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# On the existence and uniqueness of magnetohydrodynamical shock wave structures, disregarding thermal conductivity

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IN THE PRESENT paper, applying the metod described in [4], the existence of fast as well as slow shock waves structures is proved, disregarding thermal conductivity. It is also shown that by contrast with the fast shock waves, slow shock waves do not always possess a unique structure.

W pracy przeprowadzono w oparciu o metodę przedstawioną w [4] dowód istnienia struktury szybkich i struktury wolnych fal uderzeniowych; w dowodzie pominięto zjawiska przewodnictwa cieplnego. Wykazano ponadto, że — przeciwnie niż w przypadku szybkich fal uderzeniowych — fale wolne nie zawsze mają jednoznacznie określoną strukturę.

В работе доказано существование структуры быстрых и структуры медленных ударных волн в случае пренебрежимости теплопроводностью. Доказательство основано на применении метода, описанного в работе [4]. Показано, что, в противовес быстрым ударным волнам, медленные ударные волны не всегда обладают единственной структурой.

## 1. Introduction

IN PAPER [4] the existence of a slow and a fast shock wave structure for a perfect gas with shear viscosity disregarded was proved. The proof was based on the topological properties of the generalized thermodynamical potential (see [1, 3, 4]).

In the present paper, using the same method, the existence of slow and fast shock wave structures is proved but under the assumption that the coefficient of thermal conductivity is equal to zero, and the other three coefficients of dissipation are positive functions of class $C^1$ of the physical parameters. Demonstrated are the uniqueness of the fast shock wave structure and the nonuniqueness of the slow shock wave structure. The latter result is also true under the assumptions of [4] where the coefficient of shear viscosity is put equal to zero instead that of thermal conductivity.

For a perfect gas, with thermal conductivity disregarded, the set of equations of the shock layer can be written in the form:

$$\varepsilon_1 \frac{dB}{dx} = \frac{B\tau}{\mu} - c_1 v + c_2,$$

$$(1.1) \qquad \varepsilon_2 \frac{dv}{dx} = (v - c_1 B),$$

$$\varepsilon_3 \frac{d\tau}{dx} = \frac{\gamma-1}{\tau}\left(\frac{M^2\tau^2}{2} + \frac{v^2}{2} + \frac{B^2\tau}{2\mu} + c_2 B - c_1 Bv - c_3\tau + c_4\right) + M^2\tau + \frac{B^2}{2\mu} - c_3;$$

$$(1.2) \qquad T = \frac{1}{c_v}\left(\frac{M^2\tau^2}{2} + \frac{v^2}{2} + \frac{B^2\tau}{2\mu} + c_2 B - c_1 Bv - c_3\tau + c_4\right),$$

where

$$\varepsilon_1 = \frac{1}{\sigma\mu^2 M}, \quad \varepsilon_2 = \frac{\eta}{M}, \quad \varepsilon_3 = \left(\xi + \frac{4}{3}\eta\right) M,$$

$T$ — temperature, $\tau$ — specific density, $\mu = \text{const}$ — magnetic permeability, $M = \mu/\tau = \text{const}$, $[u, v, 0]$ — velocity vector, $E = [0, 0, c_2\mu M]$ — electric field vector ($c_2 = \text{const}$), $B = [c_1\mu M, B, 0]$ — magnetic induction vector ($c_1 = \text{const}$), $c_3, c_4$ — positive constants, $\gamma = c_p/c_v$ — specific heat for constant pressure and, volume, respectively, $\xi, \eta$ — coefficients of bulk and shear viscosity, respectively.

We shall define functions $F$ and $W$ similar to the generalized dissipation and generalized thermodynamic potential introduced by Germain [1]. These functions can be written in the form:

$$F = \frac{\tau^{\gamma-1}}{2}\left[\varepsilon_1\left(\frac{dB}{dx}\right)^2 + \varepsilon_2\left(\frac{dv}{dx}\right)^2 + \varepsilon_3\left(\frac{d\tau}{dx}\right)^2\right], \tag{1.3}$$

$$W = \left(\frac{M^2\tau^2}{2} + \frac{v^2}{2} + \frac{B^2\tau}{2\mu} + c_2 B - c_1 Bv - c_3\tau + c_4\right)\tau^{\gamma-1}. \tag{1.4}$$

By means of them we can rewrite the system (1.1)

$$\frac{\partial W}{\partial q_i} = \frac{\partial F}{\partial \dot{q}_i}, \quad i = 1, 2, 3, \tag{1.5}$$

where $q_i$ ($i = 1, 2, 3$) denote $B, v, \tau$, respectively, and $\dot{q}_i$ denote their derivatives with respect to $x$.

It is easy to show that $W$ is an increasing function along the integral curves of the system (1.1). On that basis we shall prove the existence of the slow and fast shock waves structure.

The system (1.1) has at most four singular points (see [1]) $P(B_i, v_i, \tau_i) = P_i$, ($i = 1, 2, 3, 4$). These points are numbered according to decreasing specific volume. This enumeration corresponds to the increase of entropy

$$S(P_1) \leqslant S(P_2) \leqslant S(P_3) \leqslant S(P_4).$$

The velocities at the points $P_1, P_2, P_3, P_4$ can be ordered as follows:

$$u_1 \geqslant c_f \geqslant u_2 \geqslant b_x \geqslant u_3 \geqslant c_s \geqslant u_4, \tag{1.6}$$

where $b_x = (M^2\mu c_1^2\tau)^{\frac{1}{2}}$ is the normal component of Alfven speed, $c_f$ and $c_s$ are speeds of fast and slow magnetoacoustic waves respectively, being the roots of the biquadratic equation:

$$u^4 - u^2(a^2 + b_x^2 + b_y^2) + a^2 b_x^2 = 0,$$

$c_s < c_f$, $b_y = \left(\frac{B^2\tau}{\mu}\right)^{\frac{1}{2}}$ — tangent component of Alfven speed, $a^2 = -\frac{1}{\tau^2}\frac{\partial p}{\partial\tau}\Big|_{s=s_0}$, $a$ — speed of sound.

The pair of points $P_1, P_2$ and the Eq. (1.2) determine the states of the fast shock wave, the pair of points $P_3, P_4$ — the states of the slow shock wave, the other pairs of points $P_i, P_j$, $i < j$, determine the states of intermediate shock waves. The integral curve of the system (1.1), joining the points $P_i$, $P_j$, $i < j$, describes the structure of that shock wave. Taking into account the physical character of the variables $B$, $v$, $\tau$, $T$, our considerations will be limited to the domain $Z$ of the semispace $\{(B, v, \tau), \tau > 0\}$, where $T$, defined by (1.2), is greater than zero. In the domain $Z$, $W(B, v, \tau) > 0$. The first step in our considerations is a qualitative analysis of integral curves in the space immediately adjacent to the singular point. To this end we shall determine the signs of the eigenvalues at points $P_i (i = 1, 2, 3, 4)$ of the linearized equations (1.1).

## 2. Investigation of the integral curves of the system (1.1) in the neighbourhood of the singular points

The linearized system (1.1) in the neighbourhood of $P_i$ has the form:

$$\begin{aligned} \varepsilon_{1i}\frac{d\bar{B}}{dx} &= \frac{\tau_i}{\mu}\bar{B} - c_1\bar{v} + \frac{B_i}{\mu}\bar{\tau}, \\ \varepsilon_{2i}\frac{d\bar{v}}{dx} &= -c_1\bar{B} + \bar{v}, \\ \varepsilon_{3i}\frac{d\bar{\tau}}{dx} &= \frac{B_i}{\mu}\bar{B} + \left[(\gamma+1)M^2 + \frac{\gamma B_i^2}{2\mu\tau_i} - \gamma\frac{c_3}{\tau_i}\right]\bar{\tau}, \end{aligned} \tag{2.1}$$

where

$$\varepsilon_{ki} = \varepsilon_k(B_i, v_i, \tau_i) = \varepsilon_k(P_i), \quad k = 1, 2, 3;\ i = 1, 2, 3, 4,$$
$$B = B_i + \bar{B}, \quad v = v_i + \bar{v}, \quad \tau = \tau_i + \bar{\tau}.$$

Following the considerations of [1] or [4], we can state that the number of positive (negative) eigenvalues is equal to the number of positive (negative) coefficients in the diagonal form of the coefficients matrix $A$ of the system (2.1). The quadratic form corresponding to the matrix $A$, for the case $\tau_i \neq \mu c_1^2 = \tau_*$, may be written:

$$\begin{aligned} g(XX) &= \frac{\tau_i}{\mu}x_1^2 - 2c_1x_1x_2 + \frac{2B_i}{\mu}x_1x_3 + x_2^2 \\ &+ \left[(\gamma+1)M^2 + \frac{\gamma B^2}{2\mu\tau_i} - \gamma\frac{c_3}{\tau_i}\right]x_3^2 \equiv (x_2 - c_1x_1)^2 + \frac{\tau_i - \tau_*}{\mu}\left(x_1 + \frac{B_i}{\tau_i - \tau_*}x_3\right)^2 \\ &+ \left[(\gamma+1)M^2 + \frac{\gamma}{2\mu}\frac{B_i^2}{\tau_i} - \gamma\frac{c_3}{\tau_i} - \frac{B_i^2}{\mu(\tau_i - \tau_*)}\right]x_3^2. \end{aligned} \tag{2.2}$$

From the last-given identity it follows that the linear transformation

$$\begin{aligned} y_1 &= -c_1x_1 + x_2, \\ y_2 &= x_1 + \frac{B_i}{\tau_i - \tau_*}x_3, \\ y_3 &= x_3, \end{aligned} \tag{2.3}$$

transforms the form $g(XX)$ to the diagonal form:

$$g(XX) = y_1^2 + \frac{\tau_i - \tau_*}{\mu}y_2^2 + \left[(\gamma+1)M^2 + \frac{\gamma}{2\mu}\frac{B_i^2}{\tau_i} - \gamma\frac{c_3}{\tau_i} - \frac{B_i^2}{\mu(\tau_i - \tau_*)}\right]y_3^2. \tag{2.4}$$

The matrix $B$ of the transformation (2.3) transforms the matrix $A$ to the diagonal form $D$, i.e.

$$B\,A\,B^T = D.$$

The coefficients of the matrix $D$ are equal to the coefficients of the form $g(YY)$. The coefficient at $y_1^2$ is positive, the coefficient at $y_2^2$, in view of the inequality (1.6), is positive at the points $P_1$, $P_2$ and negative at the points $P_3$, $P_4$. The coefficient of $y_3^2$ may be written:

$$\frac{1}{\tau_i^2(u_i^2-b_{xi}^2)}\,[(u_i^2-a_i^2)(u_i^2-b_{xi}^2)-b_{yi}^2u_i^2].$$

Since $u_i^2-b_{xi}^2$ is positive at the points $P_1$, $P_2$, negative at $P_3$, $P_4$ and $[(u_i^2-a_i^2)(u_i^2-b_{xi}^2)-$ $-b_{yi}^2\,u_i^2]$ is positive at $P_1$, $P_4$, negative at $P_2$, $P_3$, the coefficient of $y_3^2$ is positive at $P_1$, $P_3$ and negative at $P_2$, $P_4$.

From the above considerations follows:

THEOREM 1. *At point $P_1$ all eigenvalues are positive, at points $P_2$, $P_3$ two eigenvalues are positive and one negative, at $P_4$ — one positive and two negative.*

*If $c_2 = 0$, then $\tau_i = \tau_*$ at point $P_i$ $(i = 2, 3)$. In this case, applying the nonsingular transformation*

$$y_1 = x_2-c_1x_1,$$

$$y_2 = x_3+\frac{B_ix_1}{\mu\left[(\gamma+1)M^2+\dfrac{\gamma B_i^2}{2\mu\tau_i}-\gamma\dfrac{c_3}{\tau_i}\right]},$$

$$y_3 = x_1,$$

*we can transform the form* (2.2) *to the diagonal form*:

$$g(YY) = y_1^2+\left[(\gamma+1)M^2+\frac{\gamma B^2}{2\mu\tau_i}-\gamma\frac{c_3}{\tau_i}\right]y_2^2-\frac{B_i^2}{\mu^2\left[(\gamma+1)M^2+\dfrac{\gamma B^2}{2\mu\tau_i}-\gamma\dfrac{c_3}{\tau_i}\right]}y_3^2.$$

This proves that for $c_2 = 0$, Theorem 1 is also true.

## 3. Qualitative analysis of the surface $W$ = const

We shall analyse the surface $W(B, v, \tau) = A$, where $W(B, v, \tau)$ is the function defined by (1.4) and $A$ is a positive constant, $(B, v, \tau) \in Z$. The gradient of $W(B, v, \tau)$ is equal to zero only at the singular points of the system (1.1) [this results from the equivalence of the systems (1.1) and (1.5)] hence, the surface $W(B, v, \tau)$ = const has the only singularities at the points $P_i$ $(i = 1, 2, 3, 4)$. On repeating the same considerations as in [4], the following corollaries may be proved:

COROLLARY 1. *In the neighbourhood of the point $P_1$ the surface $W(B, v, \tau) = A_1$ is reduced to the point $P_1$, in the neighbourhood of the point $P_i$ $(i = 2, 3, 4)$ the surface $W(B, v, \tau) = A$ is topologically equivalent to a cone.*

COROLLARY 2. *The surface* $W(B, v, \tau) = A_1+\delta$ *is in the neighbourhood of* $P_1$ *topologically equivalent to a sphere; the surfaces* $W(B, v, \tau) = A_i+\delta$, *in the neighbourhood of* $P_i (i = 2, 3)$ — *topologically equivalent to a hyperboloid of one sheet, and in the neighbourhood of* $P_4$ *the surface* $W(B, v, \tau) = A_4+\delta$ *is topologically equivalent to a hyperboloid of two sheets.*

COROLLARY 3. *The surface* $W(B, v, \tau) = A_i-\delta$, *is in the neighbourhood of* $P_i$ *an empty set for* $i = 1$, *a set topologically equivalent to a hyperboloid of two sheets for* $i = 2, 3$, *and a set topologically equivalent to a hyperboloid of one sheet for* $i = 4$, *where* $A_i = W(P_i)$ $(i = 1, 2, 3, 4)$ *and* $\delta$ *is a sufficiently small and positive constant.*

As a result of the orthogonal projection of the surface $W(B, v, \tau) = A$ into the plane $(B, \tau)$, we get a set $G_A$ in the semiplane $\tau > 0$, the boundary of which consists of the $B$ axis and the curve $Q_A$. The equations:

$$W(B, v, \tau) = A, \quad \frac{\partial W(B, v, \tau)}{\partial v} = 0, \tag{3.1}$$

describe the curve $Q_A$.

Making use of the formulae describing the function $W(B, v, \tau)$ [see (1.4)] and then eliminating $v$ from the system (3.1), we obtain the equation for $Q_A$:

$$\tau^{\gamma-1}\left(\frac{M^2\tau^2}{2}-\frac{c_1^2B^2}{2}+\frac{B^2\tau}{2\mu}+c_2B-c_3\tau+c_4\right) = A. \tag{3.2}$$

The left-hand side of (3.2) is the function $\bar{K}(B, \tau)$ known from [4]. The Eq. (3.2) describes the family of the curves $Q_A$ discussed in [4].

Each of the points belonging to the interior of the domain $G_A$ is an orthogonal projection of two different points on the surface $W(B, v, \tau) = A$ into the plane $(B, \tau)$. Each of the points belonging to the curve $Q_A$ is an orthogonal projection of one point on the surface $W(B, v, \tau) = A$.

## 4. Proof of the existence of fast and slow shock wave structures

Let us analyse changes of the surface $W(B, v, \tau) = A$ for $A > A_1$. On the basis of the interpretation of the domains $G_A$ and on the properties of the curves $Q_A$, proved in [4], we state that the surface $W(B, v, \tau) = A_1$ consists of the surface topologically equivalent to a plane and of the isolated point $P_1$. For $A > A_1$ and $A$ close to $A_1$, the closed part of the surface $W(B, v, \tau) = A$ will be formed, with the point $P_1$ being in the interior of the surface. With $A$ increasing, the closed part of the surface $W(B, v, \tau) = A$ will enclose a greater and greater domain, approaching the other part of the surface. Both parts of the surface $W(B, v, \tau) = A$ will be in touch, at the point $P_2$ for $A = A_2$.

From the considerations in 2, we have the result that all the integral curves of the system (1.1) passing through the neighbourhood of $P_1$ leave the point $P_1$. Along each integral curve, $W(B, v, \tau)$ is increasing. Thus through each point of the closed part of the surface $W(B, v, \tau) = A (A_1 < A < A_2)$ there passes an integral curve leaving the point $P_1$. Because both parts of the surface $W(B, v, \tau) = A$ have a common point $P_2$ (for $A = A_2$), then there must exist an integral curve joining $P_1$ and $P_2$. The second integral curve of the system (1.1) reaches the point $P_2$ in the opposite sense and it cannot leave $P_2$ (there are only two integral curves reaching $P_2$). Thus is proved the existence and uniqueness of the fast shock wave structure.

To prove the existence of the integral curve of the system (1.1) joining $P_3$ and $P_4$, let us notice that according to Hadamard-Perron's lemma [2], the integral curves leaving $P_3$ form in the neighbourhood of $P_3$ the manifold diffeomorphic to a plane. The manifold will intersect the surface $W(B, v, \tau) = A$, ($A > A_3$ and $A$ close to $A_3$), along the closed curve $L_A$, that cannot (without leaving the surface $W = A$) be continuously transformed into a point. The curve $L_A$ must, as was shown in paper [4] for $A = A_4$, pass through $P_4$. Thus is proved the existence of the slow shock wave.

## 5. On the non-uniqueness of the slow shock wave structure

Contrary to the fast shock waves, the system (1.1) determines not always uniquely the slow shock wave structures. This fact can be stated as follows.

THEOREM 2. *There are sets of positive, class* $C^1$, *coefficients* $\varepsilon_i(B, v, \tau)$ $(i = 1, 2, 3)$, *for which there exist at least two integral curves of the system* (1.1) *joining the singular point* $P_3$ *with the singular point* $P_4$.

Let us define a set of positive functions $\varepsilon_i(B, v, \tau)$ in the semiplane $\{(B, v, \tau): \tau > 0\}$, $\varepsilon_i \in C^1$ $(i = 1, 2, 3)$. To facilitate the considerations, we take $\varepsilon_i = \varepsilon_i^0 = \text{const} > 0$. To these $\varepsilon_i$ corresponds the integral curve $C$ of the system (1.1) joining $P_3$ with $P_4$. Then, let us restrict the domain of these functions so as to form two closed domains $D_3, D_4$, $D_3 \cap D_4 = 0$, sufficiently bounded by the regular surfaces $\Sigma_3, \Sigma_4$.

Let the singular point $P_i$ belong to the interior of the domain $D_i (i = 3, 4)$. Thus the system (1.1) is uniquely defined in $D_i$ and the solutions of (1.1) are defined in $D_i (i = 3, 4)$. According to the results of 2, the integral curves of the system (1.1) leaving the point $P_3$, as well as the integral curves reaching $P_4$, form two-dimensional manifolds. The points of these manifolds belonging to $D_i (i = 3, 4)$ are well defined because $\varepsilon_i$ are known. Let us join point $P_3$ with $P_4$ by means of two regular arcs $C_1, C_2 \in Z$ (having a parametrical representation of class $C^2$) non intersecting with each other in such a way that in the domain $D_3$ the arcs form two integral curves leaving $P_3$ and that in the domain $D_4$ the arcs form two integral curves reaching $P_4$. Thus to each point belonging to one of the arcs $C_1, C_2$ can be attached a well defined direction (the direction of the arc at this point). Let us define on the arcs $C_1$ and $C_2$ coefficients $\varepsilon_i(B, v, \tau)$ $(i = 1, 2, 3)$ of the system (1.1) in such a way that every point of each arc the direction of the arc is the same as the field of directions defined by (1.1). Moreover, it can be guaranteed that such defined $\varepsilon_i(B, v, \tau)$ are positive functions of the arc parameter of class $C^1$. Indeed, as $C_1$ may be taken the curve $C$ defined at the beginning of 5. In the neighbourhood of $C$ the arc $C_2$ can be constructed corresponding to positive $\varepsilon_i (i = 1, 2, 3)$. This is the result of the continuity of the right-hand sides of (1.1). The regularity of the right-hand sides of (1.1) and the assumptions adopted on the regularity of the arcs $C_1$ and $C_2$ guarantee the continuity of the derivatives $\varepsilon_i$ along the arc. Thus we have $\varepsilon_i$ defined on the set $D_3 \cup D_4 \cup C_1 \cup C_2$. It remains to continue them as a function of class $C^1$ in the semispace $\tau > 0$. It is evident that the adopted assumptions enable the continuation to the function of class $C^1$ in the neighbourhood of $D_3 \cup D_4 \cup C_1 \cup C_2$, and by Whitney's theorem [5] follows the possibility of continuation to the space $(B, v, \tau)$.

We have proved that there exist sets of positive $\varepsilon_i$, $\varepsilon_i \in C^1$, to which two integral curves of the system (1.1) correspond.

The result obtained is also true under the assumptions adopted in [4].

## 6. Conclusion

The results obtained concern the limiting case corresponding to the thermal conductivity equal to zero. The thermal conductivity always causes some effects, but in some problems of wave structure they may be negligibly small compared with those due to other dissipation mechanisms. In such cases, to describe the wave structure it is convenient to adopt thermal conductivity as equal to zero. The results obtained guarantee the existence of the description of the fast and slow wave structures in the class of differentiable functions.

The results of Sec. 5 show that for certain sets of dissipation coefficients the system of magnetohydrodynamic equations does not determine uniquely the structure of slow shock waves. Perhaps some additional conditions imposed on the dissipation coefficients, following from physical arguments might eliminate this nonuniqueness.

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