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Some problems of double waves in magnetohydrodynamics

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IT WAS shown in the paper, that an interaction process of two magnetoacoustic simple waves for the MHD system can be described always by a system of Riemann invariants. The hodograph of this interaction process, being the solution of the MHD equations, is determined by a partial differential equation of second order for a density fluid function with independent variables as components of the fluid velocity. In the case of the cylindrycal hodograph this problem is reduced to analysis of the ordinary differential equations. The posibility of the linear interaction process of two magnetoacoustic simple waves was presented.

W pracy wykazano, że proces oddziaływania dwóch fal prostych magnetoakustycznych dla układu MHD może być zawsze opisany przez układ inwariantów Riemanna. Hodograf tego oddziaływania, będącego rozwiązaniem równań MHD, jest określony przez równanie różniczkowe cząstkowe rzędu drugiego dla funkcji gęstości cieczy, gdzie zmiennymi niezależnymi są składowe wektora prędkości cieczy. W przypadku hodografu cylindrycznego problem sprowadza się do badania równań różniczkowych zwyczajnych. Ponadto przedstawiono możliwość liniowego oddziaływania fal prostych magnetoakustycznych.

В работе показано, что процесс взаимодействия двух магнитоакустических простых волн для системы МГД может быть всегда описан при помощи системы инвариантов Риманна. Годограф этого взаимодействия, являющегося решением уравнений МГД, определен дифференциальным уравнением в частных производных второго порядка относительно плотности жидкости, где независимыми переменными являются составляющие вектора скорости течения жидкости. В случае цилиндрического годографа задача сводится к исследованию обыкновенных дифференциальных уравнений. Показана возможность линейного взаимодействия простых магнитоакустических волн.

ϱ	density of the fluid,
p	pressure of the fluid,
$\mathbf{u} = (v_1, v_2)$	velocity of the fluid,
H	a magnetic field,
E	physical space,
$\mathscr{H}$	hodograph space,
$x = (t, x, y)$	coordinates of E,
$u = (\varrho, p, v_1, v_2, H)$	coordinates of $\mathscr{H}$,
$\gamma = (\gamma_\varrho, \gamma_p, \gamma_1, \gamma_2, h)$	characteristic vector from $\mathscr{H}$,
$\lambda = (\lambda_0, \lambda_1, \lambda_2) \equiv (\lambda_0, \boldsymbol{\lambda})$	characteristic covector from E,
$\delta = \lambda_0 + \mathbf{u} \cdot \boldsymbol{\lambda}$	velocity of magnetoacoustic wave regard to a moving media,
R^1, R^2	Riemann invariants,
$\varkappa$	adiabatic exponent,
σ	direction in which every double wave is constant.

1. Introduction

IT HAS been shown [7] that the process of interaction of two, in general non parallel ([1]), acoustic simple waves in gasdynamics may be described by an appropriate system of two

([1]) The case in which the directions of propagation are parallel was solved by Riemann (1869), [10].

Riemann invariants (by double waves). This fact may be physically expressed in this that the result of the interaction of two simple waves being also two simple waves of this same kind. Thus the process of interaction changes only the profiles and directions of propagation of the waves. In the present paper it is shown that for interaction of two magnetoacoustic waves in the case in which magnetic field is perpendicular to the direction of propagation of the waves, there exists also a system of two Riemann invariants describing the interaction process. We have assumed that the flow is an ideal isentropic flow with infinite conductivity. Similarly as in paper [7], the hodograph of this interaction process is described by a second-order partial differential equation. In this paper is analysed a particular solution of this equation for a cylindrical hodograph. On the cylindrical hodograph will be exemplified the procedure for constructing a solution of this equation. In this paper we apply the method developed in [1, 2, 3, 4, 5, 10].

2. Simple elements

We have to deal with equations of MHD in the case in which a vector of magnetic field has a constant direction and is perpendicular to the motion plane. Under these assumptions, the equations of magnetohydrodynamics have the form:

$$\begin{aligned}&\frac{d}{dt}\varrho+\varrho\operatorname{div}\mathbf{u}=0, \quad \varrho\frac{d\mathbf{u}}{dt}+\nabla\left(p+\frac{H^2}{8\pi}\right)=0,\\ &\frac{dH}{dt}+H\operatorname{div}\mathbf{u}=0, \quad \frac{d}{dt}\left(\frac{p}{\varrho^{\varkappa}}\right)=0,\end{aligned}\tag{2.1}$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla \text{ and } \varrho=\varrho(t,x,y),\ \mathbf{u}=\mathbf{u}(t,x,y),$$

$p=p(t,x,y)$, $H=H(t,x,y)$. By $u=(\varrho, p, v^1, v^2, H)$ we denote coordinates of hodograph space $\mathscr{H}$ and by $x=(t,x,y)$ coordinates of physical space E, so that $\dim\mathscr{H}=5$ and $\dim E=3$. At first we seek equations for simple elements, and hence we put $\gamma^j\lambda^\nu$, $j=1,\ldots,5$, $\nu=1,2,3$ into the Eqs. (2.1) instead of the derivatives $u^j_{x^\nu}$. Then we get the system of equations for characteristic vectors γ from the hodograph space and characteristic covectors λ from the space E^* dual to physical space. The coordinates of vector γ and covector λ we shall denote as follows: $\gamma=(\gamma_\varrho, \gamma_p, \gamma_1, \gamma_2, h)$. $\lambda=(\lambda_0, \lambda_1, \lambda_2)$. Finally, we introduce the denotation $\delta=\lambda_0+\mathbf{u}\cdot\boldsymbol{\lambda}$, where $\mathbf{u}=(v^1, v^2)$, $\boldsymbol{\lambda}=(\lambda^1,\lambda^2)$ and hence the system of Eqs. (2.1) will receive the form:

$$\begin{aligned}&\delta\gamma_\varrho+\varrho\boldsymbol{\gamma}\cdot\boldsymbol{\lambda}=0, \quad \varrho\delta\boldsymbol{\gamma}+\boldsymbol{\lambda}\left(\gamma_p+\frac{Hh}{4\pi}\right)=0,\\ &\delta h+H\boldsymbol{\gamma}\cdot\boldsymbol{\lambda}=0, \quad \delta\left(\gamma_p-\frac{\varkappa p}{\varrho}\gamma_\varrho\right)=0.\end{aligned}\tag{2.2}$$

The condition of existence of solutions for the vector γ has the form:

$$\delta^3\left[\delta^2-\left(\frac{H^2}{4\pi\varrho}+\frac{\varkappa p}{\varrho}\right)\right]=0.\tag{2.3}$$

It is the characteristic determinant for the Eqs. (2.2). The Eq. (2 3) gives us two families of simple elements:

A. Entropic elements if $\delta = 0$. Then

$$\gamma = \left(\gamma_\varrho, -\frac{Hh}{4\pi}, \lambda_2, -\lambda_1, h\right), \quad \lambda = (-\mathbf{u}\cdot\boldsymbol{\lambda}, \boldsymbol{\lambda}). \tag{2.4}$$

B. Magnetoacoustic fast elements if $\delta = \pm\sqrt{\dfrac{\varkappa p}{\varrho} + \dfrac{H^2}{4\pi\varrho}}$

$$\gamma = \left(\varrho, \varkappa p, -\delta\frac{\boldsymbol{\lambda}}{|\boldsymbol{\lambda}|}, H\right), \quad \lambda = (\delta|\lambda| - \mathbf{u}\cdot\boldsymbol{\lambda}, \boldsymbol{\lambda}). \tag{2.5}$$

Some simple waves correspond to these simple elements. The method of constructing simple waves from simple elements is described in paper [5].

3. The hodograph problem

Now we restrain the interaction between the magnetoacoustic simple waves. The hodograph of double wave (the image of transformation $u: E \to \mathscr{H}$), $u = u(R^1, R^2)$, where $u = (\varrho, p, v^1, v^2, H)$ and R^1, R^2 are Riemann invariants, parametrized by two Riemann invariants, is described by the following system of partial differential equations:

$$\begin{gathered} \varrho_{,R^i} = \nu_i\varrho, \quad p_{,R^i} = \nu_i \varkappa p, \\ \mathbf{u}_{,R^i} = -\nu_i\delta\frac{\boldsymbol{\lambda}_i}{|\lambda_i|}, \quad H_{,R^i} = \nu_i H, \end{gathered} \tag{3.1}$$

$$\begin{gathered} (\delta|\boldsymbol{\lambda}_i| - \mathbf{u}\cdot\boldsymbol{\lambda}_i)_{,R^j} = \mu_{ij}(\delta|\boldsymbol{\lambda}_i| - \mathbf{u}\cdot\boldsymbol{\lambda}_i) + \nu_{ij}(\delta|\boldsymbol{\lambda}_j| - \mathbf{u}\cdot\boldsymbol{\lambda}_j), \\ \boldsymbol{\lambda}_{i,R^j} = \mu_{ij}\boldsymbol{\lambda}_i + \nu_{ij}\boldsymbol{\lambda}_j, \end{gathered} \tag{3.2}$$

where $i \neq j$; $i, j = 1,2$ and $\nu_i, \mu_{ij}, \nu_{ij}$ are arbitrary functions. From the Eqs. (3.1) we obtain the following relations:

$$p = A\varrho_\varkappa, \quad H = C\varrho, \tag{3.3}$$

$$\boldsymbol{\lambda}_i = -\frac{\varrho}{\delta\varrho_{,R^i}}\mathbf{u}_{,R^i}, \quad i = 1, 2, \quad |\boldsymbol{\lambda}_i| = 1, \tag{3.4}$$

where A, C are constants of integration.

The Eqs. (3.4) imply the conditions:

$$\frac{\delta\varrho_{,R^i}}{\varrho} = |\mathbf{u}_{,R^i}|, \tag{3.5}$$

$$\left|\frac{\varrho^2\mathbf{u}_{,R^1}\cdot\mathbf{u}_{,R^2}}{\delta^2\varrho_{,R^1}\varrho_{,R^2}}\right| \leqslant 1, \tag{3.6}$$

where $|\mathbf{u}_{,R^i}|$ are the magnitudes of the vectors $\mathbf{u}_{,R^i}$. The conditions of involution Eqs. (3.2) for covectors

$$\lambda_i = \left(\frac{\delta^2}{\varrho}\varrho_{,R^i} + \mathbf{u}\cdot\mathbf{u}_{,R^i} - \mathbf{u}_{,R^i}\right), \quad i = 1, 2$$

receive the form:

$$\left(\frac{\delta^2}{\varrho}\varrho_{,R^i}+\mathbf{u}\cdot\mathbf{u}_{,R^i}\right)_{,R^j}=\mu_{ij}\left(\frac{\delta^2}{\varrho}\varrho_{,R^i}+\mathbf{u}\cdot\mathbf{u}_{,R^i}\right)+\nu_{ij}\left(\frac{\delta^2}{\varrho}\varrho_{,R^j}+\mathbf{u}\cdot\mathbf{u}_{,R^j}\right),\tag{3.7}$$

$$\mathbf{u}_{,R^iR^j}=\mu_{ij}\mathbf{u}_{,R^i}+\nu_{ij}\mathbf{u}_{,R^j},\tag{3.8}$$

$i \neq j$, $i, j = 1, 2$ and the summation convention is not used.

From the Eqs. (3.8) it follows that $\mu_{12}=\nu_{21}$, $\mu_{21}=\nu_{12}$, so the set of Eqs. (3.7), (3.8) reduce to three independent equations.

After eliminating the arbitrary functions μ_{ij}, ν_{ij}, the Eqs. (3.7), (3.8) are reduced to one equation of the form:

$$\begin{aligned}\left(\frac{\delta^2}{\varrho}\right)_{,\varrho}\varrho_{,R^1}\varrho_{,R^2}+\frac{\delta^2}{\varrho}\varrho_{,R^1R^2}+\mathbf{u}_{,R^1}\cdot\mathbf{u}_{,R^2}+\mathbf{u}\cdot\mathbf{u}_{,R^1R^2}=\\=\frac{\det\|\mathbf{u}_{,R^1R^2},\mathbf{u}_{,R^2}\|}{\det\|\mathbf{u}_{,R^1},\mathbf{u}_{,R^2}\|}\frac{\delta^2}{\varrho}\varrho_{,R^1}+\frac{\det\|\mathbf{u}_{,R^1},\mathbf{u}_{,R^1R^2}\|}{\det\|\mathbf{u}_{,R^1},\mathbf{u}_{,R^2}\|}\frac{\delta^2}{\varrho}\varrho_{,R^2}\end{aligned}\tag{3.9}$$

where

$$\delta^2=A\varkappa\varrho^{\varkappa-1}+\frac{C^2\varrho}{4\pi}.$$

The system of Eqs. (3.5), (3.9) describes the hodograph interaction of two magneto-acoustic waves, for which there exist Riemann invariants and the condition (3.6) must be satisfied.

From the Eqs. (3.5), (3.9) we eliminate the Riemann invariants R^1, R^2 by exchange of independent variables $R^1, R^2 \to v^1, v^2$; thus we obtain one equation of the form:

$$\begin{aligned}\frac{1}{2}\left[\frac{\delta^2}{\varrho^2}-\left(\frac{\varrho^2}{\varrho}\right)_{,\varrho}\right](\varrho_{,v1}^2+\varrho_{,v2}^2)-2+\frac{\delta^2}{\varrho}\left[\varrho^2_{,v1v1}\left(\varrho^2_{,v2}\frac{\delta^2}{\varrho^2}-1\right)\right.\\\left.-2\varrho_{,v1}\varrho_{,v2}\varrho_{,v1v2}\frac{\delta^2}{\varrho^2}+\varrho_{,v2v2}\left(\varrho^2_{,v1}\frac{\delta^2}{\varrho^2}-1\right)\right]=0\end{aligned}\tag{3.10}$$

for the unknown function ϱ.

The condition (3.6), which assures the existence of characteristics, takes the form:

$$\frac{\delta^2}{\varrho^2}(\varrho_{v1}^2+\varrho_{,v2}^2)\geqslant 1.\tag{3.11}$$

Thus the hodograph of the double wave determined by the Eqs. (3.1), (3.2) is described by a system of functions:

$$v^1=v^1,\quad v^2=v^2,\quad \varrho=\varrho(v^1,v^2),\quad p=A\varrho^{\varkappa}(v^1,v^2),\quad H=C\varrho(v^1,v^2),\tag{3.12}$$

where A, C are constants and ϱ satisfy the Eq. (3.10) and the condition (3.11). Equation

$$\left(\varrho_{v1}^2\frac{\delta^2}{\varrho^2}-1\right)(\dot v^1)^2+2\varrho_{v1}\varrho_{v2}\dot v^1\dot v^2-\left(\varrho_{v2}^2\frac{\delta^2}{\varrho^2}-1\right)(\dot v^2)^2=0$$

determining a characteristic curve $(\varrho(s), v^1(s), v^2(s))$ from the Eq. (3.5) has the same form as the equation defining characteristic curves for the Eq. (3.10). Then the Eq. (3.10) is a hyperbolic one on a hodograph surface determined by the Eq. (3.12). This suggests the Theorem 5.1 from paper [7].

THEOREM. *If Γ_1, Γ_2 are two characteristic curves in the hodograph space (with the exception of the case in which Γ_1, Γ_2 lies on a plane $Av^1+Bv^2-C=0$) passing through a point* $(\varrho_0, v_0^1, v_0^2)$ i.e. $(\varrho_0, v_0^1, v_0^2) \in \Gamma_1 \cap \Gamma_2$ *— then*

1) *in some neighbourhood of the point* $(\varrho_0, v_0^1, v_0^2)$ *there exists a solution of the Eq.* (3.10) *which passes through the lines Γ_1, Γ_2.*

2) *this solution is uniquely determined.*

This theorem may be understood as the solution of the problem of simple wave — simple wave interaction for system (2.1).

It is useful sometimes to seek solutions in the implicit form:

$$F(\varrho, v^1, v^2) = 0. \tag{3.13}$$

Expressing the Eq. (3.10) and the condition (3.11) in terms of $F(\varrho, v^1, v^2)$, we arrive at

$$\begin{aligned}\frac{1}{2}\left[\frac{\delta^2}{\varrho^2}-\left(\frac{\delta^2}{\varrho}\right)_{,\varrho}\right](F_{v1}^2+F_{v2}^2)F_\varrho-2F_\varrho^3+\frac{\delta^2}{\varrho}\,[F_{\varrho\varrho}(F_{v1}^2+F_{v2}^2)-2F_\varrho(F_{\varrho v1}F_{11}\\+F_{\varrho v2}F_{v2})-(F_{v1v1}F_{v2}^2-2F_{v1}F_{v2}F_{v1v2}+F_{v2v2}F_{v1}^2)+(F_{v2v1}+F_{v2v2})F_\varrho^2]=0\end{aligned} \tag{3.14}$$

and

$$F_\varrho^2 \leqslant \frac{\delta^2}{\varrho^2}(F_{v1}^2+F_{v2}^2). \tag{3.15}$$

The set of solutions of the Eq. (3.14) contains the set of solutions of the Eq. (3.10) and solutions of the form: $F = \varphi(v^1, v^2)-\varrho$. If we put $C = 0$ (otherwise $H \to 0$) in the Eqs. (3.14), (3.15), we obtain analogous equations for gasdynamics, which can be found in paper [7].

The equations, which possess only magnetoacoustic simple elements, have the form:

$$\frac{d\varrho}{dt}+\varrho\,\mathrm{div}\,\mathbf{u}=0, \quad \varrho\frac{d\mathbf{u}}{dt}+\nabla\left(\frac{C^2\varrho^2}{8\pi}+A\varrho^\varkappa\right)=0, \quad \mathrm{rot}\,\mathbf{u}=0. \tag{3.16}$$

The system (3.16) is the Q_1-system; hence every solution of one may be constructed from simple magnetoacoustic elements. Moreover, the double wave is a solution of this system and also a set of solutions of this system contains some non-linear superposition of three and more magnetoacoustic waves.

It is easy to see that the Eqs. (2.1) are invariant under the following homotetia transformation:

$$\begin{aligned}(\varrho, p, \mathbf{u}, H) &\to (\sigma\varrho, \delta p, \sigma^{\frac{\varkappa-1}{2}}\delta^{1/2}\mathbf{u}, \delta^{1/2}\sigma^{\varkappa/2}H),\\ (t, x^1, x^2) &\to (\beta t, \beta\delta^{1/2}\sigma^{\frac{\varkappa-1}{2}}x^1, \beta\delta^{1/2}\sigma^{\frac{\varkappa-1}{2}}x^2),\end{aligned} \tag{3.17}$$

where $\sigma > 0$, $\delta > 0$, $\beta \neq 0$ are constants.

For the Eqs. (3.16), the general homotetia transformation (3.17) reduces for $\varkappa \neq 2$ to one parametr group:

$$\begin{aligned}(\varrho, p, \mathbf{u}, H) &\to (\varrho, p, \mathbf{u}, H),\\ (t, x^1, x^2) &\to (\beta t, \beta x^1, \beta x^2),\end{aligned} \tag{3.18}$$

and for $\varkappa = 2$ to:

$$(3.19)\qquad \begin{aligned} &(\varrho, p, \mathbf{u}, H) \to (\alpha\varrho, \alpha^2 p, \sqrt{\alpha}\mathbf{u}, \alpha H), \\ &(t, x^1, x^2) \to (\beta t, \beta x^1, \beta x^2). \end{aligned}$$

For $\varkappa = 2$ the Eqs. (3.16) reduce to hydrodynamic equations of the form:

$$(3.20)\qquad \frac{da}{dt} + \frac{1}{2} a \operatorname{div}\mathbf{u} = 0, \qquad \frac{d\mathbf{u}}{dt} + 2a\nabla a = 0$$

where $a = 2\left(A + \frac{C^2}{8\pi}\right)\varrho$, $p = A\varrho^2$, $H = C\varrho$. The Eqs. (3.20) were analysed in paper [7], where a plays the role of sound velocity.

4. Cylindrical hodographs

Now we shall seek solutions to the basic Eqs. (2.1), hodographs of which are described by the Eq. (3.14), and which have the form of cylindrical surfaces. First, we restrict our considerations to the function F of the form $F = v^1 - \varphi(v^2)$, substitution of which into the Eq. (3.14) yields:

$$(4.1)\qquad \ddot{\varphi} = 0.$$

Here, dots denote differentiation with respect to v^2. Thus, besides the planes (describing one-dimensional flows), there are no hodographs in the form $F(v^1, v^2) = 0$.

In a manner similar to [7], we may consider the hodograph given by the relation $F \equiv \equiv v^1 - \varphi(\varrho)$; substituting $v^1 - \varphi(\varrho)$ for the function $F(\varrho, v^1, v^2)$ into the Eq. (3.14), we obtain the following ordinary differential equation:

$$(4.2)\qquad \left[\frac{c^2}{4\pi} + A\varkappa\varrho^{\varkappa-2}\right]\ddot{\varphi} + \left[\frac{c^2}{4\pi} - A\varkappa(\varkappa-3)\varrho^{\varkappa-2}\right]\frac{\dot{\varphi}}{\varrho} - 2\dot{\varphi}^2 = 0,$$

where the dot denotes differentiation with respect to ϱ. This equation which is known as an Abel equation of the first kind, has solution:

$$(4.3)\qquad \begin{aligned} &v^1 = \varphi = \pm \int \frac{\alpha + A\varkappa\varrho^{\varkappa-2}}{\varrho} \frac{d\varrho}{\sqrt{C_1 + \frac{4\alpha}{\varrho} - \frac{4A\varkappa}{\varkappa-3}\varrho^{\varkappa-3}}} + C_2 \quad \text{for} \quad \varkappa \neq 3 \\ &v^1 \equiv \varphi = \pm \int \frac{\alpha + 3A\varrho}{\varrho} \frac{d\varrho}{\sqrt{C_1 + \frac{\alpha}{\varrho} - 3A\ln\varrho}} + C_2 \quad \text{for} \quad \varkappa = 3, \end{aligned}$$

where $\alpha = C^2/4\pi$ and C_1, C_2 are arbitrary constants of integration. This solution contains two arbitrary constants. We may put $C_2 = 0$, since the equations are invariant under the Galillelian transformation.

The characteristic vectors in the hodograph space for solutions (4.3) may be expressed in terms of $\varphi(\varrho)$:

$$\gamma_1 = \left(1, \dot{\varphi}, +\sqrt{\frac{\alpha\varrho + A\varkappa\varrho^{\varkappa-1}}{\varrho^2} - \dot{\varphi}^2}\right), \tag{4.4}$$
$$\gamma_2 = \left(1, \dot{\varphi}, -\sqrt{\frac{\alpha\varrho + A\varkappa\varrho^{\varkappa-1}}{\varrho^2} - \dot{\varphi}^2}\right),$$

and the characteristics covectors $\underset{1}{\lambda}, \underset{2}{\lambda}$ in physical space knotted with $\underset{1}{\gamma}, \underset{2}{\gamma}$, respectively, take the form:

$$\underset{1}{\lambda} = \left(\alpha + A\varkappa\varrho^{\varkappa-2} + v^1\dot{v}^1 + v^2\sqrt{\frac{\alpha\varrho + A\varkappa\varrho^{\varkappa-1}}{\varrho^2} - (\dot{v}^1)^2}, -\dot{v}^1, -\sqrt{\frac{\alpha\varrho + A\varkappa\varrho^{\varkappa-1}}{\varrho^2} - (\dot{v}^1)^2}\right), \tag{4.5}$$
$$\underset{2}{\lambda} = \left(\alpha + A\varkappa\varrho^{\varkappa-2} + v^1\dot{v}^1 - v^2\sqrt{\frac{\alpha\varrho + A\varkappa\varrho^{\varkappa-1}}{\varrho^2} - (\dot{v}^1)^2}, -\dot{v}^1, +\sqrt{\frac{\alpha\varrho + A\varkappa\varrho^{\varkappa-1}}{\varrho^2} - (\dot{v}^1)^2}\right),$$

where the dot denotes differentiation with respect to ϱ.

The direction $\sigma(\sigma \| \underset{1}{\lambda} \times \underset{2}{\lambda})$ on which the solution must be constant is given by

$$\sigma = \left(1, \frac{\alpha + A\varkappa\varrho^{\varkappa-2}}{\dot{\varphi}} + v^1, v^2\right). \tag{4.6}$$

The construction presented has sense if the following conditions are satisfied:

$$4C_1\varrho + 3\alpha + \varkappa\,\frac{\varkappa-1}{\varkappa-3}\,\varrho^{\varkappa-2} > 0 \quad \text{for} \quad \varkappa \neq 3, \tag{4.7}$$
$$4C_1 + 3\alpha - \varrho(1 + 4\ln\varrho) > 0 \quad \text{for} \quad \varkappa = 3,$$

where C_1 is an arbitrary real constant and α is an arbitrary nonnegative constant. The conditions (4.7) assure the existence of two families of characteristics and thus define the area of hyperbolicity of the Eq. (3.14).

The double waves considered are constant over the direction σ; hence the operator

$$\sigma \cdot \nabla = \frac{\partial}{\partial t} + \left(\frac{\alpha + A\varkappa\varrho^{\varkappa-2}}{\dot{\varphi}} + v^1\right)\frac{\partial}{\partial x} + v^2\frac{\partial}{\partial y} = \frac{d}{dt} + \frac{\alpha + A\varkappa\varrho^{\varkappa-2}}{\dot{\varphi}}\frac{\partial}{\partial x} \tag{4.8}$$

must vanish on the solutions.

If F is the solution of the Eq. (4.2), then a double wave with hodograph F is described by the Eq. (3.16) and equations:

$$\sigma \cdot \nabla\varrho = 0, \quad \sigma \cdot \nabla v^1 = 0, \quad \sigma \cdot \nabla v^2 = 0. \tag{4.9}$$

Therefore, using the Eq. (4.9), the Eqs. (3.16) take the form:

$$-\frac{\alpha + A\varkappa\varrho^{\varkappa-2}}{\dot{\varphi}(\varrho)}\frac{\partial\varrho}{\partial x} + \varrho\left(\frac{\partial v^1}{\partial x} + \frac{\partial v^2}{\partial y}\right) = 0, \tag{4.10}$$
$$-\frac{\alpha + A\varkappa\varrho^{\varkappa-2}}{\dot{\varphi}(\varrho)}\varrho\frac{\partial v^1}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\alpha}{2}\varrho^2 + A\varrho^{\varkappa}\right) = 0,$$
$$-\frac{\alpha + A\varkappa\varrho^{\varkappa-2}}{\dot{\varphi}(\varrho)}\varrho\frac{\partial v^2}{\partial x} + \frac{\partial}{\partial y}\left(\frac{\alpha}{2}\varrho^2 + A\varrho^{\varkappa}\right) = 0.$$

The Eqs. (4.10) are linearly dependent; hence we obtain two equivalent forms:

$$\left[-\frac{\alpha+A\varkappa\varrho^{\varkappa-2}}{\dot{\varphi}^2\varrho}+1\right]v^1_{,x}+v^2_{,y}=0,\quad\text{or}\quad\left[1-\frac{\alpha+A\varkappa\varrho^{\varkappa-2}}{\dot{\varphi}^2\varrho}\right]\dot{\varphi}\varrho_{,x}+v^2_{,y}=0,$$
$$v^1_{,y}-v^2_{,x}=0,\quad \dot{\varphi}\varrho_{,y}-v^2_{,x}=0. \tag{4.11}$$

After hodograph transformation, instead of the Eq. (4.11) we have:

$$\left[1-\frac{\alpha+A\varkappa\varrho^{\varkappa-2}}{\dot{\varphi}^2\varrho}\right]\dot{\varphi}y_{,v2}+x_{,\varrho}=0,\quad \dot{\varphi}x_{,v2}-y_{,\varrho}=0 \tag{4.12}$$

and further

$$[\varrho\dot{\varphi}^2-(\alpha+A\varkappa\varrho^{\varkappa-2})]\,\dot{\varphi}y_{,v2v2}=\varrho\ddot{\varphi}y_{,\varrho}-\varrho\dot{\varphi}y_{,\varrho\varrho}. \tag{4.13}$$

Using the Eq. (4.3), we obtain:

$$\left(\frac{\varkappa+1}{\varkappa-3}A\varkappa\varrho^{\varkappa-2}-C_1\varrho-3\alpha\right)(\alpha+A\varkappa\varrho^{\varkappa-2})\,y_{,v2v2}=\varrho\Bigg[A\varkappa\varrho^{\varkappa-2}(\varkappa-3)C_1$$
$$+4A\varkappa\alpha\frac{(\varkappa-1)(\varkappa-4)}{\varkappa-3}\varrho^{\varkappa-3}-2\varkappa^2A^2\varrho^{2\varkappa-5}-2\alpha^2\varrho^{-1}-C_1\alpha\Bigg]y_{,\varrho}$$
$$-(\alpha+A\varkappa\varrho^{\varkappa-2})\left(C_1+4\alpha\varrho^{-1}-\frac{4A\varkappa}{\varkappa-3}\varrho^{\varkappa-3}\right)y_{,\varrho\varrho}\quad\text{for}\quad \varkappa\neq 3. \tag{4.14}$$

This is the linear equation of second order for the function y, and for $\varkappa=3$, we obtain the analogous linear equation:

$$(\alpha+3A\varrho)^2[C_1\varrho+(\ln\varrho-1)3A\varrho]y_{,v2v2}=[2\alpha(C_1\varrho+\alpha-3A\varrho\ln\varrho)$$
$$-(\alpha+3A\varrho)^2]y_{,\varrho}+2\varrho(\alpha+3A\varrho)(C_1\varrho+\alpha-3A\varrho\ln\varrho)y_{,\varrho\varrho}. \tag{4.15}$$

The Eqs. (4.11) describe the interaction of two fast magnetoacoustic waves with a hodograph which can be represented by the expressions:

$$H=C\varrho,\quad p=A\varrho^{\varkappa},\quad v^1=\varphi(\varrho), \tag{4.16}$$

where v^2 is arbitrary and φ is a solution of the Eq. (4.2) (see Eq. (4.3)[9]). Therefore, we have a three-parametr (C, A, C_1) family of hodographs. Each solution of the Eqs. (4.11) presents the interaction process of two fast magnetoacoustic waves for the fixed moment of time. Dependence on time is described by the Eqs. (4.9). The Eqs. (4.11) are of mixed type; hence we have:

hyperbolic type if $\dfrac{\alpha+A\varkappa\varrho^{\varkappa-2}}{\varrho\dot{\varphi}^2}-1>0,$

elliptic type if $\dfrac{\alpha+A\varkappa\varrho^{\varkappa-2}}{\varrho\dot{\varphi}^2}-1<0,$

The method presented makes sense in the area of the hyperbolicity of the Eqs. (4.11), or equivalently if characteristics vectors exist γ_i, λ_i $i=1,2$.

Now shall describe the manner of constructing solutions with cylindrical hodographs. First, we solve the system (4.11) from which we obtain $v^2=v^2(x,y)$, $\varrho=\varrho(x,y)$ and the other hodograph functions we have from the relations (4.16). Subsequently, this solution may be prolongated along direction σ using the Eq. (4.9), because the solution is constant in that direction; hence we obtain time dependence $v^2 = v^2(x, y, t)$, $\varrho = \varrho(x, y, t)$. This method is described precisely in paper [6].

We know in gasdynamics the case in which two simple acoustic waves interact linearly (8, 9). This interaction is possible if and only if $\cos\varphi = -\frac{\varkappa-1}{2}$, where φ is the angle between directions of propagations of the interacting waves. The physical meaning of the linear interaction lies in that one wave does not disturb another. The mathematical meaning denotes that the hodograph is of the form:

$$u = u_1(R^1)+u_2(R^2),$$

where $u_{,R^i} = \underset{i}{\gamma}$ and covectors from the physical space $\underset{k}{\lambda} = \underset{k}{\lambda}(R\,)$, so that $R^k = \varphi(\underset{k}{\lambda}\cdot x)$. The case of linearly interacting waves appears in magnetohydrodynamics for two simple magnetoacustic waves only for $\varkappa = 2$, but in this case equations of magnetohydrodynamics reduce to equations of hydrodynamics [see (3.20)], [8, 9].

Obviously, the angle between directions of propagations of waves is determined by $\cos\varphi = -{}^1/_2$. Of interest is the possibility of this interaction being independent of the profiles and amplitudes of the waves.

5. Entropic waves in the interaction

Now we are interested in the problem of interaction of two entropic waves. Using the form of entropic simple elements in the Eq. (2.4), it is easy to see that ϱ is an arbitrary function and $p+H^2/8\pi = C$, where C is a constant of integration. Further, from the characteristic vectors in hodograph space, we have the following form for covectors of entropic waves:

$$\underset{i}{\lambda} = (-v^1v^2_{,R^i}+v^2v^1_{,R^i},\ v^2_{,R^i},\ -v^1_{,R^i}), \quad i = 1, 2, \tag{5.1}$$

where R^1, R^2 are Riemann invariants for entropic waves.

From the condition of involution, we have the equation

$$v^1_{,R^2}v^2_{,R^1}-v^1_{,R^1}v^2_{,R^2} = 0, \quad \text{so} \quad v^2 = \psi(v^1) \text{ there } \underset{1}{\lambda}\|\underset{2}{\lambda}. \tag{5.2}$$

Thus interaction waves are not described by a double wave.

The process of interaction of entropic and magnetoacoustic waves is more complicated and not yet solved in general, except for a few special examples.

The results presented in this paper may be used for constructing particular solutions of the basic equations and for numerical analysis of solutions.

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