, Y, Z is given by
$$P\Omega^2 = -a^2(y-Y)(z-Z) - b^2(z-Z)(x-X) - c^2(x-X)(y-Y)$$
and
$$A\Omega^2 = -a^2(0-Y)(0-Z) - b^2(0-Z)(1-X) - c^2(1-X)(0-Y)$$
$$= -a^2YZ - b^2ZX - c^2XY + b^2Z + c^2Y,$$
$$B\Omega^2 = -b^2ZX - c^2XY - a^2YZ + c^2X + a^2Z,$$
$$C\Omega^2 = -c^2XY - a^2YZ - b^2ZX + a^2Y + b^2X;$$
$$\therefore\ P\Omega^2 - xA\Omega^2 - yB\Omega^2 - zC\Omega^2 = -a^2yz - b^2zx - c^2xy.$$

Now, if $S \equiv a^2yz + b^2zx + c^2xy$, $S=0$ is the equation of the circumcircle, and S is equal to minus the square of the tangent from the point (x, y, z) to the circle $S=0$ if the point lie without the circle, or to the rectangle of the segments of any chord through x, y, z if within. Therefore with this meaning for S,
$$[P] = [A]x + [B]y + [C]z - \tfrac{1}{2}\pi S.$$

PROBLEMS ON QUADRATURE.

1. Interpret geometrically $\int_{p_0}^{p_1} \sqrt{r^2 - p^2}\, dp$ in the case of the curve $r = f(p)$

Prove that the value of $\int \sqrt{r^2 - p^2}\, dp$, taken all round an ellipse whose semiaxes are a, b, and whose centre is the pole, is $\pi(a-b)^2$.

[OXFORD I. P., 1903.]

2. Use the pedal equation of an ellipse, viz. $\frac{a^2b^2}{p^2} = a^2 + b^2 - r^2$, to show that the area of the portion of an ellipse included between the curve, the semi-major axis and a central radius vector r, is

$$\frac{ab}{2} \tan^{-1} \sqrt{\frac{a^2 - r^2}{r^2 - b^2}},$$

a, b being the semiaxes of the ellipse. [COLLEGES, 1882.]

3. Find the area of the part of the ellipse $p^2(2a - r) = b^2 r$ included between two focal radii vectores drawn, one to an extremity of the minor axis and the other to the nearer extremity of the major axis.

[OXFORD I. P., 1889.]

4. Find the area included between an ellipse and its evolute and bounding radii of curvature, the one coinciding with the major axis and the other inclined at an angle of $\frac{\pi}{4}$ to it.

[COLLEGES, 1884, AND β, 1888.]

5. Through every point of an ellipse a line is drawn outwards normal to the ellipse and equal to the radius of curvature at the point. Show that the area of the curve thus obtained is

$$\pi \frac{9a^4 + 14a^2b^2 + 9b^4}{2ab}.$$

[COLLEGES α, 1891.]

6. Show that the area of that part of the evolute of an ellipse $\left(\text{eccentricity} > \frac{1}{\sqrt{2}}\right)$ which lies outside the ellipse is

$$a^4b^4 \int_{b^2}^{\frac{a^2+b^2}{3}} \frac{(a^2 + b^2 - 3\rho)^2}{\rho^3(a^2 + b^2 - \rho)^2} \frac{d\rho}{\sqrt{(\rho - b^2)(a^2 - \rho)}}.$$

[COLLEGES, 1882.]

7. Find the area of the pedal of the curve

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}},$$

the origin being taken at $x = \sqrt{a^2 - b^2}$, $y = 0$. [OXFORD I. P., 1888.]

8. Show that the area of the space between the epicycloid $p=A\sin B\psi$ and its pedal curve taken from cusp to cusp is $\frac{1}{4}\pi A^2B$. [COLLEGES, 1878.]

9. Show that the area between an epicycloid and the arc of the fixed circle included between two consecutive cusps is

$$\frac{\pi b^2}{a}(3a+2b),$$

where a and b are the radii of the fixed and rolling circles respectively. [COLLEGES α, 1884.]

Show also that the area of the corresponding sector of the fixed circle is that of an ellipse with semiaxes the radii of the two circles. [OXFORD I. P., 1913.]

10. Show that the p-ψ equation to a cycloid when one of the cusps is taken as origin is

$$p=2a(\sin\psi-\psi\cos\psi),$$

where a is the radius of the generating circle; and find the area between the curve from cusp to cusp and the corresponding arc of the pedal with regard to a cusp. [OXFORD II. P., 1903.]

11. Show that the area bounded by that portion of the cardioide $r^{\frac{1}{2}}=a^{\frac{1}{2}}\sin\frac{1}{2}\theta$, which lies in the first quadrant, the terminal tangents, and the corresponding portion of the locus of the extremity of the polar subtangent, is

$$3a^2(10-3\pi)/16.$$ [MATH. TRIPOS, 1896.]

12. Show that in the curve in which the area bounded by the curve and the radii vectores from a certain fixed point varies as the square of the length of the bounding arc, the radius of curvature varies as the projection of the radius vector on the tangent. [COLLEGES α, 1891.]

13. The pedal of a cycloid with regard to any point on its axis meets the cycloid at the vertex A and cuts the tangent at the cusp in Q; find the area between it and the chord AQ; and prove that this area is least when the origin is the middle point of the axis. [ST. JOHN'S, 1883]

14. An elliptic wire is pushed in one plane through a very short straight tube; find the equation to the locus of the centre, and prove that the area of each loop is $\frac{\pi}{2}(a-b)^2$, where a and b are the semiaxes. [COLLEGES, 1886].

15. A point Q is taken on the normal drawn outward at a point P of a catenary, the parameter of which is c. Prove that if PQ is equal to the length of the arc of the catenary measured from the vertex to P, the area between the locus of Q and the catenary, and bounded by the normal at the vertex and by another normal inclined at an angle ψ to this, is

$$\frac{c^2}{2}(\tan^2\psi + \tan\psi - \psi).$$

[COLLEGES γ, 1882.]

16. Prove that the pedal of the cardioide $r = a\cos^2\dfrac{\theta}{2}$ with respect to the cusp consists of two closed regions of areas A and B, A consisting of the inner loop and B being external to A and bounded by the outer line of the curve and such that $2A + B = \dfrac{15\pi a^2}{32}$.

[COLLEGES γ, 1899.]

17. Prove that the area of the pedal of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ with respect to the point $(a, 0)$ is five times as great as the area of its pedal with respect to the origin. [OXFORD II. P., 1899.]

18. The tangent at a point P of a lemniscate cuts the curve again at Q, R. Prove that the middle point of QR is at the same distance from the nodal point as P; and that the equation to its locus is

$$a^{10}(x^2 - y^2) = r^4\{a^8 + 4(a^4 - r^4)(a^4 - 4r^4)\},$$

where $$r^2 \equiv x^2 + y^2.$$

Show that it can be written $$r^2 = a^2\cos\tfrac{2}{5}\theta.$$

Trace the curve completely, and prove that the portion corresponding to the upper half of one branch of the lemniscate divides the other branch into two parts whose areas are in the ratio of

$$6 - 3\sqrt{3} : 3\sqrt{3} - 4.$$

[ST. JOHN'S, 1884.]

19. Show that the area of a loop of the curve

$$(x^2 - a^2)^2 + (y^2 - 3a^2)^2 = a^4$$

is $$a^2\sqrt{2}\left(\frac{\pi}{3} - \log_e\frac{\sqrt{3}+1}{2}\right).$$

[MATH. TRIPOS, 1882.]

20. The tangent at every point P of a closed finite curve is produced to Q so that PQ is constant. Find the area between the locus of Q and the original curve. How is the result to be explained, (i) if the curvature of the first curve is sometimes in one direction, sometimes in the opposite direction; (ii) if the curve cuts itself a given number of times. [ST. JOHN'S COLL., 1881.]

21. A straight line of constant length c is drawn from each point of a closed oval curve making a given angle α with the normal at that point. Prove that the area of the curve traced out by the end of the line is

$$S+\pi c^2 \pm lc\cos\alpha,$$

where S is the area of the given oval curve and l is its length.

[COLL. γ, 1893.]

22. Show that the area of the polar reciprocal of a curve whose equation is given in rectangular coordinates is

$$\tfrac{1}{2}k^4\int\frac{\dfrac{d^2y}{dx^2}}{\left(y-x\dfrac{dy}{dx}\right)^2}dx,$$

x, y being the coordinates of a point on the original curve.

Apply this to find the area of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

[COLLEGES, 1886.]

23. The area of a given closed oval curve is A; the bisectors of the internal and external angles between tangents to it which meet at a given constant angle 2α envelop curves whose areas are A_1 and A_2; show that

$$A_1\cos^2\alpha+A_2\sin^2\alpha=A.$$

[COLLEGES γ, 1888.]

24. Prove that for any closed curve which has a centre, the area of the locus of intersection of tangents at right angles, and the area of the locus of intersection of normals at right angles differ by twice the area of the curve. [MATH. TRIPOS, 1888.]

25. O being a fixed point, OP a radius vector of any curve, OP is produced to Q so that $OP\,.\,PQ=a^2$, and A is the area between the locus of Q and the given curve. If A' be the area of the inverse of the curve with respect to O, the constant of inversion being a, show that $A'-A$ is independent of the form of the curve.

If the given curve be a circle, and O a point on its circumference, find the area of any part bounded by the locus of Q, the circle and two radii vectores from O. [ST. JOHN'S, 1891.]

26. A circle rolls on the outside of an oval curve, the pedals of the curve, of the locus of the centre of the circle and of the envelope of the circle are of areas A_0, A_1, A_2, respectively; prove that $A_2-2A_1+A_0$ depends only on the rolling circle.

Show that if the area of the oval curve, of the locus of the centre of the circle and of the envelope of the circle be S_0, S_1, S_2 respectively,

$$A_2-2A_1+A_0=S_2-2S_1+S_0.$$

[TRINITY, 1878.]

27. One of the curves given by the equation

$$y=a^2\frac{d}{ds}\left\{\frac{dy}{dx}+\frac{1}{3}\left(\frac{dy}{dx}\right)^3\right\}$$

cuts the axis of x twice at the angle α. Prove that the area between the curve and the axis is

$$a^2\{\tan\alpha\sec\alpha+\log(\sec\alpha+\tan\alpha)\}.$$ [OXF. I. P., 1912.]

28. A curve concave to the axis of x is such that the product of the ordinate and radius of curvature at any point is constant and equal to c^2 (The Elastica, or Bent Bow). Prove that the maximum value of the ordinate is $2c\sin\frac{\alpha}{2}$, where α is the angle at which the curve crosses the axis of x. [OX. I. P., 1903.]

Show that the area which lies between the bow and the bow-string is $2c^2\sin\alpha$.

29. Show that the area of a closed curve, which is the envelope of the line $x\cos\psi+y\sin\psi=p$, is the value of the integral

$$-\frac{1}{2}\int\left(\frac{dp}{d\psi}+p\right)\left(\frac{dp}{d\psi}-p\right)d\psi$$

taken completely round the curve. [MATH. TRIP., 1898.]

30. The integral $-\frac{1}{2}\int\left(\frac{dp}{d\psi}+np\right)^2 d\psi$ is taken round a closed curve, n being taken equal to $\tan\psi$ or to $-\cot\psi$, according as the one or the other is numerically less than unity. Show that the value of the integral differs from the area of the curve by the sum of the squares of the perpendiculars from the origin upon the tangents at the points where the integral changes form. [MATH. TRIP., 1898.]

31. In the cycloid prove that the conic locus of points with regard to which the area of the pedal is constant, is in general a circle, and find the point for which the area of the pedal is a minimum. [OX. I. P., 1900.]

32. In a catenary, A is the vertex, P any point on the curve, AO, PN perpendiculars upon the directrix, PY a tangent and NY perpendicular to it. Show that the area of the figure $ONPA$ is double that of the triangle YNP.

33. Show that the area of the first positive pedal of the curve $p=f(r)$ may be obtained by the formula

$$\frac{1}{2}\int\frac{p^2}{\sqrt{r^2-p^2}}\frac{dp}{dr}\,dr,$$

where the letters p and r are the pedal coordinates of a point on the original curve.

Apply this method to find the area of the cardioide, which is the first positive pedal of the circle $r^2=ap$.

34. Employ the formula

$$\frac{1}{2}\int \frac{pr}{\sqrt{r^2-p^2}}\,dr$$

to find the area of

$$r^2+a^2=b^2+2ap \quad (a>b).$$

To what curve does this pedal equation belong?

35. In the epicycloid $\quad p^2=a^2\dfrac{r^2-a^2}{c^2-a^2},$

where a and $\dfrac{c-a}{2}$ are the radii of the fixed and rolling circles respectively, obtain a formula for the area of any sectorial portion with centre of the sector at the origin. Hence show that the area between one foil of the curve and the fixed circle is

$$\pi(c-a)^2(c+2a)/4a.$$

36. When $a<b$ the conchoid of Nicomedes, viz.

$$x^2y^2=(a+y)^2(b^2-y^2) \quad \text{or} \quad r=a\operatorname{cosec}\theta \pm b$$

has a loop. Find its area.

37. Let S be the focus of a parabola, SP_1, SP_2 two focal radii vectores of lengths r_1, r_2. The latus rectum is $4a$ and $P_1P_2=c$. Prove Lambert's expression for the sectorial area SP_1P_2, viz.

$$\frac{\sqrt{a}}{3}\left[s^{\frac{3}{2}}-(s-c)^{\frac{3}{2}}\right],$$

where $2s=r_1+r_2+c$.

Show that the segment cut off by a focal chord of length c is

$$\frac{1}{3}\,a^{\frac{1}{2}}c^{\frac{3}{2}}.$$

38. In the case of the Cotes's spirals, whose equations are of the form

$$\frac{1}{p^2}=\frac{A}{r^2}+B,$$

show that the area of the sectorial portion bounded by the curve and the radii vectores r_1 and r_2 is

$$\frac{1}{2B}\left\{(Br_1^{\,2}+A-1)^{\frac{1}{2}} \sim (Br_2^{\,2}+A-1)^{\frac{1}{2}}\right\},\ B\neq 0.$$

Examine in detail the particular cases of

(i) the equiangular spiral;

(ii) the reciprocal spiral;

(iii), (iv) and (v) the cases which reduce to the polar forms,

$u = a \sinh n\theta$, $u = a \cosh n\theta$, $u = a \sin n\theta$, respectively.

39. Riccati's *Syntractory* * is generated as follows. The tractory is an involute of a common catenary of parameter c, starting from the vertex. PT is a tangent at any point P of the tractory, cutting the directrix of the catenary at T. Q is a point on PT or PT produced such that $QT = c'$. The locus of Q is the syntractory.

Show that the areas between the two branches and the directrix are

$$\frac{\pi}{2} c'(2c \pm c').$$

40. If A be the area of the 'Helmet,'

$$(k+1)\{(x^2+ka^2)y^2 - 2ay(a^2-x^2)\} + (a^2-x^2)^2 = 0, \quad (k \neq -1),$$

and V the volume formed by its revolution about the y-axis, prove that

$$A = \frac{\pi a^2}{\sqrt{k(k+1)}}\,[2(k+1)^{\frac{3}{2}} - (2k+3)k^{\frac{1}{2}}],$$

$$V = \frac{2\pi a^3}{3\sqrt{k+1}}\left[3(k+1)^{\frac{3}{2}} \log \frac{\sqrt{k+1}+1}{\sqrt{k+1}-1} - 2(3k+4)\right]$$

[For the first part of the example, and for several others of similar character, see Wolstenholme's *Problems*, Nos. 1886 to 1870.]

* *Comment. Bononensia*, Tom. iii., 1755.