CHAPTER IX.

GENERAL THEOREMS.

319. **Various Limiting Forms expressed as Definite Integrals.** The definition of an integral, viz.

$$\int_a^b \phi(x)dx = Lt_{h=0} h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b)]$$

where $b=a+nh$ may be expressed as

$$Lt_{n=\infty} \sum_{\frac{r}{n}=0}^{\frac{r}{n}=1} \frac{b-a}{n}\phi\left(a+r\frac{b-a}{n}\right),$$

and can be used for the evaluation of a certain class of limiting forms.

Ex. Find the value of

$$Lt_{n=\infty}\left[\frac{1^2}{n^3+1^3}+\frac{2^2}{n^3+2^3}+\frac{3^2}{n^3+3^3}+\ldots+\frac{n^2}{n^3+n^3}\right].$$

This may be written as

$$Lt_{n=\infty} \sum_{\frac{r}{n}=\frac{1}{n}}^{\frac{r}{n}=\frac{n}{n}=1} \frac{1}{n}\cdot\frac{\frac{r^2}{n^2}}{1+\frac{r^3}{n^3}},$$

and taking $\frac{r}{n}$ as x and $\frac{1}{n}$ as dx

$$=\int_0^1 \frac{x^2}{1+x^3}dx = \tfrac{1}{3}\Big[\log(1+x^3)\Big]_0^1 = \tfrac{1}{3}\log_e 2.$$

320. In the same way

$$Lt_{n=\infty}\{\phi(a)\phi(a+h)\phi(a+2h)\ldots\phi(a+nh)\}^{\frac{1}{n}},$$

where $h=\dfrac{b-a}{n}$ may be evaluated.

Let $$u=\{\phi(a)\phi(a+h)\phi(a+2h)\ldots\phi(a+nh)\}^{\frac{1}{n}};$$

then $$\log u=\frac{1}{n}\{\log\phi(a)+\log\phi(a+h)+\ldots+\log\phi(a+rh)$$
$$+\ldots+\log\phi(b)\}$$

$$=\sum_{\frac{r}{n}=\frac{0}{n}}^{\frac{r}{n}=\frac{n}{n}=1}\frac{1}{n}\log\phi\left\{a+(b-a)\frac{r}{n}\right\};$$

and therefore if we write
$$a+(b-a)\frac{r}{n}=x$$

and $$(b-a)\frac{1}{n}=dx,$$

the limit of $\log u$ is $$\int_a^b\frac{\log\phi(x)}{b-a}\,dx.$$

Hence $$Lt_{n=\infty}\{\phi(a)\phi(a+h)\phi(a+2h)\ldots\phi(a+nh)\}^{\frac{1}{n}},$$

where $h=\dfrac{b-a}{n}$,
$$=e^{\frac{1}{b-a}\int_a^b\log\phi(x)\,dx}$$

[see *Diff. Calc.*, p. 6, Ex. 3].

Ex. Find the limit when $n=\infty$ of
$$\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right\}^{\frac{1}{n}}.$$

Calling this expression u,
$$\log u=\frac{1}{n}\left\{\log\left(1+\frac{1^2}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\ldots+\log\left(1+\frac{n^2}{n^2}\right)\right\}$$
$$=\sum_{\frac{r}{n}=\frac{1}{n}}^{\frac{r}{n}=\frac{n}{n}=1}\frac{1}{n}\log\left(1+\frac{r^2}{n^2}\right),$$

and $$Lt\log u=\int_0^1\log(1+x^2)\,dx$$
$$=\Big[x\log(1+x^2)\Big]_0^1-2\int_0^1\frac{x^2}{1+x^2}dx$$
$$=\log 2-2\int_0^1\left(1-\frac{1}{1+x^2}\right)dx$$
$$=\log 2-2+2\,.\,\frac{\pi}{4}=\frac{\pi}{2}+\log 2-2;$$
$$\therefore\ Lt\,u=e^{\log 2+\frac{\pi-4}{2}}=2e^{\frac{\pi-4}{2}}$$

Examples.

1. Determine by integration the limiting values of the sums of the following series when n is infinitely great:

(i) $\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\dots+\frac{1}{n+n}$. [$\alpha$, 1884.]

(ii) $\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\dots+\frac{n}{n^2+n^2}$. [Oxford, 1888.]

(iii) $\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\frac{1}{\sqrt{6n-3^2}}+\dots+\frac{1}{\sqrt{2n^2-n^2}}$.
[Clare, etc., 1882.]

(iv) $\frac{1}{n}\left\{\sin^{2k}\frac{\pi}{2n}+\sin^{2k}\frac{2\pi}{2n}+\sin^{2k}\frac{3\pi}{2n}+\dots+\sin^{2k}\frac{\pi}{2}\right\}$,

k being a positive integer. [St. John's, 1886.]

2. Show that the limit when n is increased indefinitely of

$$\frac{(n-m)^{\frac{1}{3}}}{n}+\frac{(2^2n-m)^{\frac{1}{3}}}{2n}+\frac{(3^2n-m)^{\frac{1}{3}}}{3n}+\dots+\frac{(n^3-m)^{\frac{1}{3}}}{n^2} \quad \text{is} \quad \frac{3}{2}.$$

[Colleges, 1892.]

3. Find the limit when n is indefinitely great of the series

$$\frac{\sqrt{n-1}}{n}+\frac{\sqrt{2n-1}}{2n}+\frac{\sqrt{3n-1}}{3n}+\dots+\frac{\sqrt{n^2-1}}{n^2}.$$

[Colleges, 1890.]

4. Evaluate

$$Lt_{n=\infty}\left[\frac{1}{\sqrt{2a^2n-1}}+\frac{1}{\sqrt{4a^2n-1}}+\frac{1}{\sqrt{6a^2n-1}}+\dots+\frac{1}{\sqrt{2a^2n^2-1}}\right].$$

5. Evaluate

$$Lt_{n=\infty}\left[\frac{n^2}{(n^2+1^2)^{\frac{3}{2}}}+\frac{n^2}{(n^2+2^2)^{\frac{3}{2}}}+\dots+\frac{n^2}{\{n^2+(n-1)^2\}^{\frac{3}{2}}}\right].$$

[C. S., 1901.]

General Theorems on Integration.

321. Various Propositions.

There are certain general propositions on integration, many of which are almost self-evident from the definition of integration or from geometrical considerations, the truth of some of which the student will have noticed for himself, but which require to be definitely stated. It will be assumed that all functions occurring in the following theorems are finite and continuous between the limits ascribed, unless the contrary be specified:

322. I. $$\int_a^b \phi(x)\,dx = \int_a^b \phi(z)\,dz,$$

for if $\psi(x)$ be such that

$$\phi(x) = \frac{d}{dx}\psi(x),$$

and therefore such that

$$\phi(z) = \frac{d}{dz}\psi(z),$$

each integral is equal to $\psi(b) - \psi(a)$.

In other words, the result being necessarily eventually independent of x or z, it is plainly immaterial whether the letter x or the letter z is used in the process of obtaining the indefinite integral previous to the substitution of the limits.

323. II. $$\int_a^b \phi(x)\,dx = \int_a^c \phi(x)\,dx + \int_c^b \phi(x)\,dx.$$

For if $\psi(x)$ be the indefinite integral of $\phi(x)$,

the left side is $\psi(b) - \psi(a)$

and the right side is

$$\{\psi(c) - \psi(a)\} + \{\psi(b) - \psi(c)\},$$

which is the same thing.

Further, it is equally clear that

$$\int_a^b \phi(x)\,dx = \int_a^c \phi(x)\,dx + \int_c^d \phi(x)\,dx + \int_d^e \phi(x)\,dx + \ldots + \int_k^b \phi(x)\,dx,$$

where $c, d, e, f, \ldots k$ are any real quantities which lie in the region from a to b for which $\phi(x)$ has been assumed to be finite and continuous.

Let us illustrate the fact geometrically.

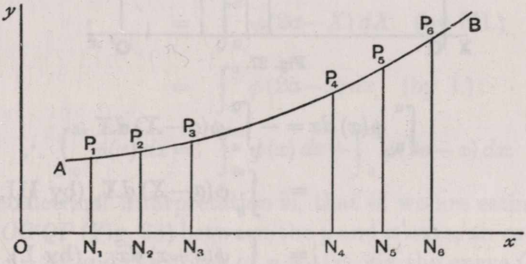

Fig. 26.

Let the curve drawn be the graph of $y=\phi(x)$, and let the equations of the ordinates

$$N_1P_1,\ N_2P_2,\ N_3P_3,\ \dots\ N_5P_5,\ N_6P_6$$

be $$x=a,\ x=c,\ x=d,\ \dots\ x=k,\ x=b$$

respectively.

Then the above theorem in integration expresses the obvious fact that

$$\text{Area } N_1N_6P_6P_1 = \text{Area } N_1N_2P_2P_1 + \text{Area } N_2N_3P_3P_2 + \dots + \text{Area } N_5N_6P_6P_5.$$

324. III. $$\int_a^b \phi(x)\,dx = -\int_b^a \phi(x)\,dx.$$

For, with the same notation as before,

the left side is $\psi(b)-\psi(a)$

and the right side is $-\{\psi(a)-\psi(b)\}$.

An interchange of the limits, therefore, changes the sign of the integral.

325. IV. $$\int_0^a \phi(x)\,dx = \int_0^a \phi(a-x)\,dx.$$

For if we put $x=a-X$, we have $dx=-dX$; and

if $x=a$, $X=0$;

if $x=0$, $X=a$.

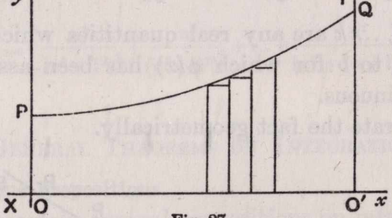

Fig. 27.

Hence $$\int_0^a \phi(x)\,dx = -\int_a^0 \phi(a-X)\,dX$$

$$= \int_0^a \phi(a-X)\,dX, \text{ (by III.)},$$

$$= \int_0^a \phi(a-x)\,dx, \quad \text{(by I.)}.$$

Geometrically this expresses the obvious fact that, in estimating the area $OO'QP$ (Fig. 27) between the y and x-axes, an ordinate $O'Q$, and the curve PQ, which is the graph of $y=\phi(x)$, we *may if we like take our origin at* O', $O'Q$ *as our* Y*-axis and* $O'X$ *as our* X*-axis*, as it cannot affect the result, whether the elements of area are added up from left to right, or from right to left.

326. V. $\displaystyle\int_0^{2a}\phi(x)\,dx=\int_0^a\phi(x)\,dx+\int_0^a\phi(2a-x)\,dx.$

For, by II.,

$$\int_0^{2a}\phi(x)\,dx=\int_0^a\phi(x)\,dx+\int_a^{2a}\phi(x)\,dx,$$

and if in the second term we put $x=2a-X$, we have $dx=-dX$, and

$$\text{when } x=a,\quad X=a;$$
$$\text{when } x=2a,\ X=0.$$

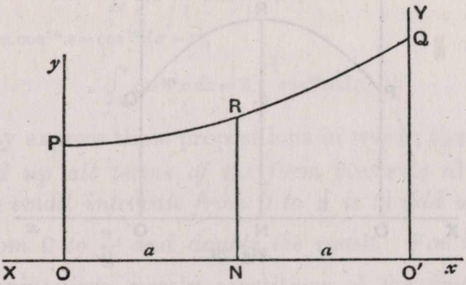

Fig. 28.

Thus the second integral on the right side, viz.

$$\begin{aligned}\int_a^{2a}\phi(x)\,dx&=-\int_a^0\phi(2a-X)\,dX\\&=\int_0^a\phi(2a-X)\,dX\quad\text{(by III.)}\\&=\int_0^a\phi(2a-x)\,dx\quad\text{(by I.)};\end{aligned}$$

$$\therefore\ \int_0^{2a}\phi(x)\,dx=\int_0^a\phi(x)\,dx+\int_0^a\phi(2a-x)\,dx.$$

The geometrical interpretation is, that if we are estimating the area $OO'QP$ (Fig. 28) between the y and x axes, an ordinate $O'Q$, viz. $x=2a$, and the graph of $y=\phi(x)$, viz. the curve QP, we may if we like take Ox and Oy for our axes for the portion $ONRP$, NR being the mid-ordinate, and $O'X$, $O'Y$ for axes in the second portion, thus finding each part separately, and then adding together, a fact obviously true.

327. VI. Plainly, if $\phi(x)$ be such that

$$\phi(2a-x)=\phi(x),$$

this proposition takes the form

$$\int_0^{2a}\phi(x)\,dx=2\int_0^a\phi(x)\,dx\,;$$

and if $\phi(x)$ be such that

$$\phi(2a-x)=-\phi(x),$$

$$\int_0^{2a}\phi(x)\,dx=0.$$

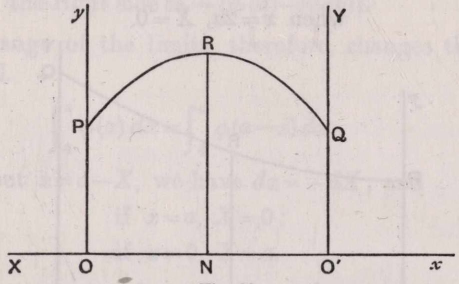

Fig. 29.

In the first case there is symmetry about the mid-ordinate NR (Fig. 29), and the whole area $OO'QRP$ in such a case is double that of $ONRP$.

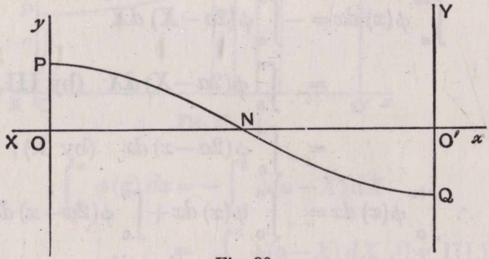

Fig. 30.

In the second case $\phi(a)=-\phi(a)$, *i.e.* $\phi(a)=0$, and the curve cuts the x-axis at N (Fig. 30), viz. where $x=a$, and though the regions ONP, $O'NQ$ are equal in absolute area, the second integral of Art. 326, viz. $\int_0^a \phi(2a-x)dx$, which is referred to $O'X$ and $O'Y$ as axes, represents (—the area $O'NQ$), for all the ordinates are affected by a negative sign.

Hence, the algebraic sum of the two is zero, the one cancelling the other.

There is now symmetry about the point N.

328. This principle is very useful in the integrals of the trigonometric or of any periodic functions.

Thus, since $\sin^n x = \sin^n(\pi - x)$,

$$\int_0^\pi \sin^n x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^n x\,dx.$$

And since $\cos^{2n+1}x = -\cos^{2n+1}(\pi - x)$,

$$\int_0^\pi \cos^{2n+1}x\,dx = 0;$$

so also since $\cos^{2n}x = \cos^{2n}(\pi - x)$,

$$\int_0^\pi \cos^{2n}x\,dx = 2\int_0^{\frac{\pi}{2}} \cos^{2n}x\,dx.$$

We may express these propositions in words, thus:

To add up all terms of the form $\sin^n x\,dx$ at equal indefinitely small intervals from 0 *to* π *is to add up all such terms from* 0 *to* $\frac{\pi}{2}$ *and double the result.* For the second quadrant sines are merely repetitions of the first quadrant sines in the reverse order.

Or geometrically, the curve $y = \sin^n x$ being symmetrical about the ordinate $x = \frac{\pi}{2}$, the whole area between the ordinates 0 and π is double that between 0 and $\frac{\pi}{2}$.

Similarly, the second quadrant cosines are repetitions of the first quadrant cosines with opposite signs, and therefore a term of form $\cos^{2n+1}x\,dx$ in the first quadrant is cancelled by the corresponding term in the second quadrant, but a term $\cos^{2n}x\,dx$, the index being now even, is duplicated by the corresponding term in the second quadrant.

Similar remarks and geometrical illustrations apply to other cases and for wider limits of integration.

Thus $$\int_0^{2\pi} \sin^{2n+1} x\, dx = 0,$$
for the third and fourth quadrant elements cancel those from the first and second.

$$\int_0^{2\pi} \sin^{2n} x\, dx = 4\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx,$$

$$\int_0^{2\pi} \cos^{2n+1} x\, dx = 0,$$

$$\int_0^{2\pi} \cos^{2n} x\, dx = 4\int_0^{\frac{\pi}{2}} \cos^{2n} x\, dx,$$

and so on.

329. VII. A Periodic Function.

If $$\phi(x) = \phi(a+x),$$
$$\int_0^{na} \phi(x)\, dx = n\int_0^a \phi(x)\, dx.$$

For, drawing the graph of $y=\phi(x)$, it is clear that it consists of an infinite series of repetitions of the part lying between the ordinates OP_0, $(x=0)$, and N_1P_1, $(x=a)$, (Fig. 31), for
$$\phi(x) = \phi(x+a),$$
and therefore writing $x+a$ for x,
$$\phi(x+a) = \phi(x+2a) = \phi(x+3a) = \text{etc.}$$
Also the areas bounded by the successive portions of the curve, the corresponding ordinates and the x-axis are all equal.

Thus $$\int_0^a \phi(x)dx = \int_a^{2a} \phi(x)dx = \int_{2a}^{3a} \phi(x)dx = \text{etc.}$$

and $$\int_0^{na} \phi(x)dx = \int_0^a \phi(x)dx + \int_a^{2a} \phi(x)dx + \dots + \int_{(n-1)a}^{na} \phi(x)\, dx$$
$$= n\int_0^a \phi(x)dx.$$

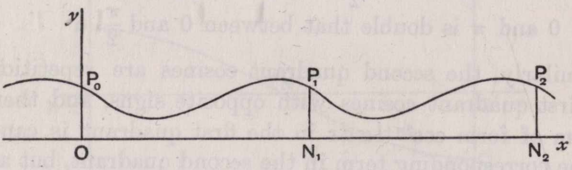

Fig. 31.

Thus, for instance, since $\sin^{2n} x = \sin^{2n}(\pi + x)$,

$$\int_0^{4\pi} \sin^{2n} x\, dx = 4\int_0^{\pi} \sin^{2n} x\, dx = 8\int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx = 8\,\frac{2n-1}{2n}\,\frac{2n-3}{2n-2}\dots\frac{1}{2}\cdot\frac{\pi}{2}.$$

330. VIII. **Arbitrary Change of the Limits.**

In estimating $\int_a^b \phi(x)dx$, the limits may be altered arbitrarily to p, q, provided x be transformed linearly in a suitable manner.

Take $x=A+B\xi$. Let A and B be chosen so that

$$\left.\begin{array}{l} a=A+Bp, \\ b=A+Bq, \end{array}\right\} \text{ whence } A=\frac{aq-bp}{q-p}, \quad B=\frac{b-a}{q-p},$$

i.e. $$x=\frac{aq-bp}{q-p}+\frac{b-a}{q-p}\xi$$

and $$dx=\frac{b-a}{q-p}d\xi.$$

Then $$\int_a^b \phi(x)dx=\frac{b-a}{q-p}\int_p^q \phi\left(\frac{aq-bp}{q-p}+\frac{b-a}{q-p}\xi\right)d\xi$$
$$=\frac{b-a}{q-p}\int_p^q \phi\left(\frac{aq-bp}{q-p}+\frac{b-a}{q-p}x\right)dx \quad \text{(by I.)}.$$

The geometrical significance of this is that instead of finding the area of $y=\phi(x)$ from $x=a$ to $x=b$, we may find the area of

$$\eta=\frac{b-a}{q-p}\phi\left(\frac{aq-bp}{q-p}+\frac{b-a}{q-p}\xi\right)$$

from $\xi=p$ to $\xi=q$.

Let the two graphs be drawn (Fig. 32), and let AA', BB', two ordinates, viz. $x=a$, $x=b$ in the one, correspond to PP',

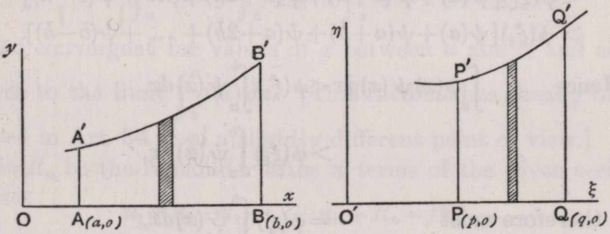

Fig. 32.

QQ', two ordinates, viz. $\xi=p$, $\xi=q$ in the other; then each element of the distance AB is reduced to a corresponding element of PQ in the ratio $\frac{q-p}{b-a}$, whilst there is a transference of the origin a distance $\dfrac{aq-bp}{q-p}$ in the positive direction of the x-axis if this quantity be positive, or in the opposite direction if negative. This alteration in the graph leaves the number of units of area in the portion of the graph considered unaltered, the effect being merely that of drawing the graph on a different scale, the ordinates being altered in the ratio $\dfrac{b-a}{q-p}$, whilst the breadths of the elementary strips are altered in the inverse ratio, leaving the areas unchanged.

331. IX. If $\phi(x)$, $\psi(x)$ be single-valued continuous and finite functions of x, of which the latter retains the same sign between a and b, then

$$\int_a^b \phi(x)\,\psi(x)\,dx = \phi(\xi)\int_a^b \psi(x)\,dx,$$

where $$a<\xi<b.$$

For $\int_a^b \phi(x)\psi(x)\,dx$, by the definition of an integral (Art. 11),

$$=Lt_{h=0}\,h\,[\phi(a)\,\psi(a)+\phi(a+h)\,\psi(a+h)+\phi(a+2h)\,\psi(a+2h)+\dots$$
$$+\phi(b-h)\,\psi(b-h)].$$

Now, of all the expressions

$$\phi(a),\quad \phi(a+h),\quad \phi(a+2h),\ \dots\ \phi(b-h),$$

let $\phi(\xi_1)$ be the greatest and $\phi(\xi_2)$ the least.

Then $\phi(a)\,\psi(a)+\phi(a+h)\,\psi(a+h)+\dots+\phi(b-h)\,\psi(b-h)$

$$<\phi(\xi_1)[\psi(a)+\psi(a+h)+\psi(a+2h)+\dots+\psi(b-h)]$$

and $$>\phi(\xi_2)[\psi(a)+\psi(a+h)+\psi(a+2h)+\dots+\psi(b-h)].$$

Hence $$\int_a^b \phi(x)\,\psi(x)\,dx<\phi(\xi_1)\int_a^b \psi(x)\,dx$$

and $$>\phi(\xi_2)\int_a^b \psi(x)\,dx,$$

and therefore must $$=\phi(\xi)\int_a^b \psi(x)\,dx,$$

where $\phi(\xi)$ is intermediate between $\phi(\xi_1)$ and $\phi(\xi_2)$. And ξ is a value of x somewhere between a and b.

It has been assumed that $\psi(x)$ is positive for the range from a to b. If $\psi(x)$ be negative throughout, the order of the inequalities is reversed, but the final result remains the same.

332. Cor. I. As a case of this theorem write $\phi'(x)$ for $\phi(x)$, and 1 for $\psi(x)$.

Then $$\int_a^b \phi'(x)\,dx = \phi'(\xi)\int_a^b 1\,dx = (b-a)\,\phi'(\xi),$$

i.e. $$\phi(b)-\phi(a)=(b-a)\,\phi'(\xi);$$

or putting $b=a+h$ and $\xi=a+\theta h$, where θ is a positive proper fraction,

$$\phi(a+h)=\phi(a)+h\phi'(a+\theta h),$$

subject to the condition that $\phi(x)$ and $\phi'(x)$ are finite and continuous functions of x for the whole range of values of x from a to $a+h$. [See *Diff. Calc.*, Art. 139.]

333. Cor. II. If $\phi(x)$ has a finite value for all values of x, $a<x<b$, it follows that $I\equiv\int_a^b \phi(x)\,dx$ is finite if a and b are finite, for if $\phi(\xi_1)$ be the greatest and $\phi(\xi_2)$ the least of the values of $\phi(x)$, I lies between $\phi(\xi_1)(b-a)$ and $\phi(\xi_2)(b-a)$, and is therefore finite.

334. Cor. III. If $u_1, u_2, u_3, \ldots$ be all single-valued functions of x, finite and continuous for all values of x between a and b, and if the series $u_1+u_2+u_3+u_4+\ldots$ to an infinite number of terms be uniformly and unconditionally convergent for all values of x between these limits, and $f(x)$ the limit towards which it converges, then the series

$$\int_a^x u_1\,dx+\int_a^x u_2\,dx+\int_a^x u_3\,dx+\ldots$$

is also convergent for values of x between a and b, and converges to the limit $\int_a^x f(x)\,dx$. [This theorem has already been proved in Art. 34 from a slightly different point of view.]

Let R_n be the remainder after n terms of the given series, so that

$$u_1+u_2+u_3+\ldots+u_n+R_n=f(x).$$

Then

$$\int_a^x u_1\,dx+\int_a^x u_2\,dx+\int_a^x u_3\,dx+\ldots+\int_a^x R_n\,dx=\int_a^x f(x)\,dx.$$

Now, by supposition, R_n is finite. Let R_n' and R_n'' be the greatest and least values of R_n as x changes continuously from a to b.

Then $\int_a^x R_n\,dx$ lies between $R'_n(x-a)$ and $R''_n(x-a)$.

Moreover, R_n vanishes by hypothesis when n is indefinitely increased, whence R'_n and R''_n also vanish in the limit;

$\therefore \int_a^x R_n\,dx$ vanishes in the limit.

Hence $$\int_a^x u_1\,dx+\int_a^x u_2\,dx+\int_a^x u_3\,dx+\dots$$

converges to the limit $\int_a^x f(x)\,dx$.

[SERRET, *Calcul Intég.*, p. 108.]

335. Cor. IV. If a continuous function $f(x)$ can be expanded in a series of powers of x convergent for values of x between 0 and a,

say, $$A_0+A_1x+A_2x^2+\dots,$$

then $$A_0x+A_1\frac{x^2}{2}+A_2\frac{x^3}{3}+\dots$$

is also a continuous and convergent series tending to the limit

$$\int_0^x f(x)\,dx. \qquad \text{[Cf. Art. 34.]}$$

336. Cor. V.

$$\int_0^x f(x)\,dx=\int_0^x \left[f(0)+xf'(0)+\frac{x^2}{\lfloor 2}f''(0)+\dots\right]dx$$

$$=xf(0)+\frac{x^2}{\lfloor 2}f'(0)+\frac{x^3}{\lfloor 3}f''(0)+\dots,$$

convergent between the same limits for which Maclaurin's series, which has been used, is convergent.

This gives a means of expressing an integration by means of a series.

337. Lemma. A Theorem due to Abel. If S_r be the sum of the first r terms, and S'_r the sum of the last r terms of the series

$$u_1+u_2+u_3+\dots+u_r+\dots+u_n,$$

each term being real and finite, but not necessarily all of the same sign, and if

Σ and σ be the greatest and least values of S_r,

and Σ' and σ' be the greatest and least values of S'_r, and if $a_1, a_2, a_3, \dots a_n$ be n positive finite quantities arranged in descending order of magnitude, and if

$$S = a_1u_1 + a_2u_2 + a_3u_3 + \dots + a_nu_n,$$

then we shall have $\quad a_1\Sigma > S > a_1\sigma;$

and if $a_1, a_2, a_3, \dots a_n$ be arranged in ascending order of magnitude, then

$$a_n\Sigma' > S > a_n\sigma'.$$

For

$$\begin{aligned} S &= a_1u_1 + a_2u_2 + a_3u_3 + \dots + a_nu_n \\ &= a_1(S_1) + a_2(S_2 - S_1) + a_3(S_3 - S_2) \\ &\qquad + \dots + a_{n-1}(S_{n-1} - S_{n-2}) + a_n(S_n - S_{n-1}) \\ &= S_1(a_1 - a_2) + S_2(a_2 - a_3) + S_3(a_3 - a_4) \\ &\qquad + \dots + S_{n-1}(a_{n-1} - a_n) + S_na_n, \end{aligned}$$

and $a_1 - a_2,\ a_2 - a_3, \dots a_{n-1} - a_n,\ a_n$ are all positive quantities;

$$\therefore\ S < \Sigma[(a_1-a_2)+(a_2-a_3)+(a_3-a_4)+\dots+(a_{n-1}-a_n)+a_n]$$

and $\quad > \sigma[(a_1-a_2)+(a_2-a_3)+(a_3-a_4)+\dots+(a_{n-1}-a_n)+a_n],$

i.e. $\quad S < a_1\Sigma$ and $S > a_1\sigma$, *i.e.* $a_1\Sigma > S > a_1\sigma$.

In the same way, writing the series from the other end, and if $a_n, a_{n-1}, a_{n-2}, \dots a_1$ be in descending order of magnitude,

$$a_n\Sigma' > S > a_n\sigma'.$$

This theorem in inequalities is due to ABEL.

We note also that if $a_1, a_2, a_3, \dots a_n$ were all negative, the same theorems would still hold, except that the inequalities would have been reversed, viz.

$$a_1\Sigma < S < a_1\sigma \quad \text{and} \quad a_n\Sigma' < S < a_n\sigma'.$$

338. X. Applying Abel's inequality theorem to the case of the integral

$$\int_a^b \phi(x)\,\psi(x)\,dx,$$

where $\phi(x)$ and $\psi(x)$ are finite and continuous functions of x for all values of x between the limits a and b, and $\phi(x)$ positive and continually decreasing throughout that range, and writing

$$\phi(a),\ \phi(a+h),\ \phi(a+2h),\ \dots\ \phi(b-h)$$

respectively for $\quad a_1, \quad a_2, \quad a_3, \quad \dots \quad a_n,$

and $\quad h\psi(a),\ h\psi(a+h),\ h\psi(a+2h),\ \dots\ h\psi(b-h)$

for $\quad u_1, \quad u_2, \quad u_3, \quad \dots \quad u_n,$

and taking the limit when h is indefinitely small, we have

$$S=\int_a^b \phi(x)\psi(x)\,dx,$$

$$a_1\Sigma=\phi(a)\int_a^{\xi_1}\psi(x)\,dx,$$

$$a_1\sigma=\phi(a)\int_a^{\xi_2}\psi(x)\,dx,$$

where ξ_1, ξ_2 are the limits corresponding to the greatest and least values of $\int_a^\xi \psi(x)\,dx$ for different values of ξ between a and b;

$$\therefore\ \phi(a)\int_a^{\xi_1}\psi(x)\,dx>\int_a^b\phi(x)\psi(x)\,dx>\phi(a)\int_a^{\xi_2}\psi(x)\,dx,$$

and therefore $\int_a^b\phi(x)\psi(x)\,dx=\phi(a)\int_a^\xi\psi(x)\,dx$

for some value of ξ intermediate between a and b.

Similarly, if $\phi(x)$ be a continually increasing function,

$$\phi(b)\int_{\xi_1'}^b\psi(x)\,dx>\int_a^b\phi(x)\psi(x)\,dx>\phi(b)\int_{\xi_2'}^b\psi(x)\,dx,$$

where ξ_1', ξ_2' are the values of ξ which make $\int_\xi^b\psi(x)\,dx$ greatest or least, and therefore

$$\int_a^b\phi(x)\psi(x)\,dx=\phi(b)\int_\xi^b\phi(x)\,dx,$$

where ξ is intermediate between a and b.

339. From the last remark of Art. 337 it appears that the same theorem will be true when $\phi(x)$ is negative throughout. That is, that provided $\phi(x)$ be continually positive or continually negative from $x=a$ to $x=b$, and $\phi'(x)$ retains the same sign throughout this range,

$$\int_a^b\phi(x)\psi(x)\,dx=\phi(a)\int_a^\xi\psi(x)\,dx$$

or
$$\int_a^b\phi(x)\psi(x)\,dx=\phi(b)\int_\xi^b\psi(x)\,dx,$$

according as $\phi'(x)$ is negative or positive, where ξ is some value of x between a and b, *i.e.* $\xi=a+\theta(b-a)$, where θ is some positive proper fraction.

340. A Theorem due to Ossian Bonnet.

If $\phi'(x)$ be negative, *i.e.* $\phi(x)$ decreasing, but $\phi(x)$ changing sign in the interval from $x=a$ to $x=b$, and therefore $\phi(b)$ negative and $\phi(a)$ positive, write

$$\phi(x)-\phi(b)=\chi(x);$$

then $\chi'(x)$ is negative and $\chi(x)$ is positive from a to b.

$$\begin{aligned}\therefore\ \int_a^b \phi(x)\psi(x)\,dx &= \int_a^b [\phi(b)+\chi(x)]\psi(x)dx\\ &= \phi(b)\int_a^b \psi(x)\,dx + \chi(a)\int_a^\xi \psi(x)\,dx\\ &= [\phi(b)\int_a^\xi \psi(x)\,dx + \phi(b)\int_\xi^b \psi(x)\,dx] + \chi(a)\int_a^\xi \psi(x)\,dx\\ &= [\phi(b)+\chi(a)]\int_a^\xi \psi(x)\,dx + \phi(b)\int_\xi^b \psi(x)\,dx\\ &= \phi(a)\int_a^\xi \psi(x)\,dx + \phi(b)\int_\xi^b \psi(x)\,dx.\end{aligned}$$

341. Finally, if $\phi'(x)$ be positive, *i.e.* $\phi(x)$ increasing, but changing sign in the interval between a and b, and therefore $\phi(a)$ negative and $\phi(b)$ positive, write

$$\phi(x)-\phi(a)=\chi(x);$$

then $\chi'(x)$ is positive and $\chi(x)$ is positive from a to b.

$$\begin{aligned}\therefore\ \int_a^b \phi(x)\psi(x)dx &\\ &= \int_a^b [\phi(a)+\chi(x)]\psi(x)dx\\ &= \phi(a)\int_a^b \psi(x)dx + \chi(b)\int_\xi^b \psi(x)dx\\ &= \phi(a)\left[\int_a^\xi \psi(x)dx + \int_\xi^b \psi(x)dx\right] + \chi(b)\int_\xi^b \psi(x)dx\\ &= \phi(a)\int_a^\xi \psi(x)dx + [\phi(a)+\chi(b)]\int_\xi^b \psi(x)dx\\ &= \phi(a)\int_a^\xi \psi(x)dx + \phi(b)\int_\xi^b \psi(x)dx.\end{aligned}$$

Hence, in all cases where the differential coefficient of $\phi(x)$ is a continuous function, retaining one sign between the limits, though $\phi(x)$ itself may change sign,

$$\int_a^b \phi(x)\psi(x)dx = \phi(a)\int_a^\xi \psi(x)dx + \phi(b)\int_\xi^b \psi(x)dx$$

for some value of ξ intermediate between a and b, ϕ and ψ being finite and continuous throughout.

This theorem is due to OSSIAN BONNET.

342. XI. (i) Since

$$(a_1^2+a_2^2+a_3^2+\ldots+a_n^2)(b_1^2+b_2^2+\ldots+b_n^2)h^2 \\ \nless (a_1b_1+a_2b_2+\ldots+a_nb_n)^2h^2,$$

we have upon putting

$$a_1=\phi(a),\quad a_2=\phi(a+h)\ldots a_n=\phi(b-h),$$
$$b_1=\psi(a),\quad b_2=\psi(a+h)\ldots b_n=\psi(b-h),$$

and taking the limit when h is indefinitely small,

$$\int_a^b[\phi(x)]^2dx\int_a^b[\psi(x)]^2dx \nless \left[\int_a^b\phi(x)\psi(x)dx\right]^2.$$

(ii) If $a_1,\ a_2,\ a_3,\ \ldots\ a_n,$

and $b_1,\ b_2,\ b_3,\ \ldots\ b_n,$

be two sets of positive quantities, both in descending or both in ascending order of magnitude,

$$\Sigma a_r\Sigma a_r^2b_r-\Sigma a_r^2\Sigma a_rb_r \nless 0$$

[for $\Sigma a_ra_s(a_r-a_s)(b_r-b_s)$ is positive].

And it follows as in (i) that if $\phi(x)$ and $\psi(x)$ be finite, continuous, and positive, and $\phi'(x)$ and $\psi'(x)$ be both positive or both negative from $x=a$ to $x=b$, then

$$\int_a^b\phi(x)dx\int_a^b[\phi(x)]^2\psi(x)dx \nless \int_a^b[\phi(x)]^2dx\int_a^b[\phi(x)][\psi(x)]dx.$$

If ϕ' and ψ' are of opposite signs the order of the inequality is reversed.

GENERAL AND PRINCIPAL VALUES OF AN INTEGRAL. CAUCHY.

343. XII. **The Definition of Integration. Modifications.**

In our summation definition of integration, as

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b-h)],$$

which has been denoted by

$$\int_a^b\phi(x)dx,$$

we have assumed

(1) $\phi(x)$ finite and continuous and single-valued for the whole range from $x=a$ to $x=b$.

(2) a and b to be both finite quantities.

This definition will fail when these conditions are not satisfied, and will require modification.

We have also (Art. 18) extended our notation so as to let $\int_a^\infty \phi(x)dx$ stand for the limit when b is indefinitely increased of $\psi(b)-\psi(a)$ where $\frac{d\psi(x)}{dx}=\phi(x)$, with a similar extension when the lower limit becomes infinitely large. The subject of integration itself, viz. $\phi(x)$, has been so far, however, in all cases, understood to be *finite*, single-valued, and continuous for the whole range of integration from a to b, whether that range be finite or infinite.

344. Infinities of the Integrand. General and Principal Values. Cauchy.

When $\phi(x)$ becomes infinite between the limits of integration, say at the point $x=c$, where $a<c<b$, and nowhere else between a and b, our definition holds

$$\text{from } x=a \quad \text{to } x=c-\epsilon$$

and

$$\text{from } x=c+\eta \text{ to } x=b,$$

where ϵ and η are two positive quantities which may be taken as small as we please.

The integral $\int_a^b \phi(x)dx$ is now to be understood as meaning

$$Lt_{\substack{\epsilon=0\\ \eta=0}}\left[\int_a^{c-\epsilon} \phi(x)dx+\int_{c+\eta}^b \phi(x)dx\right].$$

This limit may be finite, infinite, or of undetermined value.

It is called the General Value of the Integral.

When $\eta=\epsilon$, Cauchy has named the limiting form derived, the Principal Value of the Integral, viz.

$$Lt_{\epsilon=0}\left[\int_a^{c-\epsilon} \phi(x)dx+\int_{c+\epsilon}^b \phi(x)dx\right],$$

which may be finite or infinite.

A similar modification of the original definition will obviously be necessary when the subject of integration, viz. $\phi(x)$, attains an infinite value more than once between the extreme limits of the integration, viz. between a and b.

If the infinity of $\phi(x)$ occurs *at* one of the limits, say at the upper one, then the integral $\int_a^b \phi(x)\,dx$ is to be understood to mean

$$Lt_{\epsilon=0}\int_a^{b-\epsilon} \phi(x)\,dx.$$

Again when the upper *limit* is infinite we shall understand $\int_a^\infty \phi(x)\,dx$ to mean

$$Lt_{\epsilon=0}\int_a^{\frac{1}{\epsilon}} \phi(x)\,dx$$

and when the lower *limit* is infinite we shall understand $\int_\infty^b \phi(x)\,dx$ to mean

$$Lt_{\epsilon=0}\int_{\frac{1}{\epsilon}}^{b} \phi(x)\,dx.$$

When the integration is from $-\infty$ to $+\infty$ we shall consider the integration $\int_{-\infty}^{\infty} \phi(x)\,dx$ to mean

$$\mathop{Lt}_{\substack{\epsilon=0\\ \eta=0}}\int_{-\frac{1}{\eta}}^{+\frac{1}{\epsilon}} \phi(x)\,dx,$$

which we shall refer to as its **General** value; *i.e.*

$$\mathop{Lt}_{\substack{\epsilon=0\\ \eta=0}}\left[\psi\left(\frac{1}{\epsilon}\right)-\psi\left(-\frac{1}{\eta}\right)\right], \quad \text{where } \frac{d\psi}{dx}=\phi(x),$$

ϵ and η being small positive quantities independent of each other; and when $\eta=\epsilon$ we shall refer to

$$Lt_{\epsilon=0}\int_{-\frac{1}{\epsilon}}^{\frac{1}{\epsilon}} \phi(x)\,dx$$

as its **Principal** value; *i.e.*

$$Lt_{\epsilon=0}\left[\psi\left(\frac{1}{\epsilon}\right)-\psi\left(-\frac{1}{\epsilon}\right)\right].$$

345. **Geometrical Illustrations.**

Let a graph be drawn of $y=\phi(x)$, and let $OA=a$, $OC=c$, $OB=b$. Then at C $(x=c)$ there is an asymptote parallel to the y-axis. The graph may be such as to approach the asymptote from opposite sides at the same extremity (Fig. 33), or from opposite sides at opposite extremities (Fig. 34). In the first case there is no change of sign of $\phi(x)$ as x passes leave the choice of these relative speeds till after integration, and thereby retain command of the mode in which the ordinates are made to close up.

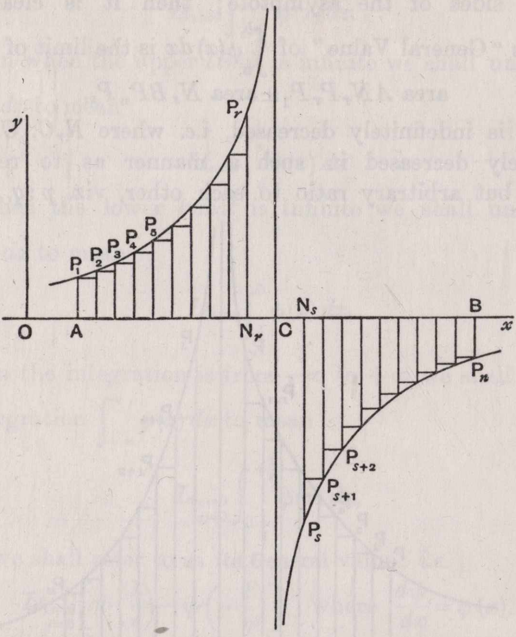

Fig. 34.

In understanding $\int_a^b \phi(x)\,dx$ to mean

$$Lt_{\substack{\epsilon=0\\ \eta=0}}\left[\int_a^{c-\epsilon}\phi(x)\,dx+\int_{c+\eta}^{b}\phi(x)\,dx\right],$$

where ϵ, η are two positive quantities, we can ultimately make $\frac{\epsilon}{p}=\frac{\eta}{q}$ in our investigations of the "General Value," and if we take $p=q$, that is $\epsilon=\eta$, we shall have Cauchy's "Principal Value."

346. When the inscribed and circumscribed rectangles are drawn in the Newtonian manner (Art. 11), the pairs in immediate contiguity with the asymptote are in area [Fig. 35]

$$\epsilon\phi(c-\epsilon),\ \epsilon\phi(c)\quad\text{and}\quad\eta\phi(c+\eta),\ \eta\phi(c).$$

The circumscribed rectangles are numerically greater than the inscribed ones. They are of infinite length $\phi(c)$, and of infinitesimal breadths ϵ and η respectively (Fig. 35).

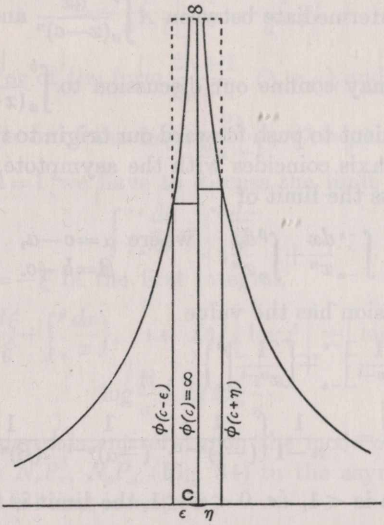

Fig. 35.

These areas then are "undetermined" quantities until we know the nature of $\phi(c)$. If the orders of the infinitesimals ϵ, η be higher than the order of the infinity $\phi(c)$ their limits are zero. If of lower order their limits are infinite. But, in the latter case, if $\phi(x)$ change sign as x passes through the value c, we may be only concerned with the difference of these infinities, which may be finite.

347. If $\phi(x)$ becomes infinite at a point $x=c$, the general way in which it does so is by the vanishing of a factor in its denominator.

Let $\phi(x)=\dfrac{F(x)}{(x-c)^n}$, where $F(x)$ contains no factor $x-c$, and therefore retains the same sign as x increases through the value c, and n is positive.

We are only concerned to discuss the behaviour of this function in the immediate vicinity of the asymptote.

Therefore we may take our limits a, b so near to $x=c$ that $F(x)$ retains the same sign throughout, and if A and B are the greatest and least values of $F(x)$ in this interval,

$\int_a^b \phi(x)\,dx$ is intermediate between $A\int_a^b \frac{dx}{(x-c)^n}$ and $B\int_a^b \frac{dx}{(x-c)^n}$.

Hence we may confine our discussion to $\int_a^b \frac{dx}{(x-c)^n}$. And it will be convenient to push forward our origin to the point $(c, 0)$, so that the y-axis coincides with the asymptote, and we then have to discuss the limit of

$$\int_{-\alpha}^{-\epsilon}\frac{dx}{x^n}+\int_{\eta}^{\beta}\frac{dx}{x^n}, \quad \text{where } \alpha=c-a, \quad \beta=b-c.$$

This expression has the value

$$-\frac{1}{n-1}\left\{\left[\frac{1}{x^{n-1}}\right]_{-\alpha}^{-\epsilon}+\left[\frac{1}{x^{n-1}}\right]_{\eta}^{\beta}\right\}$$

$$=-\frac{1}{n-1}\left\{\frac{1}{(-\epsilon)^{n-1}}-\frac{1}{(-\alpha)^{n-1}}+\frac{1}{(\beta)^{n-1}}-\frac{1}{(\eta)^{n-1}}\right\}.$$

(a) When n is <1, *i.e.* $0<n<1$, the limit is finite, viz.

$$-\frac{1}{n-1}[-(-\alpha)^{1-n}+\beta^{1-n}],$$

and is independent of the limiting value of $\frac{\epsilon}{\eta}$. This is then both the "General Value" and the "Principal Value."

The first and last elements in the summations, viz. $\epsilon\cdot\frac{1}{\epsilon^n}$ and $\eta\cdot\frac{1}{\eta^n}$, being respectively ϵ^{1-n} and η^{1-n} $(n<1)$ vanish *independently of each other*.

(b) If $n>1$, the limit to be discussed is that of

$$-\frac{1}{n-1}\left\{\frac{1}{(-\epsilon)^{n-1}}-\frac{1}{(-\alpha)^{n-1}}+\frac{1}{\beta^{n-1}}-\frac{1}{\eta^{n-1}}\right\},$$

which is infinite in general, when ϵ and η diminish independently and ultimately vanish in any arbitrary ratio of inequality. Hence the "General Value" is infinite.

But when n is odd or of the form $\frac{2\lambda+1}{2\mu+1}$, ($\lambda$ and μ being integers and $\lambda > \mu$), the infinities will cancel each other when ϵ, η ultimately vanish in a ratio of equality.

The Principal Value is therefore finite, and

$$= -\frac{1}{n-1}\left[\frac{1}{\beta^{n-1}} - \frac{1}{\alpha^{n-1}}\right],$$

when n is odd or of the form $\frac{2\lambda+1}{2\mu+1}$, $(\lambda > \mu)$, and infinite if n is an even integer or of the form $\frac{2\lambda}{2\mu+1}$, $(\lambda > \mu)$.

(*c*) When $n=1$ we have to discuss the limit of

$$\int_{-\alpha}^{-\epsilon} \frac{dx}{x} + \int_{\eta}^{\beta} \frac{dx}{x},$$

or putting $x = -\xi$ in the first integral,

$$Lt\left\{\int_{\alpha}^{\epsilon} \frac{d\xi}{\xi} + \int_{\eta}^{\beta} \frac{dx}{x}\right\}, \quad i.e.\ Lt\left\{\Big[\log \xi\Big]_{\alpha}^{\epsilon} + \Big[\log x\Big]_{\eta}^{\beta}\right\};$$

i.e. $$\log \frac{\beta}{\alpha} + Lt \log \frac{\epsilon}{\eta}.$$

This limit depends entirely upon the mode of approach of the ordinates N_rP_r, N_sP_s (Fig. 34) to the asymptote, and is undetermined till that is settled.

When $\frac{\epsilon}{p} = \frac{\eta}{q}$, where p, q are any finite quantities to be chosen, the limit is $\log \frac{\beta}{\alpha} + \log \frac{p}{q}$, and is arbitrary, depending upon the choice of p and q.

When p and q have been chosen equal, that is when ϵ, η vanish in a ratio of equality, the limit becomes $\log \frac{\beta}{\alpha}$.

Hence the General Value is an arbitrary quantity; the Principal Value is $\log \frac{\beta}{\alpha}$.

If n be of the form $\frac{2\lambda+1}{2\mu}$, $\frac{1}{x^n}$ becomes unreal when x is negative and the first integral is unreal, from $-\alpha$ to $-\epsilon$. Excluding this we are then only concerned with

$$Lt_{\eta=0}\int_{\eta}^{\beta} \frac{dx}{x^n}, \quad i.e.\ -\frac{1}{n-1}\, Lt\left[\frac{1}{x^{n-1}}\right]_{\eta}^{\beta},$$

or $$-\frac{1}{n-1}\, Lt\left[\frac{1}{\beta^{n-1}} - \frac{1}{\eta^{n-1}}\right],$$

integers and $\lambda > \mu$), the infinities will cancel each other when ϵ, η ultimately vanish in a ratio of equality.

The Principal Value is therefore finite, and

$$=-\frac{1}{n-1}\left[\frac{1}{\beta^{n-1}}-\frac{1}{a^{n-1}}\right],$$

when n is odd or of the form $\frac{2\lambda+1}{2\mu+1}$, $(\lambda > \mu)$, and infinite if n is an even integer or of the form $\frac{2\lambda}{2\mu+1}$, $(\lambda > \mu)$.

(c) When $n=1$ we have to discuss the limit of

$$\int_{-a}^{-\epsilon}\frac{dx}{x}+\int_{\eta}^{\beta}\frac{dx}{x},$$

or putting $x=-\xi$ in the first integral,

$$Lt\left\{\int_{a}^{\epsilon}\frac{d\xi}{\xi}+\int_{\eta}^{\beta}\frac{dx}{x}\right\},\ i.e.\ Lt\left\{\Big[\log\xi\Big]_{a}^{\epsilon}+\Big[\log x\Big]_{\eta}^{\beta}\right\};$$

i.e.
$$\log\frac{\beta}{a}+Lt\log\frac{\epsilon}{\eta}.$$

This limit depends entirely upon the mode of approach of the ordinates N_rP_r, N_sP_s (Fig. 34) to the asymptote, and is undetermined till that is settled.

When $\frac{\epsilon}{p}=\frac{\eta}{q}$, where p, q are any finite quantities to be chosen, the limit is $\log\frac{\beta}{a}+\log\frac{p}{q}$, and is arbitrary, depending upon the choice of p and q.

When p and q have been chosen equal, that is when ϵ, η vanish in a ratio of equality, the limit becomes $\log\frac{\beta}{a}$.

Hence the General Value is an arbitrary quantity; the Principal Value is $\log\frac{\beta}{a}$.

If n be of the form $\frac{2\lambda+1}{2\mu}$, $\frac{1}{x^n}$ becomes unreal when x is negative and the first integral is unreal, from $-a$ to $-\epsilon$. Excluding this we are then only concerned with

$$Lt_{\eta=0}\int_{\eta}^{\beta}\frac{dx}{x^n},\quad i.e.\ -\frac{1}{n-1}Lt\left[\frac{1}{x^{n-1}}\right]_{\eta}^{\beta},$$

or

$$-\frac{1}{n-1}Lt\left[\frac{1}{\beta^{n-1}}-\frac{1}{\eta^{n-1}}\right],$$

which is real and $=-\dfrac{1}{n-1}\dfrac{1}{\beta^{n-1}}$ if $n<1$, and infinite if $n>1$, and may be referred to as the Principal Value of the real part.

348. We next consider the case when the infinite value of $\phi(x)$ occurs *at* one of the limits, say b.

$\int_a^b \phi(x)\,dx$ is then to be interpreted as $Lt_{\epsilon=0}\int_a^{b-\epsilon}\phi(x)\,dx$, which is called the "Principal Value."

Let $\phi(x)=\dfrac{f(x)}{(x-b)^n}$, where $f(x)$ does not contain the factor $x-b$, and therefore does not vanish when $x=b$; and let n be positive. Then,

(*a*) if n be <1 and if we can find some quantity γ between a and b such that throughout the range of values of x from γ to b the numerical value of $f(x)$ does not exceed some finite quantity A, the Principal Value will be finite.

For
$$\int_a^{b-\epsilon}\phi(x)\,dx=\int_a^{\gamma}\phi(x)\,dx+\int_{\gamma}^{b-\epsilon}\phi(x)\,dx.$$

The first of these two integrals is finite, and in the limit the numerical value of the second is not greater than

$$Lt_{\epsilon=0}A\int_{\gamma}^{b-\epsilon}\frac{dx}{(x-b)^n};$$

moreover
$$\int_{\gamma}^{b-\epsilon}\frac{dx}{(x-b)^n}=\frac{1}{1-n}\Big[(x-b)^{1-n}\Big]_{\gamma}^{b-\epsilon}$$
$$=\frac{1}{1-n}[(-\epsilon)^{1-n}-(\gamma-b)^{1-n}],$$

the limit of which, when $\epsilon=0$, is $-\dfrac{1}{1-n}(\gamma-b)^{1-n}$ and therefore finite.

(*b*) If, however, $n>1$, and if we can find some quantity γ between a and b, such that throughout the range of values of x from γ to b the numerical value of $f(x)$ is greater than some finite quantity B throughout this range of values of x, and if $f(x)$ preserves the same sign throughout that range, the Principal Value of the integral will be infinite.

For, as before,

$$\int_a^{b-\epsilon}\phi(x)\,dx=\int_a^{\gamma}\phi(x)\,dx+\int_\gamma^{b-\epsilon}\phi(x)\,dx,$$

the first of the two integrals being finite.

But the numerical value of $Lt_{\epsilon=0}\int_\gamma^{b-\epsilon}\phi(x)\,dx$ is greater than the numerical value of

$$Lt_{\epsilon=0}B\int_\gamma^{b-\epsilon}\frac{dx}{(x-b)^n}$$

and $\displaystyle\int_\gamma^{b-\epsilon}\frac{dx}{(x-b)^n}=\frac{1}{n-1}\left[\frac{1}{(-\epsilon)^{n-1}}-\frac{1}{(\gamma-b)^{n-1}}\right],$

which becomes infinite when ϵ vanishes.

(*c*) Lastly, if $n=1$, and if, as in the last case (*b*), such a quantity γ can be found as there described, the numerical value of $Lt_{\epsilon=0}\int_\gamma^{b-\epsilon}\phi(x)\,dx$ is greater than the numerical value of

$$Lt_{\epsilon=0}B\int_\gamma^{b-\epsilon}\frac{dx}{x-b}, \quad\text{and}\quad \int_\gamma^{b-\epsilon}\frac{dx}{x-b}=\log\frac{\epsilon}{b-\gamma};$$

the numerical value of which is infinite, and therefore the Principal Value of $\int_a^b\phi(x)\,dx$ is in this case, also, infinite.

349. To sum up these Statements.*

If it be possible to find a quantity γ between a and b such that the numerical value of $\phi(x)(x-b)^n$, that is $f(x)$, does not exceed some finite quantity A throughout the range from γ to b, and if $n<1$, then the Principal Value of $\int_a^b\phi(x)\,dx$ is finite. If it be possible to find a quantity γ between a and b such that the numerical value of $\phi(x)(x-b)^n$ does exceed some finite quantity B throughout that range, and if $\phi(x)(x-b)^n$ does not change sign throughout that range, then if $n \nless 1$ the Principal Value of $\int_a^b\phi(x)\,dx$ will be infinite.

Obviously a similar rule holds for the lower limit by reversing the order of integration, *i.e.* interchanging the limits.

*Serret, *Calcul Intégral*, p. 100.

350. (α) Consider $$\int_0^1 \frac{dx}{\sqrt{1-x^2}}.$$

Here the subject of integration, viz. $\dfrac{1}{\sqrt{1-x^2}}$, is infinite at the upper limit.

We have to consider $$Lt_{\epsilon=0}\int_0^{1-\epsilon} \frac{dx}{\sqrt{1-x^2}}.$$

Let $\phi(x)=\dfrac{1}{\sqrt{1-x^2}}$. Then $\phi(x)\sqrt{1-x}=\dfrac{1}{\sqrt{1+x}}$, which is <1 for the whole range $0<x\nless 1$ or for any part of it, and the index of the factor $1-x$ is $\frac{1}{2}$, which is <1. Hence by Art. 348 (a) the Principal Value is finite.

It is of course obviously equal to $Lt_{\epsilon=0}\Big[\sin^{-1}x\Big]_0^{1-\epsilon}$,

i.e. $$Lt_{\epsilon=0}\{\sin^{-1}(1-\epsilon)-\sin^{-1}0\}=\sin^{-1}1-\sin^{-1}0=\frac{\pi}{2}.$$

(β) Consider $$\int_0^1 \frac{dx}{(1-x^2)^{\frac{3}{2}}}.$$

Here the subject of integration, viz. $\dfrac{1}{(1-x^2)^{\frac{3}{2}}}$, is infinite at the upper limit. Let $\phi(x)=\dfrac{1}{(1-x^2)^{\frac{3}{2}}}$. Then $\phi(x)(1-x)^{\frac{3}{2}}=\dfrac{1}{(1+x)^{\frac{3}{2}}}$, which is $\nless \dfrac{1}{2\sqrt{2}}$ and does not change sign for all values of x from $x=0$ to $x=1$ or for any part of that range. Also the index of the factor $1-x$ is $\frac{3}{2}$, *i.e.* >1. Hence, by Art. 348 (b), the Principal Value of this integral is ∞.

351. Consider $\int_0^1 \dfrac{\log x}{x^n}\,dx$, where $0<n<1$. (Serret, *C.I.*, p. 103.)

$$Lt_{x=0}\frac{\log x}{x^n}=\infty.$$

When x is made to approach zero indefinitely closely, the integrand, viz. $\phi(x)\equiv\log x/x^n$, increases numerically without limit. Take a quantity p lying between zero and $1-n$, so that p is positive and <1. Then $x^{p+n}\phi(x)\equiv x^p\log x$ has a turning point at $x=e^{-\frac{1}{p}}$, vanishes at $x=0$, and whilst numerically decreasing to zero as x diminishes from $e^{-\frac{1}{p}}$ to zero is always numerically less than $\dfrac{1}{pe}$. Moreover $p+n$ is a positive index less than 1.

Hence, by Art. 349, the Principal Value of this integral is finite.

352. Suppose that $\int_a^b f(x)\,dx$ has a value which is finite and determinate, when $f(x)$ becomes ∞ at $x=c$. ($a<c<b$.) Then this value must be

$$Lt_{\epsilon=0}\left\{\int_a^{c-p\epsilon} f(x)\,dx+\int_{c+q\epsilon}^b f(x)dx\right\}, \ldots\ldots\ldots\ldots\ldots\text{(A)}$$

whatever may be the ratio of $p:q$, and if this limit were not independent of $p:q$, this General Value would not be determinate.

The Principal Value is the case when $p=q=1$,

$$Lt_{\epsilon=0}\left[\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^{b} f(x)\,dx\right] \text{......................(B)}$$

The difference of these expressions A and B is

$$Lt_{\epsilon=0}\left[\int_{c-\epsilon}^{c-p\epsilon} f(x)\,dx + \int_{c+q\epsilon}^{c+\epsilon} f(x)\,dx\right],$$

and this limit must therefore vanish whatever the ratio $p:q$ may be if $\int_a^b f(x)\,dx$ is to have a finite and determinate value.

Cauchy* calls such integrals "Singular Definite" integrals [Intégrales définies singulières], viz. those in which the subject of integration becomes infinitely great at the same time that the limits differ by an infinitesimal.

In order that p and q shall disappear, the first integral must be independent of p, the second of q, when ϵ is indefinitely diminished.

For example, in the case

$$I=\int_a^b \frac{dx}{(x-c)^{\frac{1}{3}}}, \quad \text{where } a<c<b;$$

here

$$\int_{c-\epsilon}^{c-p\epsilon} \frac{dx}{(x-c)^{\frac{1}{3}}} = \frac{3}{2}\left[(x-c)^{\frac{2}{3}}\right]_{c-\epsilon}^{c-p\epsilon}$$

$$= \frac{3}{2}\left[(-p\epsilon)^{\frac{2}{3}} - (-\epsilon)^{\frac{2}{3}}\right],$$

and the limit when $\epsilon=0$ is zero and independent of p.

Similarly for $\int_{c+\epsilon}^{c+q\epsilon} \frac{dx}{(x-c)^{\frac{1}{3}}}$ the limit is independent of q, and the integral I is determinate.

See Williamson, *Int. Calc.*, pages 128-135; Moigno, *Calc. Intég.*; Serret, *Calc. Int.*, pages 91-107; Bertrand, *C.I.*, p. 117, for further information as to General and Principal Values.

353. Successive Integrations.

Successive integrations of a function may be expressed in terms of single integrals.

Let u be any function of x.

Then will

$$n!\frac{1}{D^{n+1}}u = x^n\frac{1}{D}u - {}^nC_1x^{n-1}\frac{1}{D}xu + {}^nC_2x^{n-2}\frac{1}{D}x^2u$$

$$-\ldots+(-1)^n\frac{1}{D}x^nu, \quad \text{where } D\equiv\frac{d}{dx}.$$

* Serret, *Calcul Intégral*, p. 107.

For
$$\frac{1}{D^2}u=\int\left[\int u\,dx\right]dx$$
$$=x\int u\,dx-\int xu\,dx$$
$$=x\frac{1}{D}u-\frac{1}{D}xu,$$
and the theorem is therefore true when $n=1$.

Also, integrating each term of the stated result, assumed for the moment true,
$$n!\frac{1}{D^{n+2}}u=\left[\frac{x^{n+1}}{n+1}\frac{1}{D}u-\frac{1}{D}\frac{x^{n+1}}{n+1}u\right]$$
$$-{}^nC_1\left[\frac{x^n}{n}\frac{1}{D}xu-\frac{1}{D}\frac{x^{n+1}}{n}u\right]$$
$$+{}^nC_2\left[\frac{x^{n-1}}{n-1}\frac{1}{D}x^2u-\frac{1}{D}\frac{x^{n+1}}{n-1}u\right]$$
$$+\dots$$
$$+(-1)^n\left[x\frac{1}{D}x^nu-\frac{1}{D}x^{n+1}u\right],$$
and
$$\frac{1}{n+1}-\frac{{}^nC_1}{n}+\frac{{}^nC_2}{n-1}-\dots+(-1)^n\frac{{}^nC_n}{1}$$
$$=\frac{1}{n+1}[1-{}^{n+1}C_1+{}^{n+1}C_2-\dots+(-1)^n\,{}^{n+1}C_n]$$
$$=\frac{1}{n+1}[(1-1)^{n+1}-(-1)^{n+1}]=\frac{-(-1)^{n+1}}{n+1}.$$

Hence, the right-hand members of the several brackets add up to
$$\frac{(-1)^{n+1}}{n+1}\frac{1}{D}x^{n+1}u.$$

Therefore, multiplying by $n+1$,
$$(n+1)!\frac{1}{D^{n+2}}u=x^{n+1}\frac{1}{D}u-{}^{n+1}C_1x^n\frac{1}{D}xu$$
$$+{}^{n+1}C_2x^{n-1}\frac{1}{D}x^2u-\dots+(-1)^{n+1}\frac{1}{D}x^{n+1}u,$$
i.e. if the theorem be true for the operator $\frac{1}{D^{n+1}}$, *i.e.* for $n+1$ integrations, it is true for $\frac{1}{D^{n+2}}$, *i.e.* for $n+2$ integrations; which establishes the inductive proof, for we have

shown that it is true if $n=1$, whence it is true for $n=2$, etc., and generally.

The theorem shows that a repeated integral such as

$$\iiiint u\,dx\,dx\,dx\,dx$$

can be expressed in terms of single integrations of

$$\int u\,dx,\quad \int xu\,dx,\quad \int x^2u\,dx,\quad \int x^3u\,dx.$$

This theorem is given by Todhunter, *Integral Calculus*, p. 72, *q.v.*

MISCELLANEOUS EXAMPLES.

1. Integrate (i) $\displaystyle\int\frac{\sin^2 x\,dx}{(x\cos x-\sin x)^2}$; [L.]

(ii) $\displaystyle\int\frac{\log x\,dx}{x^2(1-\log x)^2}$. [L.]

2. Prove that $\displaystyle\int_{-c}^{c}\frac{x(c^2-x^2)\,dx}{(b^2+c^2-2bx)^{\frac{3}{2}}}=\frac{4}{5}\frac{c^5}{b^4}\quad(b>c)$. [L.]

3. If $X=a+2bx+cx^2$, show that $\displaystyle\int\frac{dx}{X^n}$ can be made to depend upon $\displaystyle\int\frac{dx}{X^{n-1}}$.

Find a reduction formula for $\displaystyle\int\cos mx\sin^n x\,dx$, and apply it to the case $n=4$. [L.]

4. Evaluate $\displaystyle\int\frac{2x-3}{5x^2-16x+14}\,\frac{dx}{\sqrt{3x^2-10x+9}}$. [L.]

5. Prove that $\displaystyle\int_0^b u\frac{d^nv}{dx^n}dx$

can be made to depend upon

$$\int_0^b v\frac{d^nu}{dx^n}dx.$$

Hence show that if $f(x)$ be an arbitrary polynomial of degree $n-1$, and

$$P_n(x)=\frac{d^n(Ax^2+Bx+C)^n}{dx^n},$$

then

$$\int_\alpha^\beta f(x)P_n(x)\,dx=0,$$

where α, β are the roots, considered real, of the quadratic

$$Ax^2+Bx+C=0.$$

6. Prove that the effect of the operation $p\dfrac{d}{dt}+q$ on a periodic function $a\cos(nt+\epsilon)$ is to multiply the amplitude a by $\sqrt{p^2n^2+q^2}$, and to increase the angle $nt+\epsilon$ by $\tan^{-1}\dfrac{pn}{q}$.

Write down the effect of the operation

$$\left(p\frac{d}{dt}+q\right)\Big/\left(P\frac{d}{dt}+Q\right),$$

and generally, of the operation

$$\left(\alpha+\beta\frac{d}{dt}+\gamma\frac{d^2}{dt^2}+\dots\right)\Big/\left(A+B\frac{d}{dt}+C\frac{d^2}{dt^2}+\dots\right)$$

on the same periodic function. [INT. ARTS, LONDON.]

7. When $y^2\equiv ax^2+2bx+c$, prove that

$$\int\frac{dx}{y}=\frac{1}{\sqrt{a}}\,\mathrm{ch}^{-1}\frac{y\sqrt{a}}{\sqrt{ac-b^2}},\quad \frac{1}{\sqrt{a}}\,\mathrm{sh}^{-1}\frac{y\sqrt{a}}{\sqrt{b^2-ac}}\quad\text{or}\quad\frac{1}{\sqrt{-a}}\sin^{-1}\frac{y\sqrt{-a}}{\sqrt{b^2-ac}},$$

the real form to be chosen, and deduce the value of the integral in the degenerate case when $a=0$. [INT. ARTS, LONDON.]

8. Find the limiting value of $(n!)^{\frac{1}{n}}/n$, when n is infinite.

9. Find the limiting value when n is infinite of the n^{th} part of the sum of the n quantities

$$\frac{n+1}{n},\ \frac{n+2}{n},\ \frac{n+3}{n},\dots\frac{n+n}{n},$$

and show that it bears to the limiting value of the n^{th} root of the product of the same quantities the ratio $3e:8$, where e is the base of the Napierian logarithms. [OXFORD 1886, and I. P., 1911.]

10. If na is always equal to unity, and n is indefinitely great, show that the limiting value of the product

$$(1+a^4)\{1+(2a)^4\}^{\frac{1}{2}}\{1+(3a)^4\}^{\frac{1}{3}}\{1+(4a)^4\}^{\frac{1}{4}}\dots\{1+(na)^4\}^{\frac{1}{n}}$$

is $$e^{\frac{\pi^2}{48}}.$$ [OXFORD, 1888.]

11. Show that the limit of the sum of n terms of the series

$$\frac{n^3}{(n^2+1^2)(n^2+2\,.\,1^2)}+\frac{n^3}{(n^2+2^2)(n^2+2\,.\,2^2)}+\dots+\frac{n^3}{(n^2+n^2)(n^2+2n^2)},$$

when n is infinite, is

$$\sqrt{2}\tan^{-1}\sqrt{2}-\frac{\pi}{4}.$$ [γ, 1901.]

12. Find $Lt_{n=\infty}\left[\dfrac{\sqrt{n-a}}{n-c}+\dfrac{\sqrt{2n-a}}{2n-c}+\dfrac{\sqrt{3n-a}}{3n-c}+\dots+\dfrac{\sqrt{n^2-a}}{n^2-c}\right].$

13. Find the limiting value, when n is infinite, of

$$\left\{\tan\frac{\pi}{2n}\,.\tan\frac{2\pi}{2n}\,.\tan\frac{3\pi}{2n}\ldots\tan\frac{n\pi}{2n}\right\}^{\frac{1}{n}}.$$

[Oxford I. P., 1903.]

14. Show that the limit of the product

$$\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{\frac{1}{2}}\left(1+\frac{3}{n}\right)^{\frac{1}{3}}\ldots\left(1+\frac{n}{n}\right)^{\frac{1}{n}},$$

when n is increased indefinitely, is $e^{\frac{\pi^2}{12}}$. [Colleges, 1896.]

15. Find the limit, when n is indefinitely increased, of

$$\frac{1}{n}\left\{\sec\frac{x}{n}+\sec\frac{2x}{n}+\ldots+\sec\frac{(n-1)x}{n}\right\},$$

where x is $<\frac{\pi}{2}$.

Examine the case when $x>\frac{\pi}{2}$.

16. Find the limiting value of

$$2\log n-\log\left[(1+n^2)^{\frac{1}{n}}(2^2+n^2)^{\frac{1}{n}}\ldots(2n^2)^{\frac{1}{n}}\right],$$

when n is indefinitely increased. [Oxford I. P., 1900.]

17. Show from elementary considerations that when n increases indefinitely,

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log n$$

approaches a finite limit intermediate between $\frac{1}{2}$ and 1.

[St. John's, 1884.]

18. If $f(x)=f(a+x)$, show that

$$\int_a^{na} f(x)\,dx=(n-1)\int_0^a f(x)\,dx,$$

and illustrate geometrically. [Oxford I. P., 1888.]

19. Prove that $\displaystyle\int_0^a \phi(x)\,dx=\int_0^a \phi(a-x)\,dx,$

and show that (1) $\displaystyle\int_0^\pi \frac{x\sin^n x}{1+\cos^2 x}\,dx=\frac{\pi}{2}\int_0^\pi \frac{\sin^n x}{1+\cos^2 x}\,dx,$

and evaluate this integral when $n=1$ and when $n=3$.

(2) $\displaystyle\int_0^{\frac{\pi}{2}} \log\frac{(1+\sin^2 x)^2}{1+\frac{1}{8}\sin^2 2x}\,dx=\frac{\pi}{2}\log 2.$

20. If $\phi(x)=-\phi(2a-x)$, show that

$$\int_b^{2a}\phi(x)\,dx=-\int_0^b\phi(x)\,dx.$$

[Colleges, 1886.]

21. Prove that $\displaystyle\int_b^c \frac{\phi(x-b)}{\phi(c-x)}\,dx = \int_b^c \frac{\phi(c-x)}{\phi(x-b)}\,dx$,

provided $\phi(x)$ remains finite when x vanishes. [St. John's, 1883.]

22. Prove that if $\phi(x)$, $\psi(x)$, $\phi'(x)$, $\psi'(x)$ be continuous and finite from $x=a$ to $x=b$,

$$\int_a^b \phi'(x)\,\psi'(x)\,dx = \phi\{a+\theta(b-a)\}[\psi(b)-\psi(a)],$$

where θ is a positive proper fraction.

23. Prove that $\displaystyle\int_a^{\pi-a} x f(\sin x)\,dx = \frac{\pi}{2}\int_a^{\pi-a} f(\sin x)\,dx$.

[St. John's, 1883.]

24. Show that

$$\int_a^b f^n(x)\,\phi(c-x)\,dx - \int_a^b f(x)\,\phi^n(c-x)\,dx = \sum_{r=1}^{r=n}\Big[f^{r-1}(x)\,\phi^{n-r}(c-x)\Big]_a^b,$$

where $f^n(x)$ means the n^{th} differential coefficient of $f(x)$. [γ, 1893.]

25. Show that, if $\displaystyle\psi(x) = \int_0^x \phi(x)\,\phi'(2a-x)\,dx$,

then $\psi(2a) - 2\psi(a) = [\phi(a)]^2 - \phi(0)\,\phi(2a)$. [Trinity, 1895.]

26. If $f(x, y)$ is symmetrical in x and y, prove that

$$\int_{1-b}^b x f(x, 1-x)\,dx = \tfrac{1}{2}\int_{1-b}^b f(x, 1-x)\,dx.$$

[Colleges α, 1889.]

27. Examine under what limitations the formula

$$\int_b^a \phi(x)\,dx = \int_b^c \phi(x)\,dx + \int_c^a \phi(x)\,dx$$

holds good.

Show that $\displaystyle\int_{-\infty}^{\infty}\left(x+\frac{1}{x}\right)\phi\left(x-\frac{1}{x}\right)\frac{dx}{x} = 2\int_{-\infty}^{\infty}\phi(x)\,dx$.

[Math. Tripos, 1884.]

28. If
$$A_n = 1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n+1},$$
$$B_m = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2m},$$

show that when n and m are both infinite and the ratio $n:m$ tends to a limit k^2, $A_n - B_m = \log 2 + \log k$. [Colleges α, 1888.]

29. Show that

$$\frac{1}{D^n} e^{ax} \frac{\sin}{\cos} (bx + \alpha) = e^{ax} \frac{\frac{\sin}{\cos}\left(bx + \alpha - n\tan^{-1}\frac{b}{a}\right)}{(a^2+b^2)^{\frac{n}{2}}}$$
$$+ A_0 + A_1 x + A_2 x^2 + \ldots + A_{n-1} x^{n-1},$$

A_0, A_1, etc., being arbitrary constants, and also that it may be written

$$e^{ax} \frac{1}{(D+a)^n} \frac{\sin}{\cos} (bx + \alpha) + A_0 + A_1 x + \ldots + A_{n-1} x^{n-1},$$

and explain how the latter operation is to be conducted.

30. If $\quad I_1 = \int_0^{\frac{\pi}{2}} \log(1 + a_1 \sin^2\theta)\, d\theta,$

show that $\quad I_1 = \frac{\pi}{4}\log(1 + a_1) + \frac{1}{2} I_2,$

where $\quad I_2 = \int_0^{\frac{\pi}{2}} \log(1 + a_2 \sin^2\theta)\, d\theta$

and $\quad 4(1 + a_2)(1 + a_1) = (2 + a_1)^2.$

Hence show that

$$I_1 = \frac{\pi}{4}\log\left[(1+a_1)(1+a_2)^{\frac{1}{2}}(1+a_3)^{\frac{1}{4}}\ldots\right],$$

where $\quad 4(1 + a_{r+1})(1 + a_r) = (2 + a_r)^2.$

31. Show that if $n > 1$,

$$\int_0^1 \tanh\frac{1}{nx}\, dx < \frac{1}{n}(1 + \log n).$$ [Oxford I. P., 1911.]

32. How is the equation

$$\int_a^b f'(x)\, dx = f(b) - f(a)$$

to be interpreted when $f(x)$ is not a single-valued function?

Illustrate your answer by evaluating

$$\int_0^{\frac{1}{2}n\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta},$$

where a and b are real and n is a positive integer.

[Oxford I. P., 1912.]

33. Remembering that $\int_0^\infty$ means the limit tended to by $\int_\epsilon^\eta$ as the first of the two positive quantities ϵ, η tends to zero, and the second to infinity, prove that if $a > 1$, the value of

$$\int_0^\infty (a^n e^{-ax} - e^{-x}) x^{n-1}\, dx$$

is zero if $n > 0$, but not if $n = 0$. [Oxford I. P., 1917.]

34. If $f(x)$ be any function of x which can be put into partial fractions of the form $\dfrac{A}{a^2-x^2}$, then will

$$\int_0^\infty \frac{f(x)}{1+x^2}\,dx = \frac{\pi}{2} f(\sqrt{-1}).$$ [R. P.]

35. If $\quad 0<b<a, \quad a_1 = \tfrac{1}{2}(a+b), \quad b_1 = (ab)^{\frac{1}{2}},$

prove that $\quad b<b_1<a_1<a, \quad a_1-b_1<\tfrac{1}{2}(a-b).$

Show that if $\quad (a+b)\tan\theta\cot\phi = a - b\tan^2\theta,$

then $\displaystyle\int_0^{\frac{\pi}{2}} (a^2\cos^2\theta + b^2\sin^2\theta)^{-\frac{1}{2}}\,d\theta = \int_0^{\frac{\pi}{2}} (a_1{}^2\cos^2\phi + b_1{}^2\sin^2\phi)^{-\frac{1}{2}}\,d\phi.$

[Math. Trip., Part II., 1915.]

36. Show that

$$\int_0^{\frac{\pi}{2}} \sin\theta\tan^{-1}(\sin\theta)\,d\theta = \frac{\pi}{2}(\sqrt{2}-1).$$

[Math. Trip., 1882.]

37. Show how to evaluate $\int R(x, y)\,dx$, where $R(x, y)$ denotes any rational algebraic function of the coordinates x, y of a point on a conic. [St. John's, 1891.]

38. Show that if a be greater than unity,

$$\int_0^\pi \frac{x\,dx}{a^2-\cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2-1}}.$$ [Oxf. I. P., 1890.]

39. Prove that

$$\int_0^\infty \phi(x)\,dx = \int_0^1 \left\{\phi(x) + \frac{1}{x^2}\phi\left(\frac{1}{x}\right)\right\} dx.$$

[St. John's Coll., 1882.]

40. Prove that $\displaystyle\int_0^\pi \frac{x\sin x\,dx}{2+\cos 2x} = \frac{\pi}{\sqrt{2}}\tan^{-1}\sqrt{2}.$ [Oxf. I. P., 1889.]

41. Integrate $\displaystyle\int_a^b \frac{dx}{c^2-x^2}$, when c lies between a and b. [R. P.]

42. Prove that

$$\int_0^1 x^n(2-x)^n\,dx = 2^{2n}\int_0^1 x^n(1-x)^n\,dx.$$

[Oxf. II. P., 1886.]

43. Prove that

$$\int_a^{\frac{\pi}{2}-a} 2\cos^2 x\,\phi(\sin 2x)\,dx = \int_{2a}^{\frac{\pi}{2}} \phi(\sin x)\,dx.$$

[St. John's.]

44. Show that
$$\theta=\int_{a-c}^{r}\frac{a^2-c^2+r^2}{r\sqrt{2(a^2+c^2)r^2-(a^2-c^2)^2-r^4}}\,dr$$
is the equation of a circle. [MATH. TRIP., 1882.]

45. Find the integrals

$$(a)\ \int\left\{\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right\}\log x\,dx;$$

$$(b)\ \int\frac{dx}{x}\sqrt{x^2(x-3)^2-4x};$$

$$(c)\ \int\frac{dx}{\sin^3 x\tan^n\frac{x}{2}}.$$

[ST. JOHN'S, 1887.]

46. Prove that
$$\int_0^\alpha\log(1+\tan\alpha\tan x)\,dx=\alpha\log\sec\alpha.$$
[COLLS., 1896.]

47. Evaluate

(i) $\displaystyle\int\frac{xe^x}{(x+1)^2}dx.$ [TRIN., 1891.]

(ii) $\displaystyle\int e^x\frac{x^2+1}{(x+1)^2}dx.$ [HALL, *I.C.*]

(iii) $\displaystyle\int\frac{e^{x\sqrt{2}}}{1-x\sqrt{2}}\cdot\frac{1-x^2}{\sqrt{1-2x^2}}dx.$

(iv) $\displaystyle\int\frac{xe^x\,dx}{(e^x-1)^3}.$ [HALL, *I.C.*]

(v) $\displaystyle\int\frac{dx}{(1+x\tan x)^2}.$ [TRIN., 1891.]

(vi) $\displaystyle\int_e^{e^2}\frac{\sec x\operatorname{cosec}x}{\log\tan x}dx.$ [TRIN., 1884.]

(vii) $\displaystyle\int\frac{\log(1+x^2)}{\sqrt{1-x}}dx.$

48. If
$$I_n\equiv\int_{-1}^{1}(1-x^2)^n\cos ax\,dx,$$
show that
$$a^2I_n=2n(2n-1)I_{n-1}-4n(n-1)I_{n-2},$$
provided $n>1$.

Show also that $I_n=\dfrac{n!}{a^{2n+1}}\{f(a)\sin a+g(a)\cos a\},$

where $f(a)$ and $g(a)$ are algebraic functions of a, of degrees $\not> n$, with integral coefficients. [TRIN., 1892.]

49. Show that

(i) $\displaystyle\int x^3\sqrt{\frac{1+x^2}{1-x^2}}\,dx=\tfrac{1}{2}\tan^{-1}\sqrt{\frac{1+x^2}{1-x^2}}-\tfrac{1}{4}(2+x^2)\sqrt{1-x^4}.$

(ii) $\displaystyle\int\frac{x^2+1}{x^2-1}\frac{dx}{\sqrt{1-ax^2+x^4}}=\frac{1}{\sqrt{a-2}}\cos^{-1}\frac{x\sqrt{a-2}}{x^2-1}\quad(a>2).$

[HALL, *I.C.*, p. 325 and p. 346.]

50. Show that

$$\int_0^1 x^{cx^a}\,dx = 1 - \frac{c}{(a+1)^2} + \frac{c^2}{(2a+1)^3} - \frac{c^3}{(3a+1)^4} + \dots.$$

[ANGLIN.]

51. Prove that

(i) $$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1+\sin^2\theta}} = 1^2 - \left(\frac{1}{2}\right)^2 + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2 - \dots.$$

(ii) $$1 + \frac{1}{2^2}\left(\frac{1}{2}\right)^2 + \frac{1}{3^2}\left(\frac{1\,.\,3}{2\,.\,4}\right)^2 + \dots = \frac{11}{\pi} - 4.$$

[ANGLIN.]

52. If $$\phi(x) = a_1 x + \frac{2}{3} a_3 x^3 + \frac{2\,.\,4}{3\,.\,5} a_5 x^5 + \dots,$$

prove that

(i) $$\int_0^{\frac{\pi}{2}} \cos^2\theta\,\phi(\sin\theta)\,d\theta = \frac{1}{3}a_1 + \frac{1}{5}\left(\frac{2}{3}\right)^2 a_3 + \frac{1}{7}\left(\frac{2\,.\,4}{3\,.\,5}\right)^2 a_5 + \dots.$$

(ii) $$\frac{\pi}{2} = 1 + \frac{1}{3}\,.\,1^2 + \frac{1}{5}\left(\frac{2}{3}\right)^2 + \frac{1}{7}\left(\frac{2\,.\,4}{3\,.\,5}\right)^2 + \dots.$$

(iii) $$\pi - 3 = \frac{1}{3^2}\,.\,1^2 + \frac{1}{5^2}\left(\frac{2}{3}\right)^2 + \frac{1}{7^2}\left(\frac{2\,.\,4}{3\,.\,5}\right)^2 + \dots.$$

[ANGLIN.]