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On velocity discontinuities in elastic-plastic boundary value problems

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AN EXAMPLE is discussed of a quasi-static elastic-plastic boundary value problem where existence of a solution requires a velocity discontinuity. The type of velocity discontinuity required would normally be rejected on somewhat arbitrary grounds.

IT IS PECULIAR to plasticity theory, within the larger sphere of continuum mechanics, that the approach to discontinuity relations is made by introducing a transition zone. Thus, a velocity discontinuity as HILL states [1] "is arbitrarily regarded as a vanishingly thin layer Σ of intense strain rate, throughout which the stress and velocity vary continuously and satisfy all field equations". Curiously, the issue of whether or not to admit a less restrictive class of velocity discontinuities, in line with the general treatment of Hill as pertaining to general continuum theory, seems never to have arisen. Velocity discontinuities in quasi-static problems have been almost entirely confined to tangential jumps, consistent with the notion of a transition zone, as applied to the solution of limit load problems where a rigid-plastic model is employed [2].

We produce an example of an elastic-plastic quasi-static boundary value problem where a velocity discontinuity of a type which is non-admissible under Hill's approach occurs necessarily as part of the solution. One is therefore faced with the choice of either ruling out the more general discontinuity, in which case no solution to the example is possible, or of admitting the discontinuity, in which case non-uniqueness becomes a distinct possibility in other problems. While the particular example does not exhibit non-uniqueness, it does exhibit non-existence throughout a certain domain of the incremental loading parameters.

We consider an elastic-perfectly plastic material in axially symmetric plane stress characterized by a convex yield surface and associated flow rule. The stress-strain rate relations are

$$\dot{\varepsilon}_r = \frac{\partial \dot{u}}{\partial r} = \frac{1}{E}(\dot{\sigma}_r - \nu\dot{\sigma}_\theta) + \beta\Gamma\alpha_r, \tag{1}$$

$$\dot{\varepsilon}_\theta = \frac{\dot{u}}{r} = \frac{1}{E}(\dot{\sigma}_\theta - \nu\dot{\sigma}_r) + B\Gamma\alpha_\theta, \tag{2}$$

where $(\alpha_r, \alpha_\theta)$ are the components of a unit vector in the direction of the outer normal to the yield surface. Hence, $\beta = 1$ if $(\sigma_r, \sigma_\theta)$ satisfies the yield equation and if the dot product $[\dot{\sigma}_r, \dot{\sigma}_\theta]\cdot[\alpha_r, \alpha_\theta] = 0$; otherwise $\beta = 0$ and the incremental action is purely elastic.

If we take the approach of HILL, a narrow transition zone of arbitrarily small width Δr is assumed across which a jump in velocity $\dot{u}(r, t)$ takes place. Thus $[\dot{u}] \neq 0$[1] where, at the same time, stress and velocity vary continuously through the transition region and satisfy (1), (2) as well as the yield inequality and equilibrium

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = F(r, t), \tag{3}$$

where F is assumed to have continuous first derivatives. It is easily shown that such a discontinuity of the restricted type cannot exist in axially symmetric plane stress. This is true for either an elastic or plastic transition zone.

If, on the other hand, we consider a velocity discontinuity to exist in the more general sense of a strict discontinuity where the field equations are satisfied in the one-sided sense, we do not encounter difficulties in satisfying the field equations in the transition zone. It is possible to match solutions across the discontinuity such that $[\dot{u}] \neq 0$ and where displacement and stresses are continuous. The only extra condition which must be met is that the jump result in a positive power dissipation. The following problem provides an example of such a discontinuity.

Consider a plate annulus $1 \leqslant r \leqslant 2$ of constant thickness and an elastic-perfectly plastic material governed by the Tresca yield criterion and associated flow rule, Fig. 1. For

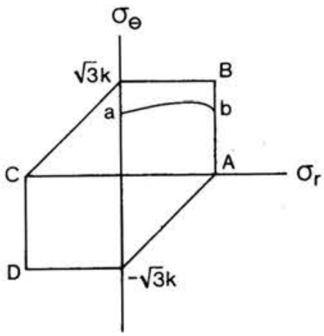

FIG. 1. The Tresca yield locus for plane stress. The initial elastic stress distribution (8), (9) lies on AB.

simplicity assume Poisson's ratio to be zero. To exhibit the discontinuity it is necessary that plastic flow takes place along side AB or CD of Fig. 1. Hence, we subject the plate to body force,

$$rF(r, t) = rf(t) - \frac{r^2}{2} q(t) \tag{4}$$

and boundary conditions

$$\sigma_r(l, t) = \sqrt{3}\, k, \quad \sigma_r(2, t) = 0 \tag{5}$$

with k the yield in shear. We choose $f(0)$ and $q(0)$ such that an elastic solution exists at $t = 0$ on the verge of plastic action, i.e., such that

$$\partial \sigma_r(1, 0)/\partial r = 0 \tag{6}$$

is satisfied.

[1] Notation $[\dot{u}]$ stands for the jump.

Choosing

$$q(0) = \frac{16}{3}\sqrt{3}\,k, \quad f(0) = 3\sqrt{3}\,k \tag{7}$$

results in the initial elastic solution

$$\sigma_r^e(r, 0) = (2r - r^2)\sqrt{3}\,k, \tag{8}$$

$$\sigma_\theta^e(r, 0) = (r - r^{2/3})\sqrt{3}\,k. \tag{9}$$

This initial stress distribution satisfies (6) and does not violate the yield inequality, Fig. 1.

The problem of interest is the existence of solutions to the incremental problem in a positive neighbourhood of $t = 0$ under arbitrarily varying loading parameters $f(t)$, $q(t)$. There are two possibilities for solutions: a purely elastic solution corresponding to unloading and an elastic-plastic solution with the elastic-plastic boundary $s(t)$, $s(0) = 1$, growing from the inner surface such that $\dot{s}(0) > 0$.

The subsequent entirely elastic solution is

$$\sigma_r^e(r, t) = \left(-1 + \frac{4}{r^2}\right)\frac{\sqrt{3}k}{3} + \left(6r - 14 + \frac{8}{r^2}\right)\frac{f(t)}{9} + \left(-3r^2 + 15 - \frac{12}{r^2}\right)\frac{q(t)}{16}, \tag{10}$$

$$\sigma_\theta^e(r, t) = -\left(1 + \frac{4}{r^2}\right)\frac{\sqrt{3}\,k}{3} + \left(3r - 14 - \frac{8}{r^2}\right)\frac{f(t)}{9} + \left(-r^2 + 15 + \frac{12}{r^2}\right)\frac{q(t)}{16}, \tag{11}$$

provided the yield inequality is not violated at (1, 0). This is the case provided that

$$\dot{q}(0) - \frac{80}{81}\dot{f}(0) \leqslant 0 \tag{12}$$

which can be considered as the initial unloading condition. Hence, an incremental elastic solution exists in the domain to the right of COD, Fig. 2.

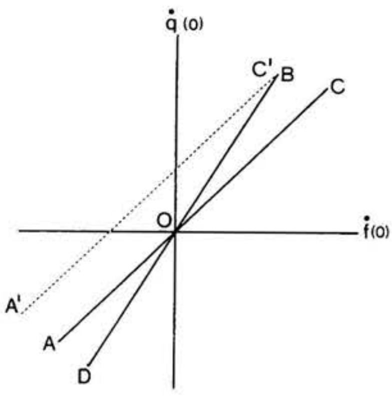

FIG. 2. Solution domains for the initial incremental problem. Elastic-plastic solutions exist to the left of AOB and elastic solutions to the right of AOC. No solutions exist in BOC. For given $\dot{f}(0)$, $q(0)$, the rate of expansion of the elastic-plastic boundary $\dot{s}(0)$ is given by $A'C'$, parallel to AC. Equations are $A'C'$, $\dot{f}(0) - \frac{81}{80}\dot{q}(0) = -\frac{9}{5}\sqrt{3}k\,\dot{s}(0)$; BD, $\dot{q}(0) = \frac{4}{3}\dot{f}(0)$.

Considering the elastic-plastic solution, in the elastic region $s(t) \leqslant r \leqslant 2$ the general solution is

$$\sigma_r^e(r, t) = C_1(t) + \frac{C_2(t)}{r^2} + \frac{2}{3}rf(t) - \frac{3}{16}r^2 q(t), \tag{13}$$

$$\sigma_\theta^e(r, t) = C_1(t) - \frac{C_2(t)}{r^2} + \frac{rf(t)}{3} - \frac{1}{16}r^2 q(t). \tag{14}$$

The plastic stress field is determined solely from (3) in addition to the yield condition

$$\sigma_r^p(r, t) = \sqrt{3}\,k. \tag{15}$$

Thus σ_r^p is a known constant and σ^p is given in terms of f and q by (3).

Rather than treating the direct problem with $f(t)$ and $q(t)$ known and unknowns $C_1(t)$, $C_2(t)$, $s(t)$ it is easier to solve an indirect problem. Regard $s(t)$ as known and $f(t)$ as unknown. Applying (5b) and continuity of σ_r and σ results in

$$f(t) = \frac{12\sqrt{3}\,k - \frac{9}{16}(4-s^2)^2 q}{-16+12s-s^3}, \tag{16}$$

$$C_1(t) = \sqrt{3}\,k + \frac{3}{8}s^2 q - sf, \tag{17}$$

$$C_2(t) = \frac{s^3}{3}f - \frac{3}{16}s^4 q, \tag{18}$$

which when substituted into (13) and (14) gives a solution representation. A plastic loading condition is obtained by differentiating (16) which reduces at the initial state to

$$\dot{f}(0) - \frac{81}{80}\dot{q}(0) = -\frac{9}{5}\sqrt{3}\,k\dot{s}(0), \tag{19}$$

where the right-hand side is an arbitrary negative real number. Hence, elastic-plastic action takes place along line $A'C'$, Fig. 2.

It may be verified that the plastic work rate is non-negative in a neighbourhood of (1, 0) or, equivalently, that $\Gamma \geqslant 0$ provided that

$$\dot{f}(0) - \frac{3}{4}\dot{q}(0) \leqslant 0, \tag{20}$$

which can be considered as a second initial loading condition. Thus, any possible initial elastic-plastic incremental solution lies to the left of AOB, Fig. 2. Secondly the requirement of non-negative power dissipation in the elastic-plastic interface implies that $[\dot{u}] \leqslant 0$. This is satisfied since

$$[\dot{u}(1, 0)] = -\frac{1}{E}\frac{\partial \dot{\sigma}_r^e}{\partial r}(1, 0) = -\frac{34}{15}\sqrt{3}\,k\dot{s}(0) < 0. \tag{21}$$

Finally, the yield inequality must be satisfied in a neighbourhood of (1, 0). This is easily established by using a Taylor expansion.

An initial elastic-plastic solution thus exists to the left of AOB, Fig. 2, while an elastic solution exists to the right of AOC. No solution is possible in the domain BOC.

The example delineates the essential necessary condition if the solution to an elastic-plastic boundary value problem be required to contain a velocity discontinuity of more general type than conventionally admitted. In general, a velocity jump will occur across the elastic-plastic boundary if the plastic velocity field is determined uniquely from the stress field which, in turn, is determined as the solution to a statically determinate stress type boundary value problem. The elastic velocity field is, of course, determined uniquely from the elastic stress field. In axially symmetric plane stress problems this situation occurs when the yield locus is normal to the σ_r axis.

References

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