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Stationary interaction of a system of two spheres resting or moving in a free-molecular medium

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THE PAPER is aimed at determination of the forces acting on a system of two spheres held at constant temperatures, resting or moving in a free-molecular medium along the axis passing through the centers of the spheres. The problem is reduced to the solution of the system of equations of continuity of the particle flux (inpermeable walls), and then to the interaction quadratures. In the case of spheres at rest, the problem is solved in part analytically, and in part — numerically; in the case of a moving system, its solution is analytical, in the large distance and small velocity approximation.

W pracy obliczono siły działające na układ dwu kul o stałych temperaturach, spoczywający lub poruszający się w ośrodku swobodnie-molekularnym wzdłuż osi łączącej środki kul. Problem sprowadza się do rozwiązania układu równań ciągłości strumienia liczby cząstek (nieprzenikalności ścianek), a następnie do wykonania kwadratur oddziaływania. W przypadku kul spoczywających zagadnienie zostało rozwiązane częściowo analitycznie, częściowo numerycznie, w przypadku układu poruszającego się — analitycznie w przybliżeniu dużych odległości ciał i małych prędkości układu.

В работе определяются силы действующие на систему двух сфер с постоянными температурами, неподвижную или движущуюся в свободно-молекулярной среде вдоль оси соединяющей центры сфер. Проблема сводится к решению системы уравнений неразрывности потока количества частиц (непроницаемость стенок), а затем к расчету квадратур воздействия. В случае неподвижных сфер проблема решена частично аналитически, частично численно, в случае движущейся системы — аналитически в приближении больших расстояний тел и малых скоростей системы.

Introduction

THE FREE-MOLECULAR medium is understood as a medium in which the mean free path λ is much greater than the dimensions of bodies R_1, R_2, and the distance d between the bodies immersed in the medium: $\lambda \gg R_1, R_2, d$. The bodies resting or moving in the medium interact with the medium and with each other (across the medium) by exchanging their momentum, spin and energy what results in changing their linear, rotary and internal motions. In this paper we shall deal with the problem of exchange of momentum and energy (the problem of forces and heat exchange). The problems concerning non-convex bodies and systems of bodies is discussed in papers [1, 2], where only the general formulae are derived. In [8], the solution is presented for the problem of two rectangular plates forming a right angle in a flow parallel to their common edge.

The present paper is aimed at exposing the interaction effects connected with the motion. In view of mathematical difficulties, the motion is limited to the axis connecting the centres of the spheres. The problem, being of a certain theoretical interest, may also prove to be of practical importance, e.g. for satellite systems placed in the space around the earth.

Let us consider a system of two spheres K_1, K_2 with radii R_1, R_2 and constant temperatures T_1, T_2, resting or moving in a free-molecular medium of temperature T_0; the motion is rectilinear and uniform, its velocity $\mathbf{q}$ is parallel to the line connecting the centres, $\mathbf{q} \| \mathbf{O}_1\mathbf{O}_2$, Fig. 1.

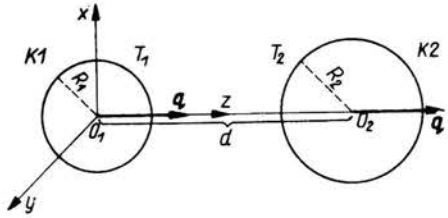

FIG. 1.

The following assumptions are made:

1) the gas is monoatomic, without internal degrees of freedom,

2) in absence of the bodies, the medium is homogeneous and isotropic, its state being described by the Maxwell-Boltzmann function $f_z^{(i)}$;

3) temperature fields of the spheres are uniform;

4) interaction of the gas particles with the surfaces of the spheres is modelled by diffusive reflection.

We shall deal with the exchange of momentum and energy; however, more detailed considerations will concern the problem of forces — the corresponding procedure in heat echange problems is similar.

1. Evaluation of forces acting on the bodies

The force $\mathbf{F}$ acting on the body is equal to the flux of momentum $\mathbf{S}$ transmitted to the body,

$$\mathbf{F} = \mathbf{S}. \tag{1.1}$$

The flux $\mathbf{S}$ equals the difference of two fluxes: incident $\mathbf{S}^{(i)}$ and "reflected" $\mathbf{S}^{(r)}$, which correspond to the incident and reflected particles

$$\mathbf{S} = \mathbf{S}^{(i)} - \mathbf{S}^{(r)}. \tag{1.2}$$

In order to detrmine the force acting on the sphere, we have to evaluate the momentum fluxes $\mathbf{S}_l^{(i)}$, $\mathbf{S}_l^{(r)}$ at an arbitrary point of the sphere, and then to integrate them over the entire surface of the sphere

$$\mathbf{S}^{(i)} = \int_{\Sigma} \mathbf{S}_l^{(i)} d\Sigma, \quad \mathbf{S}^{(r)} = \int_{\Sigma} \mathbf{S}_l^{(r)} d\Sigma. \tag{1.3}$$

The momentum flux $\mathbf{S}_l^{(i)}$ of particles striking a point of the internal surface of the sphere (internal surface denotes that part of the surface from which the other sphere, or its part, is visible) is a sum of fluxes ${}_{(m)}\mathbf{S}_l^{(i)}$, ${}_{(b)}\mathbf{S}_l^{(i)}$ arriving from the surrounding medium and from the other sphere, respectively,

$$\mathbf{S}_l^{(i)} = {}_{(m)}\mathbf{S}_l^{(i)} + {}_{(b)}\mathbf{S}_l^{(i)}. \tag{1.4}$$

According to these results, the following expressions for the forces acting on K_1 are obtained:

$$\mathbf{F}_1 = \mathbf{F}_{1(0)} + \mathbf{F}_{1(in)}, \tag{1.5}$$

$$\mathbf{F}_{1(0)} = -m \int_{\Sigma_1} \left[\int_{\Omega^c_{1/2}} \mathbf{c}_{01}(\mathbf{c}_{01}\cdot\mathbf{n}_1) f_z^{(i)(1)} d^3\mathbf{c}_{01} + \int_{\Omega^c_{1/2}} \mathbf{c}_{11}(\mathbf{c}_{11}\cdot\mathbf{n}_1) f_{z1}^{(r)} d^3\mathbf{c}_{11} \right] d\Sigma_1, \tag{1.6}$$

$$\mathbf{F}_{1(in)} = \int_{\Sigma_{2w}} \mathbf{F}^{(i)}_{K2(el)} d\Sigma_{2w} - \int_{\Sigma_{1w}} \mathbf{F}^{(i)z}_{K2(el)} d\Sigma_{1w} + \int_{\Sigma_{1w}} \mathbf{F}^{(r)}_{1(in)el} d\Sigma_{1w}, \tag{1.7}$$

$$\mathbf{F}^{(i)}_{K2(el)} = m \int_{\Omega^c_{K1(P2)}} \mathbf{c}_{21}(\mathbf{c}_{22}\cdot\mathbf{n}_2) f^{(i)}_{K2} d^3\mathbf{c}_{22}, \quad \mathbf{F}^{(i)(z)}_{K2(el)} = m \int_{\Omega^c_{K1(P2)}} \mathbf{c}_{01}(\mathbf{c}_{02}\cdot\mathbf{n}_2) f_z^{(i)(2)} d^3\mathbf{c}_{02},$$

$$\mathbf{F}^{(r)}_{1(in)el} = -m \int_{\Omega^c_{1/2}} \mathbf{c}_{11}(\mathbf{c}_{11}\cdot\mathbf{n}_1) f^{(r)}_{1w} d^3\mathbf{c}_{11} + m \int_{\Omega^c_{1/2}} \mathbf{c}_{11}(\mathbf{c}_{11}\cdot\mathbf{n}_1) f^{(r)}_{z1} d^3\mathbf{c}_{11},$$

with the notations:

- m mass of a gas particle,
- $\mathbf{c}_{21}$ velocity of a particle arriving from K_2 referred to K_1,
- $d^3\mathbf{c}$ volume element in the space of velocities,
- Σ_1 total area of the surface of K_1,
- Σ_{w1} the part of sphere K_1 which is visible from sphere K_2,
- Σ_{w2} the part of sphere K_2 which is visible from sphere K_1,
- $\Omega^c_{1/2}$ semi-space of velocities connected with the normal n_1,
- $\mathbf{n}_1$ outer normal of the sphere K_1 at the point P_1,
- $\mathbf{c}_{22}(\mathbf{c}_{11})$ velocity of a particle arriving from $K_2(K_1)$ referred to $K_2(K_1)$,
- $\mathbf{c}_{02}(\mathbf{c}_{01})$ velocity of a particle of the medium referred to the sphere $K_2(K_1)$,
- $\Omega^c_{K\alpha(P\beta)}$ the region in the space of velocities corresponding to the solid angle at which sphere K_α is visible from P_β, $\alpha, \beta = 1, 2, \alpha \neq \beta$,
- $f_z^{(i)(1)}$ function of velocity distribution of the medium particles in the reference frame of K_1,
- $f_z^{(i)(2)}$ function of velocity distribution of the medium particles in the reference frame of K_2,
- $f_{z1}^{(r)}$ function of velocity distribution of the particles "reflected" from that part of the spherical surface of K_1 which is not visible from K_2,
- $f_{w\alpha}^{(r)}$ function of velocity distribution of the particles "reflected" from that part of K_α which is visible from K_β, $\alpha, \beta = 1, 2, \alpha \neq \beta$,
- $f^{(i)}_{K2} \equiv f^{(r)}_{w2}$ function of velocity distribution of the particles reflected from K_2 (Fig. 2).

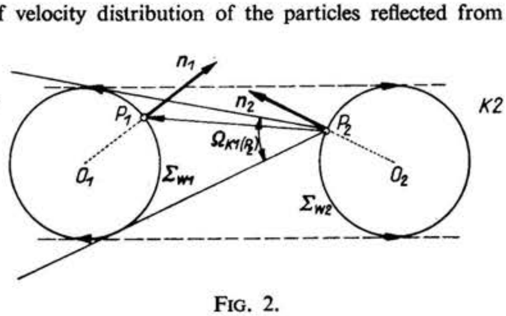

FIG. 2.

$\mathbf{F}_1^{(0)}$ may be interpreted as the drag which would act on K_1 moving at the velocity $\mathbf{q}$ through the medium in absence of the other sphere K_2; $\mathbf{F}_{1(in)}$ may be understood as the force of interaction, resulting from the presence of K_2. In the case of spheres at rest, the force $\mathbf{F}_{1(0)}$ vanishes owing to the spherical symmetry.

To make the formulation complete, let us moreover specify the functions of distribution occurring in the integrals. In compliance with the assumptions of full equilibrium of the medium and of diffusive "reflection", we obtain

$$
\begin{aligned}
&f_z^{(i)} = A_z^{(i)} e^{-B_z^{(i)}(\mathbf{c}_{01}+\mathbf{q})^2}, \qquad f_z^{(i)(1)} \equiv f_z^{(i)(2)} \equiv f_z^{(i)},\\
&f_{z1}^{(r)} = A_{z1}^{(r)} e^{-B_1 c_{11}^2}, \qquad f_{w1}^{(r)} = A_{w1}^{(r)} e^{-B_1 c_{11}^2}.\\
&f_{K2}^{(i)} = A_{K2}^{(i)} e^{-B_2 c_{22}^2} \equiv f_{w2}^{(r)},
\end{aligned} \tag{1.8}
$$

The quantities A and B are expressed by the numerical density of particles n and the temperature T,

$$B = \left(\frac{2kT}{m}\right)^{-1}, \quad A = n\left(\frac{2\pi kT}{m}\right)^{-3/2}. \tag{1.9}$$

Quantities $A_z^{(i)}$, $B_z^{(i)}$, B_1, B_2 are known, the remaining ones are determined from the continuity conditions of the particle flux.

The continuity equations written for the respective external and internal parts of the surface of K_α, $\alpha = 1, 2$, assume the following forms:

$$N_{\alpha z}^{(i)}(P_\alpha) = N_{\alpha z}^{(r)}(P_\alpha), \tag{1.10}$$

$$N_{\alpha w}^{(i)}(P_\alpha) = N_{\alpha w}^{(r)}(P_\alpha). \tag{1.11}$$

Here

$$N_{\alpha z_i}^{(i)} = \int_{\Omega_{1/2}^c} (-\mathbf{c}_{0\alpha}\mathbf{n}_\alpha) f_z^{(i)(\alpha)} d^3\mathbf{c}_{0\alpha}, \tag{1.12}$$

$$N_{\alpha z_i}^{(r)} = \int_{\Omega_{1/2}^c} (\mathbf{c}_{\alpha\alpha}\cdot\mathbf{n}_\alpha) f_{z\alpha_i}^{(r)} d^3\mathbf{c}_{\alpha\alpha}, \tag{1.13}$$

$$N_{\alpha w}^{(i)}(P_\alpha) = N_{\alpha w(0)}^{(i)} + N_{\alpha w(in)}^{(i)}, \tag{1.14}$$

$$N_{\alpha w(0)}^{(i)} = \int_{\Omega_{1/2}^c} (-\mathbf{c}_{0\alpha}\mathbf{n}_\alpha) f_z^{(i)(\alpha)} d^3\mathbf{c}_{0\alpha}, \tag{1.15}$$

$$N_{\alpha w(\mathrm{in})}^{(i)} = -\int_{\Omega_{K\beta(P\alpha)}^c} (-\mathbf{c}_{0\alpha}\mathbf{n}_\alpha) f_z^{(i)(\alpha)} d^3\mathbf{c}_{0\alpha} + \int_{\Omega_{K\beta(P\alpha)}^c} (-\mathbf{c}_{\beta\alpha}\mathbf{n}_\alpha) f_{K\beta}^{(i)}(P_\beta) d^3\mathbf{c}_{\beta\alpha}, \tag{1.16}$$

$$N_{\alpha w}^{(r)}(P_\alpha) = \int_{\Omega_{1/2}^c} (\mathbf{c}_{\alpha\alpha}\mathbf{n}_\alpha) f_{\alpha w}^{(r)} d^3\mathbf{c}_{\alpha\alpha}, \quad \beta = 1, 2;\ \beta \neq \alpha. \tag{1.17}$$

The constant $A_{1z}^{(r)}(P_1)$ (and similarly $A_{2z}^{(r)}(P_2)$) referring to the external surface is obtained directly from Eq. (1.10) by means of a threefold quadrature. Determination of the constants $A_{\alpha w}^{(r)}(P_\alpha)$ requires the solution of a very complicated system of two ($\alpha = 1, 2$) integral Eqs. (1.11) (Fredholm type, second kind) in four variables; points P_1, P_2 on the spheres are uniquely determined by means of their directions. In the general case of arbitrary system velocity and arbitrary distance between the spheres, the solution may be obtained only in an approximate manner.

It is found that in the case of spheres at rest the solutions are very simple, while in the case of a system moving along the axis connecting the centers, an exact solution may be obtained in the case in which the distance between the spheres is large in comparison with their dimensions (large distance approximation). Since the methods of solution and calculating of fivefold quadratures representing the interactions are, in the cases of spheres at rest and the system propagating along the axis, entirely different, both cases will be considered separately.

2. Spheres at rest

2.1. Solution of continuity equations

THEOREM. *Solutions of the system of Eqs.* (1.11), $A^{(r)}_{1w}(P_1)$ *and* $A^{(r)}_{2w}(P_2)$ *are the same as in the case in which each of the spheres are placed in the medium alone* (*the presence of the other sphere does* **not** *influence the solution of continuity equations*).

Proof. Let us assume the solutions $A^{(r)}_{1w}$ P_1 and $A^{(r)}_{2w}$ to be constant, independent of the points at the surface, P_1 and P_2.

$$A^{(r)}_{1w}(P_1) = \text{const}, \quad A^{(r)}_{2w}(P_2) = \text{const}. \tag{2.1}$$

It will be proved that this hypothesis does not lead to a contradiction. Performing the integrations with respect to the velocity modulus $|\mathbf{c}|$ in $N^{(i)}$ and $N^{(r)}$ and apply ing the thesis (2.1), we obtain

$$\begin{aligned}
N^{(i)}_{1w} &= b\left\{A^{(i)}_z w\left(\frac{2kT_0}{m}\right)^2 + w_1^*\left[A^{(r)}_{2w}\left(\frac{2kT_2}{m}\right)^2 - A^{(i)}_z\left(\frac{2kT_0}{m}\right)^2\right]\right\},\\
N^{(r)}_{1w} &= bwA^{(r)}_{1w}\left(\frac{2kT_1}{m}\right)^2,\\
N^{(i)}_{2w} &= b\left\{A^{(i)}_z w\left(\frac{2kT_0}{m}\right)^2 + w_2^*\left[A^{(r)}_{1w}\left(\frac{2kT_1}{m}\right)^2 - A^{(i)}_z\left(\frac{2kT_0}{m}\right)^2\right]\right\},\\
N^{(r)}_{2w} &= bwA^{(r)}_{w2}\left(\frac{2kT_2}{m}\right)^2;
\end{aligned} \tag{2.2}$$

with the notations $b = \frac{1}{2}\Gamma(2) = \frac{1}{2}$,

$$w = \int\limits_{\Omega_{1/2}} (-\mathbf{l}_{01}\cdot\mathbf{n}_1)\,d\Omega_{l01}, \quad w_1^* = \int\limits_{\Omega_{K2(P_1)}} (-\mathbf{l}_{01}\cdot\mathbf{n}_1)\,d\Omega_{l01}, \quad w_2^* = \int\limits_{\Omega_{K1(P_2)}} (-\mathbf{l}_{02}\cdot\mathbf{n}_2)\,d\Omega_{l02}. \tag{2.3}$$

It is easily verified that the assumption

$$A^{(r)}_{1w}\left(\frac{2kT_1}{m}\right) = A^{(i)}_z\left(\frac{2kT_0}{m}\right)^2 = A^{(r)}_{2w}\left(\frac{2kT_2}{m}\right)^2, \quad \mathbf{l}_{0v} \overset{dF}{=} \frac{\mathbf{c}_{0v}}{c_{0v}} \tag{2.4}$$

satisfies now the system of continuity (1.11), and this means that we have proved the existence of solutions within the range of hypothesis (2.1); that hypothesis does not lead to a contradiction. Moreover, the qualitative character of solutions is the same as in the case in which each sphere exists in the medium separately, since

$$N^{(r)}_{1w} = N^{(i)}_{1w} \equiv N^{(i)}_{1w(0)}, \quad N^{(r)}_{2w} = N^{(i)}_{2w} \equiv N^{(i)}_{2w(0)}. \tag{2.5}$$

Index 0 means that the sphere considered is placed alone in the medium (the other sphere is absent).

The result obtained may be so explained that "overshadowing", in the case of resting spheres, does not influence the particle fluxes in the diffusive model: in diffusive dispersion, the particles reflected at a given point of the sphere reproduce, at the same point, the isotropic properties of the medium (identical particle fluxes are "emitted" in all directions — for the particle flux is, by contrast with the energy flux, immaterial what is the energy spectrum within the flux). The consequences of such simplification of solutions of the continuity equations are spectacular — the calculations are considerably simplified and the possibility is created of constructing an accurate solution. Applying the above result and using the spherical symmetry properties following from that result, we obtain

$$\mathbf{F}^{(r)}_{1(in)el} = 0. \tag{2.6}$$

2.2. Quadrature of interaction integrals

Introducing the functions of distribution into the elementary forces, we transform the Eq. (1.6) to the form

$$\mathbf{F}^{(i)z}_{K2(el)} = mI_{(c)0}\mathbf{I}^{(i)(z)}_{K2(el)}A^{(i)}_z, \quad \mathbf{F}^{(i)}_{K2(el)} = mA^{(r)}_2(P_2)I_{(c)2}\mathbf{I}^{(i)}_{K2(el)}, \tag{2.7}$$

where

$$\begin{gathered} I_{(c)0} = \int\limits_0^\infty c^4_{01}e^{-B^{(i)}_z c^2_{01}}dc_{01}, \quad I_{(c)2} = \int\limits_0^\infty c^4_{22}e^{-B_2 c^2_{22}}dc_{22}, \\ \mathbf{I}^{(i)z}_{K2(el)} = \int\limits_{\Omega_{K2(P_1)}} \mathbf{l}_{01}(-\mathbf{l}_{01}\cdot\mathbf{n}_1)d\Omega_{l01}, \quad \mathbf{I}^{(i)}_{K2(el)} = \int\limits_{\Omega_{K1(P_2)}} \mathbf{l}_{21}(\mathbf{l}_{22}\cdot\mathbf{n}_2)d\Omega_{l_{22}}. \end{gathered} \tag{2.8}$$

It may be proved that

$$\int\limits_{\Sigma_{1w}} \mathbf{I}^{(i)(z)}_{K2(el)}d\Sigma_{1w} = \int\limits_{\Sigma_{2w}} \mathbf{I}^{(i)}_{K2(el)}d\Sigma_{2w}. \tag{2.9}$$

The fact expressed by the Eq. (2.9) is equivalent to the conclusion that the force $\mathbf{F}^{(i)}_{K2}$ can be written, as viewed from the K_2-sphere, as

$$F^{(i)}_{K2} = m\int\limits_{\Sigma_{1w}}\Big[\int\limits_{\Omega^c_{K2(P_1)}} (-\mathbf{c}_{21}\cdot\mathbf{n}_1)\mathbf{c}_{21}f^{(r)}_2 d^3\mathbf{c}^{(1)}_{21}\Big]\,d\Sigma_{1w}, \tag{2.10}$$

$d^3\mathbf{c}^{(1)}_{21}$ being the elementary cone with vertex at P_1, $d^3\mathbf{c}_{22}$ — the elementary cone with vertex at P_2.

Making use of Eqs. (2.1), (1.6), (2.7), (2.9), we finally obtain the expression for $\mathbf{F}_1$ [relation (2.1) enables us to write $A_2(P_2)$ and $A^{(i)}_z$, as constants, before the sign of integration]

$$\mathbf{F}_1 = -m[I_{(c)0}A^{(i)}_z - I_{(c)2}A^{(r)}_2]\int\limits_{\Sigma_{1w}} \mathbf{I}^{(i)(z)}_{K2(el)}d\Sigma_{1w}. \tag{2.11}$$

The integral $\mathbf{I}^{(i)(z)}_{K2(el)}$ is evaluated first in the local reference frame (since the calculations are then the simplest), attached to P_1, with the axes $z^{(l)}$ parallel to $P_1 0_2$ and $y^{(l)}$ so selected that $\mathbf{n}_1$ lies in the plane $(z^{(l)}, y^{(l)})$ and $\mathbf{j}^{(l)}\cdot\mathbf{n}_1 \geqslant 0$; here $\mathbf{j}^{(l)}$ denotes the unit vector in the direction of $y^{(l)}$.

Components of the vector $\mathbf{I}^{(i)z}_{K2(el)}$ in the absolute system are reconstructed by transforming the vector $\mathbf{I}^{(i)z(l)}_{K2(el)}$ from the local to the absolute system, and by applying the transformation rules for tensors according to the transformation matrix a_{km},

$$I^{(i)z}_{K2(el)k} = a_{km} I^{(i)(z)(l)}_{k2(el)m}, \tag{2.12}$$

$$a_{km} = \begin{bmatrix} -\sin\varphi_{n1} & a^{(1)}(1-k_1\cos\theta_{n1})\cos\varphi_{n1} & -a^{(1)}k_1\sin\theta_{n1}\cos\varphi_{n1} \\ \cos\varphi_{n1} & a^{(1)}(1-k_1\cos\theta_{n1})\sin\varphi_{n1} & -a^{(1)}k_1\sin\theta_{n1}\cos\varphi_{n1} \\ 0 & a^{(1)}k_1\sin\theta_{n1} & a^{(1)}(1-k_1\cos\theta_{n1}) \end{bmatrix} \tag{2.13}$$

$$a^{(1)} = (1+k_1^2-2k_1\cos\theta_{n1})^{-1/2}.$$

The magnitudes of $\mathbf{I}^{(i)z(l)}_{K2(el)}$ depend on the area of region Σ_{1w}, which is connected exclusively with the geometry, namely the region of integration — the solid angle $\Omega_{K2(P1)}$ — depends on the position of P_1 and may represent a cone or its part, larger or smaller than its half (Fig. 3, 4, 5).

Three regions may be considered, depending on θ_m: (i) $\cos\theta_{n1} \in (k_1-k_2, k_1)$; the *plane line p* (i.e. the trace of the plane tangent to K_1 at P_1 on the visible part of the surface K_2)

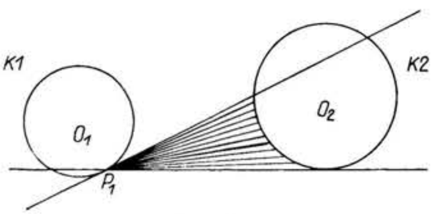

FIG. 3.

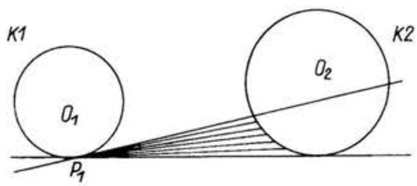

FIG. 4.

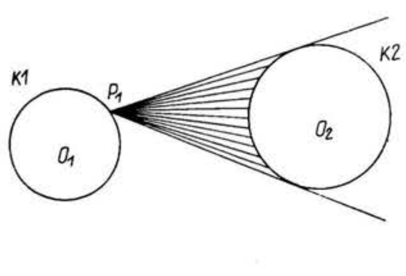

FIG. 5.

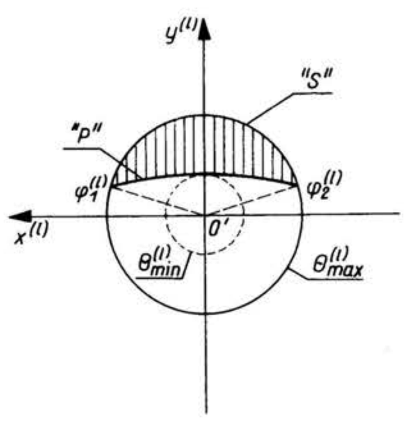

FIG. 6.

is located above the centre 0′ of the *conical line s* (the trace of a cone with the vertex at P_1 and the angle corresponding to the size of K_2 as viewed from P_1). Figure 6 also shows the traces of coordinate axes $x^{(l)}$, $y^{(l)}$ and the limit polar angles $\varphi_1^{(l)}$, $\varphi_2^{(l)}$ of the intersection of p and s, as also the azimuthal angles $\theta^{(l)}_{\max}$, $\theta^{(l)}_{\min}$, corresponding to the maximum vertex angle of the cone and to the vertex angle of a cone tangent to the line p.

Owing to the symmetry, in region (i) we have $\varphi_1^{(l)} \in (0, \pi/2)$, $\varphi_2^{(l)} \in (\pi/2, \pi)$, (ii) $\cos\theta_n \in (k_1, k_1+k_2)$; then $\varphi_1^{(l)} \in (3\pi/2, 2\pi)$, Fig. 7. $\varphi_2^{(l)} \in (\pi, 3\pi/2)$. The value $\cos\theta_{n1} = k_1$ corresponds to the situation in which the line p passes through the centre 0′ (and owing to the choice of the coordinate system — coincides with the axis $x^{(l)}$, and

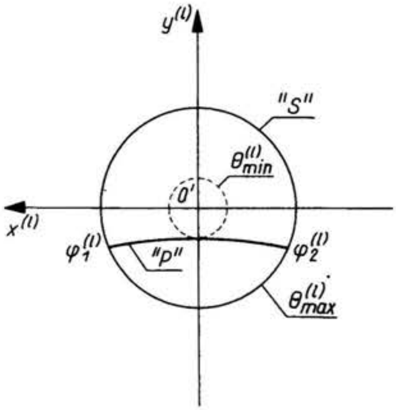

Fig. 7.

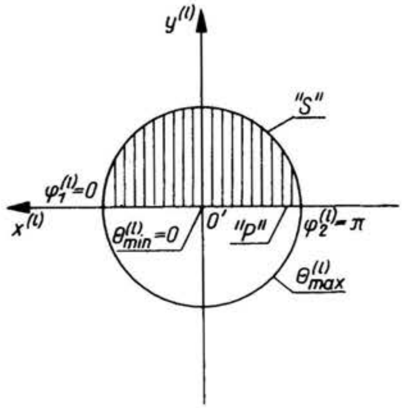

Fig. 8.

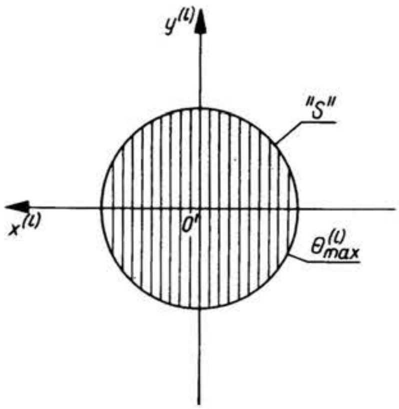

Fig. 9.

then $\varphi_1^{(l)} = 0$, $\varphi_2^{(l)} = \pi$, Fig. 8. (iii) $\cos\theta_{n1} \in (k_1+k_2, 1)$, and then the entire surface K_2 is visible within the conical line, $\varphi^{(l)} \in (0, 2\pi)$, Fig. 9.

Pursuant to the above remarks, the regions of integration within the individual segments θ_{n1} may be represented as

$$\text{(i)} \qquad \mathbf{I}_{K2(el)}^{(i)z(l)} = \int\limits_{\theta_{\min}^{(l)}}^{\theta_{\max}^{(l)}} \left[\int\limits_{\varphi_1^{(l)}}^{\varphi_2^{(l)}} \ldots d\varphi^{(l)} \right] \sin\theta^{(l)} d\theta^{(l)},$$

$$\text{(ii)} \qquad \mathbf{I}_{K2(el)}^{(i)z(l)} = \int\limits_{0}^{\theta_{\max}^{(l)}} \left[\int\limits_{0}^{2\pi} \ldots d\varphi^{(l)} \right] \sin\theta^{(l)} d\theta^{(l)} - \int\limits_{\theta_{\min}^{(l)}}^{\theta_{\max}^{(l)}} \left[\int\limits_{\varphi_1^{(l)}}^{\varphi_2^{(l)}} \ldots d\varphi^{(l)} \right] \sin\theta^{(l)} d\theta^{(l)},$$

(iii) $$\mathbf{I}^{(i)z(l)}_{K2(el)} = \int\limits_0^{\theta^{(l)}_{max}} \Big[\int\limits_0^{2\pi} \dots d\varphi^{(l)}\Big] \sin\theta^{(l)} d\theta^{(l)}.$$

In the interval (ii), the region of integration, which is a greater part of the cone (points $\varphi_1^{(l)}$, $\varphi_2^{(l)}$ are displaced under the axis $x^{(l)}$), is represented by the difference between the complete cone and its smaller part. The characteristic points $\varphi_1^{(l)}$, $\varphi_2^{(l)}$ of intersection of p with the line $\theta^{(l)} =$ const are found from the equation of the plane tangent to K_1 at P_1. In the local, spherical coordinate system, the equation has the form

(2.14) $$\sin\varphi^{(l)} = -\text{ctg}\theta^{(l)}_{n1}\text{ctg}\theta^{(l)}.$$

Here $\theta^{(l)}_{n1}$ is the angle made by the normal $\mathbf{n}_1$ with the $z^{(l)}$-axis. It is evident that for $\cos\theta_{n1} < k_1$, $\sin\varphi^{(l)} \geqslant 0$, and for $\cos\theta_{n1} > k_1$, $\sin\varphi^{(l)} < 0$. Thus we have, depending on the region, the following relations:

(i) $\varphi_1^{(l)} = \arcsin(-\text{ctg}\theta^{(l)}_{n1}\text{ctg}\theta^{(l)}), \qquad \varphi_2^{(l)} = \pi - \varphi_1^{(l)},$

(ii) $\varphi_1^{(l)} = 2\pi + \arcsin(-\text{ctg}\theta^{(l)}_{n1}\text{ctg}\theta^{(l)}), \quad \varphi_2^{(l)} = \pi - \arcsin(-\text{ctg}\theta_{n1}\text{ctg}\theta^{(l)}).$

The values of $\theta^{(l)}_{min}$ are determined by substituting in Eq. (2.14) $\varphi^{(l)} = \pi/2$ [in region (i)] or $3\pi/2$ [in region (ii)] — that is, $\sin\varphi^{(l)} = \pm 1$, since the point of minimum value of $\theta^{(l)}$ on the line "p" lies on the $y^{(l)}$-axis,

(i) $$\text{ctg}\theta^{(l)}_{min} = \frac{\sin\theta_{n1}}{k_1 - \cos\theta_{n1}}, \quad \text{(ii)} \quad \text{ctg}\theta^{(l)}_{min} = \frac{\sin\theta_{n1}}{\cos\theta_{n1} - k_1}.$$

For $\theta^{(l)}_{max}$, we obtain (Fig. 10)

$$\sin\theta^{(l)}_{max} = k_1^* \stackrel{\text{df}}{=} \frac{R_2}{P_1 O_2},$$

$$P_1 O_2 = d(1 + k_1^2 - 2k_1\cos\theta_{n1})^{1/2}.$$

The integrals $\mathbf{I}^{(i)z(l)}_{K2(el)}$ in the all three intervals of θ_{n1} are expressed by elementary functions [7].

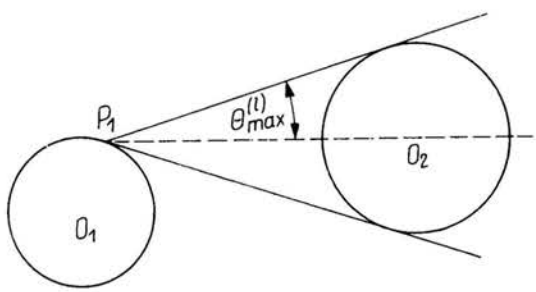

FIG. 10.

Before approaching to the integration of $\mathbf{I}^{(l)z}_{K2(el)}$ over the surface Σ_{1w}, let us first write down its components

$$I^{(l)z}_{K2(el)x} = a^{(1)}(1 - k_1\cos\theta_{n1})\cos\varphi_{n1} I^{(i)z(l)}_{K2(el)y} - a^{(1)}k_1\sin\theta_{n1}\cos\varphi_{n1} I^{(l)z(l)}_{K2(el)z},$$
$$I^{(i)z}_{K2(el)y} = a^{(1)}(1 - k_1\cos\theta_{n1})\sin\varphi_{n1} I^{(i)}_{K2(el)y} - a^{(1)}k_1\sin\theta_{n1}\sin\varphi_{n1} I^{(i)(l)}_{K2(el)z},$$
$$I^{(i)z}_{K2(el)z} = a^{(1)}k_1\sin\theta_{n1} I^{(i)z(l)}_{K2(el)y} + a^{(1)}(1 - k_1\cos\theta_{n1}) I^{(i)z(l)}_{K2(el)z}.$$

Owing to the symmetry properties the integration over φ_{n1} leads to the conclusion that

$$F_{1x} = 0, \quad F_{1y} = 0,$$

since $I^{(i)z(i)}_{K2(el)y}$ and $I^{(i)z(i)}_{K2(el)z}$ are independent of φ_{n1}. Therefore, the force $\mathbf{F}_1$ is reduced to the only z-component, and

$$\int_{\Sigma_{1w}} I^{(i)(z)}_{K2(el)z} d\Sigma_{1w} = 2\pi R_1^2 \int_0^{\arccos(k_1-k_2)} I^{(i)(z)}_{K2(el)z} \sin\theta_{n1}\, d\theta_{n1}.$$

In view of $I^{(i)z(i)}_{K2(el)z}$ being dependent on the interval of θ_{n1}, the integration may be decomposed into three regions

$$\int_{\Sigma_{1w}} I^{(i)z}_{K2(el)z} d\Sigma_{1w} = 2\pi R_1^2 \Big\{ \int_0^{\arccos(k_1+k_2)} \sin\theta_{n1}\, d\theta_{n1} + \int_{\arccos(k_1+k_2)}^{\arccos k_1} \sin\theta_{n1}\, d\theta_{n1} + \int_{\arccos k_1}^{\arccos(k_1-k_2)} \sin\theta_{n1}\, d\theta_{n1}.$$

Unfortunately, the last fifth quadrature over the angle θ_{n1} could not be performed and represented by elementary or special functions. Once the integrals $I_{(c0)}$, $I_{(c2)}$ are calculated, the force exerted by K_2 on the sphere K_1 may be represented in the form

$$\mathbf{F}_1 \equiv \mathbf{F}_{2\to 1} = \mathbf{F}_{(2\to 1)z} = maA^{(i)}_z 2\pi R_1^2 \left(\frac{2kT_0}{m}\right)^2 \left[\left(\frac{2kT_2}{m}\right)^{1/2} - \left(\frac{2kT_0}{m}\right)^{1/2}\right] \bar{I}\frac{\mathbf{r}}{r},$$

where

$$a = \frac{\Gamma\left(\frac{5}{2}\right)}{2}, \quad \mathbf{r} = \mathbf{O}_1\mathbf{O}_2, \quad \bar{I} \overset{\text{df}}{=} \Big(\int_{\Sigma_{1w}} I^{(i)(z)}_{K2(el)z} d\Sigma_{1w}\Big)(2\pi R_1^2)^{-1}.$$

An analogous expression is obtained for the force acting on K_2; it may formally be derived from the expression for $\mathbf{F}_1$ by changing the indices.

2.3. Discussion of results

The force acting on a body depends on the temperature of the other body and the temperature of the surrounding medium, and is independent of the temperature of the body itself. If the temperature of the body is higher than that of the surrounding gas, then the body exerts on the other body repulsive forces, when it is lower — the forces are attractive. Two bodies with teperatures higher than the medium temperature will then by repulsed, the bodies with lower temperatures will be attracted. Also possible is a situation that the first body will be repulsed by the second one which, in turn, will be attracted by the first one, or *vice versa* (i.e., when temperature of the first body is higher and that of the other body — lower than the gas temperature). The complete force exerted on the system is equal to

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = maA^{(i)}_z \left(\frac{2kT_0}{m}\right)^2 \Big\{\left[\left(\frac{2kT_2}{m}\right)^{1/2} - \left(\frac{2kT_0}{m}\right)^{1/2}\right] 2\pi R_1^2 \bar{I}\frac{\mathbf{r}}{r} + \left[\left(\frac{2kT_1}{m}\right)^{1/2} - \left(\frac{2kT_0}{m}\right)^{1/2}\right] \bar{I}(k_1 \rightleftarrows k_2) 2\pi R_2^2 \frac{\mathbf{O}_2\mathbf{O}_1}{O_2O_1}\Big\}.$$

In the case of spheres in equal temperatures, the force acting on the system has the same direction as the force exerted by the sphere which is larger; for equal spheres, the force is directed from the sphere with higher temperature to the colder one.

The magnitude of forces acting on the bodies or the system is simply proportiona to the geometric factor $\bar{I}$. The fact of $\bar{I}$ being dependent on k_1, k_2 may be so interpreted that the same variation of $\bar{I}(k_1, k_2)$ may be achieved by changing either R_1 and R_2 or d (increasing d corresponds to inversely proportional decrease of R_1 and R_2) provided that the variation of d is accompanied by the variation of $\mathbf{F}_1$, $(\mathbf{F}_2)$ following the change of $\bar{I}(k_1, k_2)$ $(\bar{I}(1 \rightleftarrows 2))$ only, while in varying R_1 and R_2 the forces are changed proportion-

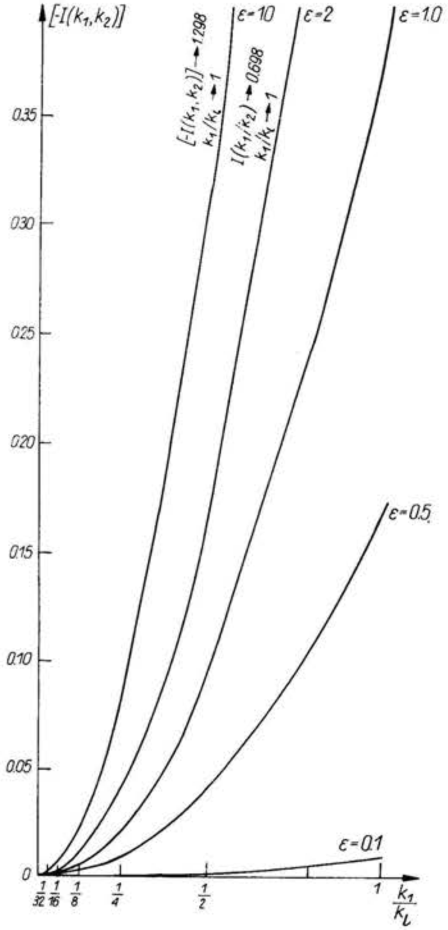

FIG. 11.

ally to the factor $\bar{I}(k_1, k_2)$ $(\bar{\mathbf{I}}(1 \rightleftarrows 2))$ and, in addition, proportionally to R_1^2 (in the case of $\mathbf{F}_1$) or to R_2^2 (in the case of $\mathbf{F}_2$).

The geometric factor $\bar{I}$ was evaluated numerically. Two kinds of graphs have been prepared: (1)-Graphs showing the dependence of $\bar{I}(k_1, k_2)$ on the distance d at fixed values of R_1 and R_2, that is $\bar{I}(R_1, \varepsilon = R_1/R_2; d)$ (or the equivalent dependence $\bar{I}(k_1, k_2)$ on the radius R_1 at fixed d and ε, Fig. 11); (2)-Graphs showing the dependence of the geometric factor $\bar{I}(k_1, k_2)$ on the ratio of radii $\varepsilon = R_1/R_2$ at fixed $k_l = (R_1 + R_2)/d$ and d, that is, $I(k_1, d; \varepsilon)$ (or the equivalent dependence $I(k_1, R_1; \varepsilon)$, Fig. 12). Detailed analysis of the numerical results proves $\bar{I}(k_1, k_2)$ to be, at constant radii, exactly inversely proportional to the square of distance d (or, equivalently, at fixed ε, d—simply proportional to the square of R_1 or R_2).

The dependence of $\bar{I}(k_1, k_2)$ on the ratio of the radii $\varepsilon = R_1/R_2$ at fixed $k_l = (R_1 + R_2)/d$ and d or R_1 is such that for $\varepsilon > 1$ the variation of $\bar{I}(k_1, k_2)$ following the changes of ε is slower than linear, while for $\varepsilon < 1$ — it is faster than linear (however, not faster

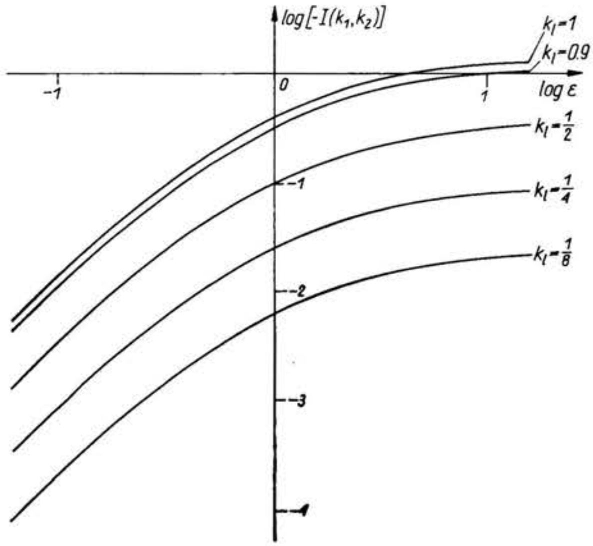

FIG. 12.

than parabolic). The values of $\bar{I}(k_1, k_2)$ become of interest when the spheres contact each other. In the case of spheres with equal radii ($\varepsilon = R_1/R_2 = 1$), we obtain $\bar{I} = -0.393$, for $R_2 = 10\,R_1$, $\bar{I} = -1.298$, and for $R_2 = R_1/10$, $\bar{I} = 0.01298$; it means that a tenfold increase of R_2 with respect to R_1 produces only a threefold increase of $\bar{I}(k_1, k_2)$, while a tenfold decrease of R_2 with respect to R_1 produces as much as a thirtyfold decrease of $\bar{I}(k_1, k_2)$ (at $\varepsilon \to 0$, $\bar{I} \to 0$, and with $\varepsilon \to \infty$, $\bar{I} \to$ const).

3. System of spheres moving along the central axis

3.1. Solution of continuity equations

The constant $A_{21}^{(r)}(P_1)$ is necessary for the determination of $F_{1(0)}$. Assuming the system velocity to be much smaller than the thermal velocity of medium particles (the hypersonic case was considered by the author in [3] and in [4] where it was solved by means of the Monte Carlo method), we may write

$$q \Big/ \left(\frac{2kT_0}{m}\right)^{1/2} \ll 1, \tag{3.1}$$

function $f_z^{(i)1}$ being approximated in the following manner:

$$f_z^{(i)(1)} = {}_{(0)}f^{(i)(1)} - {}_{(1)}f^{(i)(1)}, \tag{3.2}$$

$${}_{(0)}f^{(i)(1)} = A_z^{*(i)(1)} e^{-B_z^{(i)} c_{01}^2}, \tag{3.3}$$

$${}_{(1)}f^{(i)(1)} = \bar{A}_z^{(i)(1)} e^{-B_z^{(i)} c_{01}^2} (\mathbf{c}_{01} \cdot \mathbf{q}), \tag{3.4}$$

$$A_z^{*(i)(1)} = A_z^{(i)} e^{-B_z^{(i)} q^2}, \tag{3.5}$$

$$\bar{A}_z^{(i)(1)} = 2B_z^{(i)} A_z^{*(i)(1)}. \tag{3.6}$$

The approximation (3.2)–(3.4) is due to writing the distribution function $f_z^{(i)(1)}$ in the form of a product

$$f_z^{(i)(1)} = A_z^{(i)} e^{-B_z^{(i)}(c_{01}^2+q^2)} e^{-2B_z^{(i)}(\mathbf{c}_{01}\cdot\mathbf{q})} \tag{3.7}$$

and then to expanding the non-isotropic factor $\exp[-2B_z^{(i)}(\mathbf{c}_{01} \cdot \mathbf{q})]$ into a power series $B_z^{(i)}(\mathbf{c}_{01} \cdot \mathbf{q})$ around the point $\mathbf{c}_{01} \cdot \mathbf{q} = 0$, the linear term of the expansion being retained. In the following considerations to that approximation will be called the small velocity approximation. In the small velocity approximation, we obtain from the continuity equation

$$A_{1z}^{(r)}(P_1) = A_z^{*(i)(1)} \left[\left(\frac{2kT_0}{m}\right)^{3/2} : \left(b\left(\frac{2kT_1}{m}\right)^2\right)\right]\left[b\left(\frac{2kT_0}{m}\right)^{1/2} + \frac{4}{3}a(\mathbf{q}\cdot\mathbf{n}_1) + \dots\right] \tag{3.8}$$

$$a = \frac{1}{2}\Gamma\left(\frac{5}{2}\right), \quad b = \frac{1}{2}\Gamma(2).$$

Knowledge of the constant $A_{1w}^{(r)}$ is necessary for the determination of $\mathbf{F}_{1(in)}$. It requires the solution of a system of two integral equations. Since the attempts aimed at an accurate solution failed, the system is solved by the iteration method. The following iteration scheme is assumed:

$${}_{(n)}N_{1w}^{(r)}(P_1) = N_{1w(0)}^{(i)} + {}_{(n-1)}N_{1w(in)}^{(i)}, \tag{3.9}$$

where

$${}_{(n)}N_{1w}^{(r)} = N_{1w}^{(r)}({}_{(n)}f_{1w}^{(r)}), \quad {}_{(n-1)}N_{1w(in)}^{(i)} = N_{1w(in)}^{(i)(z)} + {}_{(n-1)}N_{1w(in)}^{(i)(K2)}, \tag{3.10}$$

$$N_{1w(in)}^{(i)(z)} \int\limits_{\Omega_{K2(P1)}^c} (-\mathbf{c}_{01}\cdot\mathbf{n}_1) f_z^{(i)(1)} d^3\mathbf{c}_{01}, \quad {}_{(n-1)}N_{1w(in)}^{(i)(K2)} \tag{3.11}$$

$$= N_{1w(in)}^{(i)(K2)}({}_{(n-1)}f_{2w}^{(r)}), \quad N_{1w(in)}^{(i)(K2)} = \int\limits_{\Omega_{K2(P1)}^c} (-\mathbf{c}_{21}\cdot\mathbf{n}) f_{K2}^{(i)}(P_2) d^3\mathbf{c}_{21}.$$

It means that the flux reflected from the sphere K_1 in the n-th iteration corresponds to the interaction flux (arriving from K_2) in the $(n-1)$ iteration.

The first iteraction is assumed in the form $(n = 1)$

$${}_{(0)}f_{2w}^{(r)} = f_{2z}^{(r)}, \quad ({}_{(0)}A_{2w}^{(r)} = A_{2z}^{(r)}), \tag{3.12}$$

i.e., the function of velocity distribution of particles in the interaction flux (the 0-iteration) is the same as if the other sphere were absent. We have then

$${}_{(1)}N_{1w}^{(r)} = N_{1w(0)}^{(i)} + {}_{(0)}N_{1w(in)}^{(i)}. \tag{3.13}$$

Identical iteration scheme is assumed for the second sphere. Applying the large distance approximation, the first iteration is effectively determined (in such a case, confining the considerations to the first iteration is sensible, since then the presence of the other sphere disturbes only but little the flux arriving to the sphere considered).

In calculating the individual fluxes occuring in the continuity equation, the function $f_z^{(i)(1)}$ is taken in the same approximation as in the evaluation of $A_{1z}^{(r)}$, that is, in the small velocity approximation. In the large distance approximation, in which $\Omega_{K2(P1)}$ may be assumed to be a cone (at every point of the surface Σ_{1w}), we obtain

$$
\begin{aligned}
{}_{(1)}N_{1w}^{r} &= {}_{(1)}A_{1w}^{r}(P_1)\pi b\left(\frac{2kT_1}{m}\right)^2,\\
N_{1w0}^{(i)} &= \pi A_z^{*(i)}\left(\frac{2kT_0}{m}\right)^{3/2}\left[b\left(\frac{2kT_0}{m}\right)^{1/2}+\frac{4}{3}a(\mathbf{q}\cdot\mathbf{n}_1)+\ldots\right],\\
N_{1w(in)}^{(i)(z)} &= A_z^{*(i)}\left(\frac{2kT_0}{m}\right)^{3/2}\left[b\left(\frac{2kT_0}{m}\right)^{1/2}W_{N1}-2aW_{N4}\right],\\
{}_{(0)}N_{1w(in)}^{(i)(K2)} &= A_z^{*(i)}\left(\frac{2kT_0}{m}\right)^{3/2}\left\{b\left(\frac{2kT_0}{m}\right)^{1/2}W_{N1}-\frac{4a}{3k_2^{*\prime}}q_z'W_{N1}\right.\\
&\qquad\left.+\frac{4a}{3k_2^{*\prime}}W_{N2}-\frac{4a}{3k_2^{*\prime}}W_{N3}\right\},
\end{aligned}
\tag{3.14}
$$

where

$$W_{N1} \stackrel{\mathrm{df}}{=} \int\limits_{\Omega_{K2(P1)}} (\mathbf{n}_1\cdot(-\mathbf{l}_{01}))\,d\Omega_{\mathbf{l}_{01}} = \pi k_2^{*\prime 2}\cos\theta_{n1}',$$

$$
\begin{aligned}
W_{N2} &\stackrel{\mathrm{df}}{=} \int\limits_{\Omega_{K2(P1)}} (\mathbf{n}_1(-\mathbf{l}_{22}))\cos\theta'(\mathbf{q}(-\mathbf{l}_{22}))\,d\Omega_{\mathbf{l}_{22}}\\
&= \frac{\pi}{2}k_2^{*\prime 4}(\mathbf{q}\cdot\mathbf{n}_1)+\frac{\pi}{2}k_2^{*\prime 2}\left(2-\frac{3k_2^{*\prime 2}}{2}\right)q_z'\cos\theta_{n1}',
\end{aligned}
$$

$$
\begin{aligned}
W_{N3} &\stackrel{\mathrm{df}}{=} \int\limits_{\Omega_{K2(P1)}} \sqrt{k_2^{*\prime 2}-\sin^2\theta'}\,(\mathbf{n}_1\cdot(-\mathbf{l}_{22}))(\mathbf{q}\cdot(-\mathbf{l}_{22}))\,d\Omega_{\mathbf{l}_{22}}\\
&= \frac{\pi}{4}\Bigg\{k_2^{*\prime}-(1-k_2^{*\prime 2})\,\mathrm{Arch}(1-k_2^{*\prime 2})^{-\frac{1}{2}}-\frac{1}{2}k_2^{*\prime 3}-\frac{1-k_2^{*\prime 2}}{4}\Big[k_2^{*\prime}-(1-k_2^{*\prime 2})\,\mathrm{Arch}(1\\
&\quad -k_2^{*\prime 2})^{-\frac{1}{2}}\Big]\Bigg\}\mathbf{q}\cdot\mathbf{n}_1+\pi q_z'\cos\theta_{n1}'\Bigg\{\Bigg\{-\frac{1}{2}[k_2^{*\prime}-(1-k_2^{*\prime 2})\,\mathrm{Arch}(1-k_2^{*\prime 2})^{-\frac{1}{2}}]\\
&\quad +3\left\{\frac{1}{4}k_2^{*\prime 3}+\frac{1-k_2^{*\prime 2}}{8}\left[k_2^{*\prime}-(1-k_2^{*\prime 2})\,\mathrm{Arch}\,(1-k_2^{*\prime 2})^{-\frac{1}{2}}\right]\right\}\Bigg\}\Bigg\}
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
W_{N4} &\stackrel{\mathrm{df}}{=} \int\limits_{\Omega_{K2(P1)}} (-\mathbf{l}_{01}\cdot\mathbf{n}_1)(\mathbf{l}_{01}\cdot\mathbf{q})\,d\Omega_{\mathbf{l}_{01}} = q_y'\sin\theta_{n1}\pi\Big[-\frac{2}{3}+\sqrt{1-k_2^{*\prime 2}}\\
&\quad -\frac{1}{3}(1-k_2^{*\prime 2})^{\frac{3}{2}}\Big]+q_z'\cos\theta_{n1}'\frac{2\pi}{3}[(1-k_2^{*\prime 2})^{\frac{3}{2}}-1]
\end{aligned}
$$

$$k_2^{*\prime} = R_2/d,\qquad d^* = P_1O_2;$$

θ'_{n1} is the azimuthal angle of the direction $\mathbf{n}_1$, θ' — of the direction-$\mathbf{l}_{01}$ or -$\mathbf{l}_{22}$, q'_x, q'_y, q'_z are components of the vector $\mathbf{q}$ in a local coordinate system.

On transforming the quantities $w_{N1}, w_{N2}, w_{N3}, w_{N4}$ to the absolute reference frame (Cartesian coordinate system with the origin at O_1 and the z-axis parallel to O_1O_2), we obtain

$$w_{N1} = \pi k_{2}^{*'2}\frac{d}{d^*}(-k_1+\cos\theta_{n1}), \quad d^* = d(1+k_1^2-2k_1\cos\theta_{n1})^{\frac{1}{2}},$$

$$w_{N2} = \frac{\pi}{2}k_{2}^{*'4}(\mathbf{q}\cdot\mathbf{n}_1)+\frac{\pi}{2}k_2^{*'2}\left(2-\frac{3}{2}k_2^{*'2}\right)\frac{d^2}{d^{*2}}q(-k_1+\cos\theta_{n1}$$

$$+k_1^2\cos\theta_{n1}-k_1\cos^2\theta_{n1}), \tag{3.16}$$

$$w_{N3} = \pi\left[\frac{1}{2}w-\frac{1}{4}k_{2}^{*'3}-\frac{1-k_2^{*'2}}{8}w\right](\mathbf{q}\cdot\mathbf{n}_1)+\pi\left[-\frac{1}{2}w+3\left(\frac{1}{4}k_2^{*'3}+\frac{1-k_2^{*'2}}{8}w\right)\right],$$

$$w_{N4} = w_{1q}\cos\theta_{n1}+q(w_2-w_1)\frac{d^2}{d^{*2}}(-k_1+\cos\theta_{n1}+k_1^2\cos\theta_{n1}-k_1\cos^2\theta_{n1}),$$

where

$$w = k_{2}^{*'}-(1-k_2^{*'2})\operatorname{Arch}(1-k_1^{*'2})^{-\frac{1}{2}},$$

$$w_1 = \pi\left[-\frac{2}{3}+\sqrt{1-k_2^{*'2}}-\frac{1}{3}(1-k_2^{*'2})^{\frac{3}{2}}\right], \quad w_2 = \frac{2\pi}{3}[(1+k_2^{*'2})^{\frac{3}{2}}-1]. \tag{3.17}$$

Here θ_{n1} — the angle made by $\mathbf{n}_1$ with the z-axis.

In the large distance approximation, the quantities w, w_1, w_2 behave as follows:

$$w \approx \tilde{w} = \frac{2}{3}k_2^{*'3}, \quad w_1 \approx \tilde{w}_1 = -\frac{\pi}{4}k_2^{*'4}, \quad w_2 \approx w_2 = \pi k_2^{*'2}\left(-1+\frac{1}{4}k_2^{*'2}\right). \tag{3.18}$$

The value of ${}_{(1)}A_{1w}^{(r)}(P_1)$ sought for in the large distance approximation and the first iteration is finally equal to

$$ {}_{(1)}A_{1w}^{(r)}(P_1) = \pi h_1 A_z^{*(i)}\left(\frac{2kT_0}{m}\right)^{3/2}\left[b\left(\frac{2kT_0}{m}\right)^{1/2}+\left(\frac{4}{3}a-\frac{26}{9}k_2^2a\right)q\cos\theta_{n1}\right], \tag{3.19}$$

$$h_1 = \left[\Pi b\left(\frac{2kT_1}{m}\right)^2\right]^{-1}.$$

Consequent application of the large distance approximation also yields the higher iterations, and by the method of induction — an arbitrary term of the iterational sequence [7]. Within the large distance approximation, convergence of that sequence may also be proved, and its limit — determined [7]. The limit iteration is shown to differ from the first one only by such terms which may (in the large distance approximation), be disregarded [7].

Exact and explicit formulae may also be derived for the iterational sequence ${}_{(n)}A_{1w}^{(r)}$, i.e., the formulae which hold true for arbitrary distances and which are expressible in terms of the initial iterations ($n = 0$ and $n = 1$). The general formulae so obtained may be applied for verification of the results derived in the large distance approximation by the direct method (consisting in the application of the large distance approximation in all consecutive iterations). Convergence of the corresponding results is complete [7].

3.2. Quadrature of interaction integrals

Let us integrate the expression for $\mathbf{F}_{1(0)}$ over the space of velocities

$$(3.20)\quad \mathbf{F}_{1(0)} = m \int_{\Sigma_1} \left[-\bar{A}_z^{(i)} d \left(\frac{2kT_0}{m}\right)^3 \mathbf{w}_{14} + A_z^{*(1)} a \left(\frac{2kT_0}{m}\right)^{5/2} \mathbf{w}_{13} - A_{1z}^{(r)} a \left(\frac{2kT_1}{m}\right)^{5/2} \mathbf{w}_9 \right] d\Sigma_1,$$

where

$$d = \frac{\Gamma(3)}{2}, \quad a = \frac{\Gamma\left(\frac{5}{2}\right)}{2},$$

$$(3.21)\quad w_{14x} = \frac{\pi}{6} q \sin\theta_{n1} \cos\theta_{n1} \cos\varphi_{n1}, \quad w_{14y} = \frac{\pi}{6} q \sin\theta_{n1} \cos\theta_{n1} \sin\varphi_{n1},$$

$$(3.22)\quad w_{14z} = \pi q \left(\frac{1}{3} + \frac{1}{6}\cos^2\theta_{n1}\right),$$

$$(3.23)\quad \mathbf{w}_{13} = -\frac{2\pi}{3}\mathbf{n}_1,$$

$$(3.24)\quad \mathbf{w}_9 = \frac{2\pi}{3}\mathbf{n}_1,$$

$\theta_{n1}, \varphi_{n1}$ are the azimuthal and polar angles of the normal $\mathbf{n}_1$ at the point P_1 in a rectangular coordinate system U with the z-axis directed parallel to $\mathbf{0}_1\mathbf{0}_2$. Vectors $\mathbf{w}_{14}$, $\mathbf{w}_{13}$, $\mathbf{w}_9$ had been evaluated first in a local coordinate system, connected locally with P_1 and with the z'-axis parallel to $\mathbf{n}_1$ (the x'-axis lies in the plane $(\mathbf{n}_1, z)$, $\mathbf{i}' \cdot \mathbf{k} \geqslant 0$, $\mathbf{i}' || x'$), and then transformed to the global system U according to the transformation matrix a_{ik},

$$(3.25)\quad a_{ik} = \begin{bmatrix} -\cos\theta_{n1}\cos\varphi_{n1} & \sin\varphi_{n1} & \sin\theta_{n1}\cos\varphi_{n1} \\ -\cos\theta_{n1}\sin\varphi_{n1} & -\cos\varphi_{n1} & \sin\theta_{n1}\sin\varphi_{n1} \\ \sin\theta_{n1} & 0 & \cos\theta_{n1} \end{bmatrix}.$$

Integration of the expression $\mathbf{F}_{1(in)el}^{(r)}$ over the region $\Omega_{1/2}^c$ of the space of velocities yields the result

$$(3.26)\quad \int_{\Omega_{1/2}^c} \mathbf{c}_{11}(\mathbf{c}_{11}\cdot\mathbf{n}_1(P_1)) f_{1w}^{(r)} d^3\mathbf{c}_{11} - \int_{\Omega_{1/2}^c} \mathbf{c}_{11}(\mathbf{c}_{11}\cdot\mathbf{n}_1(P_1)) f_{1z}^{(r)} d^3\mathbf{c}_{11}$$

$$= -\frac{26}{9}\pi\frac{a}{b}k_2^2\left(\frac{2kT_1}{m}\right)^{-2} A_z^{(*)(i)}\left(\frac{2kT_0}{m}\right)^{3/2}(\mathbf{q}\cdot\mathbf{n}_1) a \left(\frac{2kT_1}{m}\right)^{5/2}\mathbf{w}_{41},$$

where

$$\mathbf{w}_{41} \overset{df}{=} \int_{\Omega_{1/2}} \mathbf{l}_{11}(\mathbf{l}_{11}\cdot\mathbf{n}_1) d\Omega_{l11} = \frac{2\pi}{3}\mathbf{n}_1.$$

Integration of the remaining components of $\mathbf{F}_{1(in)}$ proceeds similarly to that performed in $\mathbf{F}_{1(0)}$; the difference consists in replacing the semispace of directions with the solid angle determined by the range of vision of the other sphere:

$$(3.27)\qquad \int\limits_{\Omega^c_{K1(P2)}} \mathbf{c}_{21}(\mathbf{c}_{22}\cdot\mathbf{n}_2)f^{(i)}_{K2}(P_2)d^3\mathbf{c}_{22} = A^{(i)}_{K2}(P_2)a\left(\frac{2kT_2}{m}\right)^{5/2}\mathbf{w}_1,$$

$$\mathbf{w}_1 \overset{df}{=} \int\limits_{\Omega K1(P_2)} \mathbf{l}_{22}(\mathbf{l}_{22}\cdot\mathbf{n}_2)d\Omega_{l_{22}},$$

$$(3.28)\qquad \int\limits_{\Omega_{K1(P2)}} \mathbf{c}_{21}(\mathbf{c}_{22}\cdot\mathbf{n}_2(P_2))f^{(i)(2)}_z d^3\mathbf{c}_{22} = A^{(i)}_z e^{-B_z q^2}a\left(\frac{2kT_0}{m}\right)^{5/2}\mathbf{w}_1+2A^{*(i)}_z d\left(\frac{2kT_0}{m}\right)^2\mathbf{w}_5,$$

$$\mathbf{w}_5 \overset{df}{=} \int\limits_{\Omega_{K1(P2)}} \mathbf{l}_{22}(\mathbf{l}_{22}\cdot\mathbf{n}_2)(\mathbf{l}_{22}\cdot\mathbf{q})d\Omega_{l_{22}}.$$

Vectors $\mathbf{w}_1$, $\mathbf{w}_5$ calculated in a local rectangular coordinate system with z' parallel to $\mathbf{P}_2\mathbf{O}_1$ were then transformed to the absolute system U connected with the sphere K_2, with the axis z parallel to $\mathbf{O}_2\mathbf{O}_1$. The matrix of transformation has the form

$$(3.29)\qquad a_{\alpha\beta} = \begin{bmatrix} \sin\varphi_{n2} & a^{(2)}(1-k_2\cos\theta_{n2})\cos\varphi_{n2} & -a^{(2)}k_2\sin\theta_{n2}\cos\varphi_{n2} \\ \cos\varphi_{n2} & a^{(2)}(1-k_2\cos\theta_{n2})\sin\varphi_{n2} & -a^{(2)}k_2\sin\theta_{n2}\sin\varphi_{n2} \\ 0 & a^{(2)}k_2\sin\theta_{n2} & a^{(2)}(1-k_2\cos\theta_{n2}) \end{bmatrix};$$

θ_{n2}, φ_{n2} are the azimuthal and polar angles of the normal $\mathbf{n}_2$ to the sphere K_2 (determined by P_2) in the absolute frame of reference U; $\mathbf{n}_2 \in (z', y')$ and $\mathbf{n}_2\cdot\mathbf{j}' \geqslant 0$.

In order to determine the forces $\mathbf{F}_{1(0)}$ and $\mathbf{F}_{1(in)}$, the corresponding expressions must be integrated over the surface. The integration is easily performed,

$$(3.30)\qquad \mathbf{F}_{1(0)} = -\frac{2}{9}m\pi^2R_1^2qA^{*(i)}_z\left(\frac{2kT_0}{m}\right)^{3/2}\left[5B^{(i)}_z d\left(\frac{2kT_0}{m}\right)^{3/2}+\frac{16}{3}a^2\left(\frac{2kT_1}{m}\right)^{1/2}\right]\mathbf{k}_1.$$

On evaluating the surface integral of $F_{1(in)}$, use is made of the assumption of large distances, $k_1 \ll 1$, $k_2 \ll 1$. Disregarding in the expressions for w_1, w_5 the terms containing k_1, k_2 in powers higher than 2, we obtain

$$(3.31)\qquad w_{1z} \approx \tilde{w}_{1z} = \pi k_1^2\cos\theta_{n2},$$

$$(3.32)\qquad w_{5z} \approx \tilde{w}_{5z} = -\pi k_1^2 q\cos\theta_{n2}.$$

The remaining components are of little importance since the corresponding integrals vanish owing to the symmetry properties.

From the solution of the continuity equations we obtain, like in Eqs. (3.19), the result

$$(3.33)\qquad {}_{(1)}A^{(r)}_{2w} = h_2\pi A^{*(i)}_z\left(\frac{2kT_0}{m}\right)^{3/2}\left[b\left(\frac{2kT_0}{m}\right)^{1/2}+\left(\frac{4}{3}a-\frac{26}{9}k_1^2a\right)(\mathbf{q}\cdot\mathbf{n}_2)\right],\qquad h_2 = h_1(\to 2).$$

Integration of $\mathbf{F}_{1(in)}$ over the surface yields

$$(3.34)\qquad \mathbf{F}_{1(n)} = mA^{*(i)}_z\left(\frac{2kT_0}{m}\right)^{3/2}\left\{a\left(\frac{2kT_0}{m}\right)^{1/2}\left[\left(\frac{2kT_2}{m}\right)^{1/2}-\left(\frac{2kT_0}{m}\right)^{1/2}\right]_{(0)}\bar{\tilde{\mathbf{w}}}_1\right.$$

$$\left.+\frac{2}{3}\frac{a^2}{b}\left(\frac{2kT_2}{m}\right)^{1/2}\left(2-\frac{13}{3}k_2^2\right)_{(1)}\bar{\tilde{\mathbf{w}}}_1+2d\left(\frac{2kT_0}{m}\right)^{1/2}\bar{\tilde{\mathbf{w}}}_s,\right.$$

where

$$ {}_{(0)}\bar{\tilde{\mathbf{w}}}_1 = \int\limits_{\Sigma_{w2}} \tilde{\mathbf{w}}_1 \, d\Sigma_{w2} = \pi^2 R_2^2 k_1^2 \mathbf{k}_2, \tag{3.35}$$

$$ {}_{(1)}\bar{\tilde{\mathbf{w}}}_1 = \int\limits_{\Sigma_{w2}} \tilde{\mathbf{w}}_1 (\mathbf{q}\cdot\mathbf{n}_2) \, d\Sigma_{w2} = -\frac{8}{27}\pi^2 R_2^2 q k_1^2 \mathbf{k}_2, \tag{3.36}$$

$$ \bar{\tilde{\mathbf{w}}}_s = \int\limits_{\Sigma_{w2}} \bar{\mathbf{w}}_5 \, d\Sigma_{w2} = -\pi^2 R_2^2 q k_1^2 \mathbf{k}_2, \quad \mathbf{k}_2 \stackrel{\text{df}}{=} \frac{\mathbf{O}_2\mathbf{O}_1}{O_2O_1}. \tag{3.37}$$

Summing up the results obtained we arrive at the final expression for the force acting on the sphere K_1,

$$\begin{aligned} \mathbf{F}_1 = -2mA_{z}^{*(i)} \left(\frac{2kT_0}{m}\right)^{3/2} \pi^2 R_1^2 \Bigg\{ \Bigg\{ \frac{1}{9} q \left[5\left(\frac{2kT_0}{m}\right)^{1/2} + \frac{3}{4}\pi \left(\frac{2kT_1}{m}\right)^{1/2} \right] \\ - \frac{13}{72}\pi^2 k_2^2 q \left(\frac{2kT_1}{m}\right)^{1/2} + k_2^2 \Bigg\{ \frac{3}{16}\sqrt{\pi}\left(\frac{2kT_0}{m}\right)^{1/2} \left[\left(\frac{2kT_2}{m}\right)^{1/2} - \left(\frac{2kT_0}{m}\right)^{1/2} \right] \\ - \frac{1}{18} q \left(\frac{2kT_2}{m}\right)^{1/2} - \left(\frac{2kT_0}{m}\right)^{1/2} q \Bigg\} \Bigg\} \mathbf{k}_1 . \end{aligned} \tag{3.38}$$

3.3. Interpretation of results and conclusions

The first term in brackets corresponds to the drag acting on the sphere K_1, which moves in the medium in absence of the other sphere, the other term in brackets (proportional to k_2^2) represents the interaction with the other sphere K_2, and namely: the first term in that expression, proportional to the mean thermal velocity of the medium $(2kT_0/m)^{1/2}$ corresponds to interaction of the spheres being at rest with respect to the medium; it may denote either repulsion or attraction, for $T_2 > T_0$ or $T_2 < T_0$, respectively. The second term is proportional to q and represents the contribution to the forces of interaction following from the motion of the spheres with respect to the medium; the force is attractive.

Similar expressions are obtained in calculating the force acting on the K_2 – sphere (it may formally be obtained from $\mathbf{F}_1$ by the transformations $\mathbf{k}_1 \to \mathbf{k}_2$, $q \to -q$, $T_1 \rightleftarrows T_2$, $k_1 \rightleftarrows k_2$)

$$\begin{aligned} \mathbf{F}_2 = 2mA_{z}^{*(i)} \left(\frac{2kT_0}{m}\right)^{3/2} \pi^2 R_2^2 \Bigg\{ \frac{1}{9} q \left[5\left(\frac{2kT_0}{m}\right)^{1/2} + \frac{3}{4}\pi \left(\frac{2kT_2}{m}\right)^{1/2} \right] \\ - \frac{13}{72}\pi^2 k_1^2 q \left(\frac{2kT_2}{m}\right)^{1/2} + k_1^2 \Bigg\{ -\frac{3}{16}\sqrt{\pi}\left(\frac{2kT_0}{m}\right)^{1/2} \left[\left(\frac{2kT_1}{m}\right)^{1/2} - \left(\frac{2kT_0}{m}\right)^{1/2} \right] \\ - \frac{1}{18} q \left(\frac{2kT_1}{m}\right)^{1/2} - \left(\frac{2kT_0}{m}\right)^{1/2} q \Bigg\} \Bigg\} \mathbf{k}_2 . \end{aligned} \tag{3.39}$$

Also similar to the previous results are the physical meanings of the individual terms in the expression for $\mathbf{F}_2$.

It is interesting to note that the interaction term due to the motion of the both bodies of the system with respect to the medium has the direction opposite to that of the drag.

The possibility of occuring, in the expressions for $\mathbf{F}_1$ and $\mathbf{F}_2$, of attractive and repulsive terms, gains a considerable physical importance, since—with T_0, T_1, T_2 being fixed — at certain values of the distance d between the spheres and their velocity, the forces may vanish. Under the conditions of vanishing forces and at fixed temperatures of the spheres, the system moves as a "free" system(1).

The conditions for a stationary "free" state may be written as follows:

$$\mathbf{F}_{(1)}(q, d; T_1, T_2, T_0) = 0,$$

$$\mathbf{F}_{(2)}(q, d; T_1, T_2, T_0) = 0.$$

The motion considered is one-dimensional and hence, at fixed values of T_1, T_2, T_0, the set of equations completely determines q and d. A suitable choice of T_1, T_2 may lead to the total force acting in the same direction as the velocity (the system would be accelerated and could be termed a thermal free-molecular engine).

Using the formulae derived let us estimate the values of forces acting on bodies placed in the space around the earth, where the conditions of free-molecular medium exist. To this end, the expressions for forces are represented in the form more suitable for numerical calculations:

$$\mathbf{F}_\alpha = \mathbf{F}_{\alpha(0)} + \mathbf{F}^{(0)}_{\alpha(in)} + \mathbf{F}^{(2)}_{\alpha(in)}.$$

Here

$$\mathbf{F}^{(0)}_{\alpha(in)} = -\frac{3}{4}\pi R_\alpha^2 k_\beta^2 p_0 (x_\beta - 1) e^{-q_0^2} \mathbf{k}_\alpha,$$

$$\mathbf{F}_{\alpha(0)} = \frac{4}{9}\sqrt{\pi} p_0 R_\alpha^2 (-q^0)^\alpha \left(5 + \frac{3}{4}\pi x_z\right) e^{-q_0^2} \mathbf{k}_\alpha,$$

$$\mathbf{F}_{\alpha(in)} = -\sqrt{\pi} p_0 (-q^0)^\alpha e^{-q_0^2} R_\alpha^2 k_\beta^2 \left(\frac{13}{18}\pi^2 x_\alpha + \frac{2}{9} x_\beta + 4\right) \mathbf{k}_\alpha$$

and

$$p_0 = A_z^{(i)} \left(\frac{2\pi k T_0}{m}\right)^{3/2} k T_0,$$

$$x_\alpha = \left(\frac{2kT_\alpha}{m}\right)^{1/2} : \left(\frac{2kT_0}{m}\right)^{1/2},$$

$$q^0 = q : \left(\frac{2kT_0}{m}\right)^{1/2},$$

$$\mathbf{k}_\alpha = \frac{\mathbf{O}_\alpha \mathbf{O}_\beta}{O_\alpha O_\beta}, \quad \alpha, \beta = 1,2, \alpha \neq \beta.$$

Let us assume $R_1 = R_2 = 1\,\mathrm{m}$, $X_1 = 1$, $X_2 = 2$, $q^0 = 1/3$, $k_2^2 = k_1^2 = 1/10$. The values of q^0, k_2 and k_1 assumed here are so selected that the small velocity and large dis-

(1) The system cannot be completely free, since this requires the temperatures to remain constant in the process of motion, and it is not possible in a system of that type.

tance approximations are still applicable; p_0 is assumed to be equal to the pressure existing at the elevation of 130–200 km above the earth [5]. The results are as follows:

H[km]	$F_{1(0)}\times$ $\times 10^2$ dynes $\cdot \mathbf{k}_1$	$F_{1(in)}^{(0)}\times$ $\times 10^2$ dynes $\cdot \mathbf{k}_1$	$F_{1(in)}^{(0)}\times$ $\times 10^2$ dynes $\cdot \mathbf{k}_1$	$F_1\times$ $\times 10^2$ dynes $\cdot \mathbf{k}_1$
130	−1.94	−0.24	0.65	−1.53
150	−0.83	−0.103	0.274	−0.66
200	−0.22	−0.028	0.076	−0.18

H [km]	$F_{2(0)}\times$ $\times 10^2$ dynes $\cdot \mathbf{k}_1$	$F_{2(in)}^{(0)}\times$ $\times 10^2$ dynes $\cdot \mathbf{k}_1$	$F_{2(in)}^{(0)}\times$ $\times 10^2$ dynes $\cdot \mathbf{k}_1$	$F_2\times$ $\times 10^2$ dynes $\cdot \mathbf{k}_1$	$\mathbf{F} = (F_1+F_2)\times$ $\times 10^2$ dynes $\cdot \mathbf{k}_1$
130	−2.79	0	1.32	−1.47	−3.00
150	−1.18	0	0.66	−0.51	−1.27
200	−0.325	0	0.155	−0.17	−0.325

The accelerations corresponding to the resultant force, calculated under the assumption that the spheres are homogeneous and their density is equal to ca $\cdot$ 3g/cm^3, are as follows:

$$H = 130\,\text{km} - 2.4\cdot 10^{-5}\,\text{cm/sec}^2 = 2.4\cdot 10^{-8}\,\text{g},$$

$$H = 150\,\text{km} - 1.0\cdot 10^{-5}\,\text{cm/sec}^2 = 1.0\cdot 10^{-8}\,\text{g},$$

$$H = 200\,\text{km} - 0.26\cdot 10^{-5}\,\text{cm/sec}^2 = 0.26\cdot 10^{-8}\,\text{g},$$

g denoting the acceleration of gravity. The corresponding accelerations calculated for shells may be larger by 2–3 orders of magnitude.

The accelerations are inversely proportional to R_α (the forces — to R_α^2, and mass — to R_α^3), and thus for much smaller spheres the forces may prove to be much greater. For a sphere of radius $R = 1$ cm, e. g., the acceleration is 100 times greater, for a $1\,\mu$ radius — 10^6 times greater. In the space around the earth we may thus expect rather considerable interactions between minute particles, and at lower elevations, where the pressure is much higher, also between the micrometeorites.

From the considerations concerning the spheres at rest it follows that the geometric factor — its counterpart in the present problem being the quantity $k_\alpha^2\pi/2$ — may reach values close to unity; it means that at small distances we may expect the forces to increase by one order of magnitude. The forces (accelerations) may also be changed by increasing the velocity of the system.

It should be noted that the forces of interaction connected with the motion $F_{\alpha(in)}^q$ are comparable with the drag $F_{\alpha(0)}$ even at low temperatures of the other sphere and at large distances d (we have assumed $k_\alpha^2 = 1/10$).

The effects of interaction could easily be observed in the case of free spheres (the problem is different but may be approximated by the solutions presented), since the forces would cause certain changes in distances and velocities. Owing to the cumulative character of the dependence upon the time, the effects could be detected by observation.

The estimations may also be used in the problem of spheres at rest. In order to determine the forces it is sufficient to multiply $F^{(0)}_{\alpha(in)}$ by the factor $k_\alpha^2\pi/2$ (factor $\bar{I}_\alpha$ corresponds to $\pi k_\alpha^2/2$).

3.4. Heat exchange

The problem of exchange of the energy ([2]) is solved similarly to the problem of momentum exchange. The former problem being solved, we may use the solutions of the continuity equations and certain quadratures. The general expression (before the integration) for the heat exchange is formally obtained from the expressions describing the exchange of momentum by replacing the vector quantity **c** (appearing at the scalar product) with the scalar $c^2/2$:

$$Q_{(1)} = Q_{1(0)}+Q_{(1)(in)}, \tag{3.40}$$

$$Q_{1(0)} = m\int\limits_{\Sigma_1}\Big[\int\limits_{\Omega^c_{1/2}} c^2_{01}(-\mathbf{c}_{01}\cdot\mathbf{n}_1)f^{(i)}_z d^3\mathbf{c}_{01}-\int\limits_{\Omega^c_{1/2}} c^2_{11}(\mathbf{c}_{11}\cdot\mathbf{n}_1)f^{(r)}_{21}d^3\mathbf{c}_{11}\Big]d\Sigma_1, \tag{3.41}$$

$$\begin{aligned} Q_{1(in)} = {} & -\frac{m}{2}\int\limits_{\Sigma_{1w}}\Big[\int\limits_{\Omega^c_{1/2}} c^2_{11}(\mathbf{c}_{11}\cdot\mathbf{n}_1)f^{(r)}_{1w}d^3\mathbf{c}_{11}-\int\limits_{\Omega^c_{1/2}} c^2_{11}(\mathbf{c}_{11}\cdot\mathbf{n}_1)f^{(r)}_{1z}d^3\mathbf{c}_{11}\Big] \\ & +\frac{m}{2}\int\limits_{\Sigma_{2w}}\Big[\int\limits_{\Omega^c_{K1(P2)}} c^2_{21}(-\mathbf{c}_{22}\cdot\mathbf{n}_2)f^{(i)}_z d^3\mathbf{c}_{22}+\int\limits_{\Omega^c_{K1(P2)}} c^2_{21}(\mathbf{c}_{22}\cdot\mathbf{n}_2)f^{(i)}_{K2}d^3\mathbf{c}_{22}\Big]d\Sigma_{2w}. \end{aligned} \tag{3.42}$$

It is found that, in the approximation for the distribution function $f^{(i)}_z$ used in the evaluation of **F**, no terms involving the velocity appear in the expression for $Q_{1(0)}$ which corresponds to the heat exchange due to the motion of K_1 through the medium in absence of the other sphere. This is a result of preserving in the expansion for $f^{(i)}_z$ the single term linear in **c** and that term is negligible in the problem of heat exchange between a single body and the medium (it yields a zero contribution). The terms containing the velocity of the system appear in $Q^{(1)}_{(in)}$ — that fact is connected with the solution of the continuity equations and is of the same character as in $\mathbf{F}_{1(in)}$. This explains the fact that in calculating $Q_{1(0)}$ ($Q_{2(0)}$) the function $f^{(i)}_z$ was assumed in the form of an expansion involving the term quadratic in **c**; such expansions are sensible as long as $q \ll 1$, and this assumption was made earlier:

$$f^{(i)}_z = A^{(i)}_z e^{-B_z(i)(c^2_{01}+q^2)}[1-2B^{(i)}_z(\mathbf{c}_{01}\mathbf{q})+2(B^{(i)}_z)^2(\mathbf{c}_{01}\mathbf{q})^2+\ \dots]. \tag{3.43}$$

It follows that a term proportional to the square of velocity of the system appears in $Q_{1(0)}$; the term represents the exchange of heat connected with the motion of the body in the medium. On applying the expansion (3.43) to the solution of the continuity equation and to the evaluation of $Q_{1\,(in)}$, we obtain certain terms containing q in powers higher

([2]) The problem of non-stationary energy exchange (account being taken of the radiation effects) for a simple system of two parallel plane plates in the case when one of the plates is much larger than the other was solved in [6].

than 2; these terms are consequently disregarded. Retaining the same approximations which were used in calculating **F**, we obtain the expression for the heat exchange

$$(3.44)\quad Q_1 = mA_z^{*(i)}\left(\frac{2kT_0}{m}\right)^3 2\pi^2 R_1^2\left\{2\left[1-x_1+\frac{1}{3}q_0^2\left(\frac{3}{2}-x_1\right)\right]\right.$$
$$\left.+\frac{1}{3}k_2^2\left\{2\left[x_2-1-\frac{\sqrt{\pi}}{2}q_0\left(\frac{15}{8}+\left(1-\frac{13}{6}\pi k_2^2\right)x_2\right)\right]+\frac{13}{4}\pi^{3/2}q_0x_1\right\}\right\}.$$

$$(3.45)\quad Q_2 = mA_z^{*(i)}\left(\frac{2kT_0}{m}\right)^3 2\pi^2 R_2^2\left\{2\left[1-x_2+\frac{1}{3}q_0^2\left(\frac{3}{2}-x_2\right)\right]\right.$$
$$\left.+\frac{1}{3}k_1^2\left\{2\left[x_1-1-\frac{\sqrt{\pi}}{2}q_0\left(\frac{15}{8}+\left(1-\frac{13}{6}\pi k_1^2\right)x_1\right)\right]-\frac{13}{4}\pi^{3/2}q_0x_2\right\}\right\}.$$

$$(3.46)\quad q_0 = q\Big/\left(\frac{2kT_0}{m}\right)^{1/2},\qquad x_1 = \left(\frac{2kT_1}{m}\right)^{1/2}\Big/\left(\frac{2kT_0}{m}\right)^{1/2},$$
$$x_2 = \left(\frac{2kT_2}{m}\right)\Big/\left(\frac{2kT_0}{m}\right)^{1/2}.$$

Similarly to the momentum exchange problem we may observe the possibility of reaching the states in which no heat exchange between the bodies and the medium takes place. These states are determined from the conditions

$$Q_1(T_0, T_1, T_2; q, d) = 0,$$
$$Q_2(T_0, T_1, T_2; q, d) = 0,$$

which are fulfilled for definite values of the distance d and velocity q.

3.5. Final remarks

Some of the approximations and simplifications introduced in this paper might be disregarded; by neglecting e.g. the large distance approximation we may reduce the expressions for forces and heat exchange to single integrals. The integration over the space of velocities may be performed, while the surface integral, which cannot be reduced to an analytical form, is represented in the form of a single integral owing to the cylindrical symmetry of the problem.

We may also preserve the large distance approximation and disregard the other approximation of small velocity of the system (small as compared to the thermal velocity). The distribution function $f_z^{(i)}$ of incident particles which occurs in the terms expressing interaction (the region of integration being determined by the solid angle of the range of vision of the other sphere) may be approximated by the formula

$$f_z^{(i)} \approx A_z^{(i)} e^{-B_z^{(i)}(c^2+q^2-2cq)}.$$

Here the sign $+$ refers to the particles incident at the sphere K_1, and the sign $-$ to the particles incident at the sphere K_2. In this approximation, the last quadrature should be performed numerically.

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