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Lagrangian non-linear theory of shells

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THE VIRTUAL work principle is used to derive a two-dimensionally exact non-linear theory of shells. All quantities and relations are defined in or referred to some reference shell middle surface geometry. The theory is expressed in terms of symmetric stress resultant and stress couple tensors and, when linearized, leads to the "best" linear theory of shells.

Używając zasady prac wirtualnych zbudowano dwuwymiarowo ścisłą nieliniową teorię powłok. Wszystkie wielkości i zależności zdefiniowano lub opisano w geometrii ustalonej konfiguracji porównawczej powierzchni środkowej powłoki. Przedstawiona teoria zawiera tylko symetryczne tensory sił i momentów i w przypadku linearyzacji sprowadza się do tzw. "najlepszej" liniowej teorii powłok.

На основе принципа виртуальной работы построена двумерно точная нелинейная теория оболочек. Все величины и уравнения введены или представлены в терминах геометрии срединной поверхности оболочки в некотором начальном состоянии. Предлагаемая теория оболочек выражается через симметрические тензоры внутренних сил и моментов и в случае полной линеаризации приводит к „наилучшей" линейной теории оболочек.

1. Introduction

IN THE NON-LINEAR theory of thin shells, it is desirable to distinguish at the beginning between the Eulerian (spatial) and the Lagrangian (material) formulation, as is done in three-dimensional continuum mechanics. In the Eulerian approach, all quantities are referred to the actual deformed shell configuration, while in the Lagrangian approach they are referred to some reference, usually undeformed, shell configuration.

The Lagrangian non-linear theory of shells can be constructed directly by integration of the appropriate three-dimensional continuum equations over the shell thickness in the reference configuration. Either shell middle surface displacements and rotations [1, 2] or displacements only [3, 4] can be used as independent variables. The stress resultant and stress couple tensors so defined are unsymmetric and we still need to apply some symmetrization procedure [5, 6, 7, 8] if we want to transform them into symmetric quantities.

The appropriate non-linear shell equations can also be found using the two-dimensional virtual work principle [9, 10, 11]. In this equivallent approach, the stress resultant and stress couple tensors are from the very beginning symmetric by definition.

In this report, the new Lagrangian non-linear thin shell equations are derived using the two-dimensional virtual work principle, which is written entirely with respect to the reference middle surface geometry. We assume here that the deformation of the shell space can be represented entirely by three displacement components of the shell middle surface [10]. The theory presented here is two-dimensionally exact for the shell middle surface and is formulated without any restrictions on the surface strains, displacements or rotations.

It is possible to derive many variants of the non-linear Lagrangian theory of shells, which for small elastic strains would differ only by some small terms producing negligible error in the elastic strain energy [9]. Within the linear theory there are reasonable arguments [12] to choose the "best" variant of shell equations. Although some arguments in [12] do not apply directly to the non-linear theory, in our opinion the transition between the "best" linear and any non-linear shell theory should be smooth. The Lagrangian non-linear theory of shells presented in this report is compatible with this requirement and gives us the "best" linear theory in the limiting case.

2. Notations and basic relations

We will use here, as far as possible, the system of notations used in [10, 11, 13]. Let $\mathbf{r}(\theta^\alpha)$ and $\bar{\mathbf{r}}(\theta^\alpha)$ be the position vectors of the middle surface of a shell in the reference and deformed configurations, respectively, where θ^α, $\alpha = 1, 2$, are a pair of surface convected coordinates [14]. With the reference middle surface $\mathscr{S}$ we associate the standard surface base vectors $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$, the metric tensor $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$, the unit vector normal to the surface $\mathbf{n} = \frac{1}{2}\varepsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta$ and the curvature tensor $b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{n}$. Here, comma $(\ )_{,\alpha}$ denotes partial differentiation with respect to θ^α, and $\varepsilon^{\alpha\beta}$ is the permutation tensor. Similar geometrical quantities associated with the deformed middle surface will be distinguished by a dash — for example, $\bar{\mathbf{a}}_\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{\mathbf{n}}$, $\bar{b}_{\alpha\beta}$, $\bar{\varepsilon}^{\alpha\beta}$, etc. The surface covariant differentiation with respect to the reference and deformed metric will be denoted by $(\ )|_\alpha$ and $(\ )_{:\alpha}$, respectively.

During shell deformation, the basic vectors $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ can be expressed in terms of the reference middle surface geometry and displacement vector $\mathbf{u}$ by the relations [10, 11]:

$$\bar{\mathbf{a}}_\alpha = l^\varkappa_{\cdot\alpha}\mathbf{a}_\varkappa + \varphi_\alpha \mathbf{n}, \qquad \bar{\mathbf{n}} = n^\varkappa \mathbf{a}_\varkappa + n\mathbf{n}, \tag{2.1}$$

where

$$l^\varkappa_{\cdot\alpha} = \delta^\varkappa_\alpha + u^\varkappa{}_{|\alpha} - b^\varkappa_\alpha w, \qquad \varphi_\alpha = w_{,\alpha} + b^\varkappa_\alpha u_\varkappa,$$

$$n_\varkappa = \sqrt{\frac{a}{\bar{a}}}\,\varepsilon^{\alpha\beta}\varepsilon_{\lambda\varkappa}\varphi_\alpha l^\lambda_{\cdot\beta}, \qquad n = \frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\,\varepsilon^{\alpha\beta}\varepsilon_{\lambda\varkappa} l^\lambda_{\cdot\alpha} l^\varkappa_{\cdot\beta}, \tag{2.2}$$

$$\mathbf{u} = \bar{\mathbf{r}} - \mathbf{r} = u^\alpha \mathbf{a}_\alpha + w\mathbf{n}, \qquad a = |a_{\alpha\beta}|, \qquad \bar{a} = |\bar{a}_{\alpha\beta}|.$$

The surface Lagrangian strain tensor $\gamma_{\alpha\beta}$ and the tensor of change of surface curvature $\varkappa_{\alpha\beta}$ are defined by

$$\gamma_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}), \qquad \varkappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}), \tag{2.3}$$

where in terms of displacements u^α, w these tensors can be shown to be

$$\gamma_{\alpha\beta} = \frac{1}{2}(l^\varkappa_{\cdot\alpha} l_{\varkappa\beta} + \varphi_\alpha\varphi_\beta - a_{\alpha\beta}),$$

$$\varkappa_{\alpha\beta} = -[n_\varkappa(l^\varkappa_{\cdot\alpha|\beta} - b^\varkappa_\beta\varphi_\alpha) + n(\varphi_{\alpha|\beta} + b^\varkappa_\beta l_{\varkappa\alpha}) - b_{\alpha\beta}]. \tag{2.4}$$

In what follows, instead of $\varkappa_{\alpha\beta}$ we shall use the Lagrangian tensor of change of surface curvature $\varrho_{\alpha\beta}$ defined by

$$\varrho_{\alpha\beta} = \varkappa_{\alpha\beta} + \frac{1}{2}(b_\alpha^\varkappa \gamma_{\varkappa\beta} + b_\beta^\varkappa \gamma_{\alpha\varkappa}). \tag{2.5}$$

When linearized, this tensor gives us the measure for change of surface curvature which is supposed to be the "best" for the linear theory [12].

Our definitions for $\varkappa_{\alpha\beta}$ and $\varrho_{\alpha\beta}$ differ by sign from those used in [10] and will correspond to the usual sign convention for the stress couples used, for example, also in [14, 15].

3. Eulerian theory

In the Eulerian approach to the non-linear shell theory, all quantities are defined in or referred to deformed middle surface geometry. Let the shell be in equilibrium under the surface load $\bar{\mathbf{p}}$, per unit area of deformed middle surface $\bar{\mathscr{S}}$, and boundary force $\bar{\mathbf{F}}$ and couple $\bar{\mathbf{K}}$, per unit length of deformed middle surface boundary contour $\bar{\mathscr{C}}$. Then, for any additional virtual displacement field $\delta\mathbf{u}$, referred to deformed configuration

$$\delta\mathbf{u} = \delta\bar{u}_\alpha \bar{\mathbf{a}}^\alpha + \delta\bar{w}\,\bar{\mathbf{n}}, \tag{3.1}$$

the principle of the virtual work has the form [10]

$$\iint_{\bar{\mathscr{S}}} (n^{\alpha\beta}\delta\gamma_{\alpha\beta} + m^{\alpha\beta}\delta\varrho_{\alpha\beta})\,d\bar{A} = \iint_{\bar{\mathscr{S}}} \bar{\mathbf{p}}\cdot\delta\mathbf{u}\,d\bar{A} + \int_{\bar{\mathscr{C}}} (\bar{\mathbf{F}}\cdot\delta\mathbf{u} + \bar{\mathbf{K}}\cdot\delta\boldsymbol{\Omega})\,d\bar{s}, \tag{3.2}$$

where from (2.5)

$$\delta\varrho_{\alpha\beta} = \delta\varkappa_{\alpha\beta} + \frac{1}{2}(\bar{b}_\alpha^\varkappa \delta\gamma_{\varkappa\beta} + \bar{b}_\beta^\varkappa \delta\gamma_{\alpha\varkappa}), \tag{3.3}$$

and all $\delta\gamma_{\alpha\beta}$, $\delta\varkappa_{\alpha\beta}$ and $\delta\boldsymbol{\Omega}$ are understood to be referred here to the deformed middle surface geometry. They can easily be expressed as the linear functions of $\delta\bar{u}_\alpha$, $\delta\bar{w}$ [10, 11].

The symmetric stress resultant and stress couple tensors $n^{\alpha\beta}$ and $m^{\alpha\beta}$, defined here with respect to deformed surface geometry as the coefficients in the invariant internal virtual work expression (3.2), will be called Eulerian. These Eulerian tensors can also be introduced integrating the Cauchy stress tensor over the shell thickness in deformed configuration and then using some symmetrization procedure similar to that used in the linear theory [5, 6]. Although the two ways are shown to be equivallent [10], in the present variational approach we start at once with symmetric $n^{\alpha\beta}$ and $m^{\alpha\beta}$.

Applying to (3.2) standard procedure of variational calculus, it is easy to obtain the known [10] three equations of equilibrium and four natural boundary conditions, written in terms with respect to deformed middle surface geometry.

4. Lagrangian theory

During the non-linear deformation, usually some reference (undeformed) shell configuration is the only known in advance. It is desirable then to deal from the beginning entirely with the quantities defined in and referred to the reference (undeformed) middle surface geometry. Such shell theory we call here Lagrangian.

Let the shell be in equilibrium under the surface load $\mathbf{p}$, per unit area of the reference middle surface $\mathscr{S}$, and boundary force $\mathbf{F}$ and couple $\mathbf{K}$, per unit length of the reference middle surface boundary contour $\mathscr{C}$. Then, for any additional virtual displacement field $\delta\mathbf{u}$ expressed here with respect to the reference configuration by

$$\delta\mathbf{u} = \delta u_\alpha \mathbf{a}^\alpha + \delta w \mathbf{n}, \tag{4.1}$$

the principle of the virtual work can be postulated in the form entirely with respect to the reference geometry

$$\iint_{\mathscr{S}} (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\varrho_{\alpha\beta})\,dA = \iint_{\mathscr{S}} \mathbf{p}\cdot\delta\mathbf{u}\,dA + \int_{\mathscr{C}} (F\cdot\delta\mathbf{u} + \mathbf{K}\cdot\delta\boldsymbol{\Omega})\,ds, \tag{4.2}$$

where all $\delta\gamma_{\alpha\beta}$, $\delta\varrho_{\alpha\beta}$ and $\delta\boldsymbol{\Omega}$ are understood to be referred to the chosen reference middle surface configuration. We shall show them to be linear functions of $\delta\mathbf{u}$ and will express them in terms of displacement vector $\mathbf{u}$ from the reference to deformed shell configuration.

The symmetric stress resultant and stress couple tensors $N^{\alpha\beta}$ and $M^{\alpha\beta}$, defined here with respect to the reference surface geometry as the coefficients in the invariant internal virtual work expression (4.2), will be called Lagrangian.

To transform this variational problem (4.2), let us note first that in convected coordinates the Christoffel symbols of the second kind for deformed and reference surface geometry are related by [10]

$$\bar{\Gamma}^{\varkappa}_{\alpha\beta} = \Gamma^{\varkappa}_{\alpha\beta} + \bar{a}^{\varkappa\lambda}\gamma_{\lambda\alpha\beta}, \tag{4.3}$$

where

$$\begin{aligned} \bar{a}^{\varkappa\lambda} &= \frac{a}{\bar{a}}(a^{\varkappa\lambda} + 2\varepsilon^{\varkappa\alpha}\varepsilon^{\lambda\beta}\gamma_{\alpha\beta}), \\ \frac{\bar{a}}{a} &= \frac{1}{2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}(a_{\alpha\beta} + 2\gamma_{\alpha\beta})(a_{\lambda\mu} + 2\gamma_{\lambda\mu}), \\ \gamma_{\lambda\alpha\beta} &= \gamma_{\lambda\alpha|\beta} + \gamma_{\lambda\beta|\alpha} - \gamma_{\alpha\beta|\lambda}. \end{aligned} \tag{4.4}$$

Then we can find that with respect to the reference surface geometry

$$\begin{aligned} \delta\gamma_{\alpha\beta} &= \frac{1}{2}\delta(\bar{\mathbf{a}}_\alpha\cdot\bar{\mathbf{a}}_\beta) = \frac{1}{2}(\delta\mathbf{u}_{,\alpha}\cdot\bar{\mathbf{a}}_\beta + \bar{\mathbf{a}}_\alpha\cdot\delta\mathbf{u}_{,\beta}), \\ \delta\varkappa_{\alpha\beta} &= -\delta(\bar{\mathbf{a}}_{\alpha;\beta}\cdot\bar{\mathbf{n}}) = [(\delta\mathbf{u}_{,\alpha})_{,\beta} - \bar{a}^{\varkappa\lambda}\gamma_{\lambda\alpha\beta}\delta\mathbf{u}_{,\varkappa}]\cdot\bar{\mathbf{n}}, \end{aligned} \tag{4.5}$$

where (2.1) and (4.4) should still be applied. These relations are obviously the functions of $\mathbf{u}$ and are linear in $\delta\mathbf{u}$.

Let ν and $\mathbf{t}$ be the unit vectors, outward normal and tangent, respectively, to the reference middle surface boundary contour $\mathscr{C}$. Then, using (3.3), (4.5) and divergence theorem, the left-hand side of (4.2) can be transformed to the form

$$\iint_{\mathscr{S}} (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\varrho_{\alpha\beta})\,dA = \int_{\mathscr{C}} (\mathbf{N}^\alpha\cdot\delta\mathbf{u} - M^{\alpha\beta}\bar{\mathbf{n}}\cdot\delta\mathbf{u}^{\blacksquare}_{,\beta})\nu_\alpha\,ds - \iint_{\mathscr{S}} \mathbf{N}^\alpha|_\alpha\cdot\delta\mathbf{u}\,dA, \tag{4.6}$$

where

$$\begin{aligned} \mathbf{N}^{\alpha} &= Q^{\alpha\beta}\bar{\mathbf{a}}_{\beta}+Q^{\alpha}_{\blacksquare}\bar{\mathbf{n}}, \\ Q^{\alpha\beta} &= N^{\alpha\beta}+\frac{1}{2}\bar{b}^{\alpha}_{\varkappa}M^{\varkappa\beta}-\frac{1}{2}\bar{b}^{\beta}_{\varkappa}M^{\alpha\varkappa}, \\ Q^{\alpha} &= M^{\alpha\beta}|_{\beta}+\bar{a}^{\alpha\nu}\gamma_{\nu\lambda\mu}M^{\lambda\mu}, \\ \bar{b}^{\alpha}_{\varkappa} &= \bar{a}^{\alpha\nu}(b_{\nu\varkappa}-\varkappa_{\nu\varkappa}), \end{aligned} \tag{4.7}$$

and where (2.1) and (4.4) should still be applied.

The second part of the line integral in (4.6) can be transformed further and found to be

$$-\int_{\mathscr{C}} M^{\alpha\beta}\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\beta}\nu_{\alpha}ds=\int_{\mathscr{C}}\bar{b}^{\beta}_{\varkappa}M^{\alpha\varkappa}\bar{\mathbf{a}}_{\beta}\cdot\delta\mathbf{u}\nu_{\alpha}ds+\int_{\mathscr{C}}[(M^{\alpha\beta}\nu_{\alpha}t_{\beta})_{,t}\bar{\mathbf{n}}\cdot\delta\mathbf{u} -(M^{\alpha\beta}\nu_{\alpha}\nu_{\beta})(\bar{\mathbf{n}}\cdot\delta\mathbf{u})_{,\nu}]ds, \tag{4.8}$$

where by $(\ )_{,\nu}$ and $(\ )_{,t}$ we understand directional derivatives at the reference middle surface boundary contour $\mathscr{C}$ in directions defined by $\boldsymbol{\nu}$ and $\mathbf{t}$, respectively.

The virtual rotation vector $\delta\boldsymbol{\Omega}$, with respect to the reference geometry, can be shown to be

$$\delta\boldsymbol{\Omega}=\frac{1}{2}(\bar{\mathbf{a}}^{\alpha}\times\delta\bar{\mathbf{a}}_{\alpha}+\bar{\mathbf{n}}\times\delta\bar{\mathbf{n}})=\frac{1}{2}\left[\bar{\mathbf{a}}^{\alpha}\times\delta\mathbf{u}_{,\alpha}+\sqrt{\frac{a}{\bar{a}}}\,\varepsilon^{\alpha\beta}\bar{\mathbf{n}}\times(\delta\mathbf{u}_{,\alpha}\times\bar{\mathbf{a}}_{\beta})\right], \tag{4.9}$$

where (2.1) and (4.4) should still be applied. The relation is also the function of $\mathbf{u}$ and is linear in $\delta\mathbf{u}$.

Using (4.9), the last line integral in (4.2) can be transformed to be

$$\int_{\mathscr{C}}\mathbf{K}\cdot\delta\boldsymbol{\Omega}ds=\int_{\mathscr{C}}\bar{b}^{\beta}_{\varkappa}R^{\varkappa}\bar{\mathbf{a}}_{\beta}\cdot\delta\mathbf{u}ds+\int_{\mathscr{C}}[(R^{\beta}t_{\beta})_{,t}\bar{\mathbf{n}}\cdot\delta\mathbf{u}-(R^{\beta}\nu_{\beta})(\bar{\mathbf{n}}\cdot\delta\mathbf{u})_{,\nu}]ds, \tag{4.10}$$

where

$$R^{\beta}=\sqrt{\frac{a}{\bar{a}}}\,\varepsilon^{\beta\alpha}\mathbf{K}\cdot\bar{\mathbf{a}}_{\alpha}. \tag{4.11}$$

Finally, by means of (4.6), (4.8) and (4.10) we can show that the virtual work principle (4.2) requires the following Lagrangian vector equation of equilibrium

$$\mathbf{N}^{\alpha}|_{\alpha}+\mathbf{p}=\mathbf{0} \tag{4.12}$$

to be satisfied within $\mathscr{S}$ and the following Lagrangian natural boundary conditions

$$\begin{aligned} (\mathbf{N}^{\alpha}+M^{\alpha\varkappa}\bar{\mathbf{n}}_{,\varkappa})\nu_{\alpha}+(M^{\alpha\beta}\nu_{\alpha}t_{\beta})_{,t}\bar{\mathbf{n}} &= \mathbf{F}+R^{\varkappa}\bar{\mathbf{n}}_{,\varkappa}+(R^{\beta}t_{\beta})_{,t}\bar{\mathbf{n}}, \\ M^{\alpha\beta}\nu_{\alpha}\nu_{\beta} &= R^{\beta}\nu_{\beta} \end{aligned} \tag{4.13}$$

to be satisfied at free $\mathscr{C}$.

The component form of these vector relations, with respect to the reference triad of base vectors $\mathbf{a}_{\alpha}$, $\mathbf{n}$, can be found to be the equilibrium equations

$$\begin{aligned} (Q^{\alpha\lambda}l^{\beta}_{\cdot\lambda}+Q^{\alpha}n^{\beta})_{|\alpha}-b^{\beta}_{\alpha}(Q^{\alpha\lambda}\varphi_{\lambda}+Q^{\alpha}n)+p^{\beta} &= 0, \\ (Q^{\alpha\lambda}\varphi_{\lambda}+Q^{\alpha}n)_{|\alpha}+b_{\alpha\beta}(Q^{\alpha\lambda}l^{\beta}_{\cdot\lambda}+Q^{\alpha}n^{\beta})+p &= 0, \end{aligned} \tag{4.14}$$

and the natural force boundary conditions

$$
\begin{aligned}
&\{(Q^{\alpha\beta}-\bar{b}^{\beta}_{\varkappa}M^{\alpha\varkappa})\nu_{\alpha}l^{\lambda}_{\cdot\beta}+[Q^{\alpha}\nu_{\alpha}+(M^{\alpha\beta}\nu_{\alpha}t_{\beta})_{,t}]n^{\lambda}\}\nu_{\lambda} = \{F^{\lambda}-\bar{b}^{\beta}_{\varkappa}R^{\varkappa}l^{\lambda}_{\cdot\beta}+(R^{\beta}t_{\beta})_{,t}n^{\lambda}\}\nu_{\lambda},\\
&\{(Q^{\alpha\beta}-\bar{b}^{\beta}_{\varkappa}M^{\alpha\varkappa})\nu_{\alpha}l^{\lambda}_{\cdot\beta}+[Q^{\alpha}\nu_{\alpha}+(M^{\alpha\beta}\nu_{\alpha}t_{\beta})_{,t}]n^{\lambda}\}t_{\lambda} = \{F^{\lambda}-\bar{b}^{\beta}_{\varkappa}R^{\varkappa}l^{\lambda}_{\cdot\beta}+(R^{\beta}t_{\beta})_{,t}n^{\lambda}\}t_{\lambda},\\
&(Q^{\alpha\beta}-\bar{b}^{\beta}_{\varkappa}M^{\alpha\varkappa})\nu_{\alpha}\varphi_{\beta}+[Q^{\alpha}\nu_{\alpha}+(M^{\alpha\beta}\nu_{\alpha}t_{\beta})_{,t}]n = F-b^{\beta}_{\varkappa}R^{\varkappa}\varphi_{\beta}+(R^{\beta}t_{\beta})_{,t}n,
\end{aligned}
\tag{4.15}
$$

where (4.7) and (4.4) still have to be applied.

To complete the theory, some two-dimensional constitutive equations should be given. The appropriate constitutive equations for an elastic material follow from (4.2) to be:

$$
N^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial V}{\partial\gamma_{\alpha\beta}}+\frac{\partial V}{\partial\gamma_{\beta\alpha}}\right), \qquad M^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial V}{\partial\varrho_{\alpha\beta}}+\frac{\partial V}{\partial\varrho_{\beta\alpha}}\right),
\tag{4.16}
$$

where $V = V(\gamma_{\alpha\beta}, \varrho_{\alpha\beta})$ is a two-dimensional strain energy function per unit area of the reference shell middle surface.

Thus the equations of equilibrium (4.14), the natural boundary conditions (4.15) and $(4.13)_2$, the strain-displacement relations (2.4) and (2.5), the constitutive equatious (4.16) together with the relations (2.2), (4.4) and (4.7) form the set of equations for the Lagrangian non-linear elastic shell theory to be solved.

5. Lagrangian and Eulerian quantities

The external Lagrangian and Eulerian forces and couples in (3.2) and (4.2) are supposed to be known in component form:

$$
\begin{aligned}
&\mathbf{p} = p^{\alpha}\mathbf{a}_{\alpha}+p\mathbf{n}, && \bar{\mathbf{p}} = \bar{p}^{\alpha}\bar{\mathbf{a}}_{\alpha}+\bar{p}\bar{\mathbf{n}},\\
&\mathbf{F} = F^{\alpha}\mathbf{a}_{\alpha}+F\mathbf{n}, && \bar{\mathbf{F}} = \bar{F}^{\alpha}\bar{\mathbf{a}}_{\alpha}+\bar{F}\bar{\mathbf{n}},\\
&\mathbf{K} = \varepsilon_{\alpha\beta}K^{\alpha}\mathbf{a}^{\beta}+K\mathbf{n}, && \bar{\mathbf{K}} = \bar{\varepsilon}_{\alpha\beta}\bar{K}^{\alpha}\bar{\mathbf{a}}^{\beta}.
\end{aligned}
\tag{5.1}
$$

Let us recall here the geometrical relations [16] between the reference and deformed configuration:

$$
\begin{aligned}
&d\bar{A} = \sqrt{\frac{\bar{a}}{a}}\,dA, \qquad d\bar{s} = \sqrt{1+2\gamma_{\alpha\beta}t^{\alpha}t^{\beta}}\,ds\\
&\bar{\nu}_{\alpha}d\bar{s} = \sqrt{\frac{\bar{a}}{a}}\,\nu_{\alpha}ds, \qquad \bar{t}_{\alpha}d\bar{s} = (t_{\alpha}+2\gamma_{\alpha\beta}t^{\beta})ds.
\end{aligned}
\tag{5.2}
$$

It follows from (3.2) and (4.2) that under transformations $(5.2)_1$ we have the following relations between the Lagrangian and Eulerian quantities:

$$
\begin{aligned}
&N^{\alpha\beta} = \sqrt{\frac{\bar{a}}{a}}\,n^{\alpha\beta}, \qquad M^{\alpha\beta} = \sqrt{\frac{\bar{a}}{a}}\,m^{\alpha\beta}, \qquad \bar{\mathbf{p}} = \sqrt{\frac{\bar{a}}{a}}\,\mathbf{p},\\
&\bar{\mathbf{F}} = \frac{1}{\sqrt{1+2\gamma_{\alpha\beta}t^{\alpha}t^{\beta}}}\mathbf{F}, \qquad \bar{\mathbf{K}} = \frac{1}{\sqrt{1+2\gamma_{\alpha\beta}t^{\alpha}t^{\beta}}}\mathbf{K}.
\end{aligned}
\tag{5.3}
$$

The transformation rules for various components in (5.1) now follow easily from (5.3), using the relations (2.1), (2.4), (4.4) and (4.7), [16].

6. Discussion

The Lagrangian theory presented in this report may be characterized by the following features: a) the stress resultant and stress couple tensors are symmetric and defined with respect to the reference geometry, b) the equilibrium equations and the boundary conditions are expressed entirely in terms of the reference geometry, c) when linearized, our theory leads to the "best" linear shell equations [12].

It is interesting to compare our results with some close results obtained by other authors, keeping in mind some differences in notations and sign conventions.

The Lagrangian equilibrium equations presented shortly in [7] (Eqs. $58'-60'$) were also expressed in terms of symmetric stress resultant and stress couple tensors, which are different from those used in this report. When linearized, the equilibrium equations of [7] do not lead to those of the "best" linear theory, while ours do.

The vector equilibrium equation (4.12) and boundary conditions (4.13) in our theory are resolved along the reference triad of base vectors $\mathbf{a}_\alpha, \mathbf{n}$, while in [18] they have been resolved along the deformed triad of base vectors $\bar{\mathbf{a}}_\alpha, \bar{\mathbf{n}}$, and in [16] along some intermediate triad obtained from the reference triad by a finite rotation vector. This difference seems to be quite formal when the component tensor notation in convected coordinates is used. However, using direct tensor notation, it is easy to define [17] some second Piola-Kirchhoff stress resultant and stress couple dyadics for the shell, in analogy with the second Piola-Kirchhoff stress dyadic in continuum mechanics. The natural proper components of these dyadics are only those defined in the reference basis $\mathbf{a}_\alpha, \mathbf{n}$, and from this point of view only our theory can be called fully Lagrangian.

Using the special definition for the tensor of change of surface curvature, all the relations in [8] have been expressed as polynomials in displacements. Unfortunately, the resulting equilibrium equations happen to be very complicated and have not been presented explicitly in [8]. In our theory, there is a non-rational square root function in definitions $(2.2)_2$, but the symmetry of our explicitly written equilibrium equations (4.14) is remarkable. Both theories agree within the membrane approximation and they seem to be equivalent also in the general case.

We have referred in this report to the shell middle surface rather than to the shell itself. The results presented here are thus two-dimensionally exact for the surface. It means that no restrictions have been imposed on the middle surface strains, displacements or rotations. However, any shell theory introduces unavoidable error resulting from two-dimensionality of the theory for an actually three-dimensional thin body. The theory presented here is applicable only to those non-linear shell problems for which the deformation of the shell space can really be well approximated (in some definite sense) by deformation of the shell middle surface. The full discussion of the range of validity and possible simplifications of the theory for various shell problems transcends the scope of this report. We only note here that for elastic shells undergoing small extensional strains the shell strain energy was shown [9] to be well approximated by the two-dimensional strain energy due to the bending and stretching of the shell middle surface only. In this case, the Lagrangian shell relations can be simplified according to the lines suggested in [10, 18].

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