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A rigourous derivation of the equations of compressible viscous fluid motion with gravity at low Mach number

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We exhibit two limiting forms of the Navier-Stokes equations for a heavy, compressible, viscous and thermally conducting fluid, when the characteristic Mach number tends to zero. Usually in aerodynamics, gravity is neglected, but, if to take gravity into account, the study of the flow is *fundamentally different* from the classical study. This asymptotic theory presented here, permits, to obtain not only the classical Boussinesq equations but also to define the limits of validity of the approximations through which these equations are obtained.

W pracy przedstawiono dwie graniczne postacie równań Naviera-Stokesa dla ciężkiego, lepkiego i przewodzącego ciepło płynu ściśliwego przy charakterystycznej liczbie Macha, dążącej do zera. Zazwyczaj ciężkość jest pomijana przy badaniach aerodynamicznych, lecz wzięcie jej pod uwagę *zmienia radykalnie* obraz przepływu w porównaniu do obrazu uzyskiwanego za pomocą badań klasycznych. Przedstawiona tu asymptotyczna teoria pozwala nie tylko uzyskać klasyczne równania Boussinesqa, lecz również określić granice, w których można stosować przybliżenia niezbędne do ich uzyskania.

Представлены две граничные формы уравнений Навье-Стокеса для тяжелой, вязкой и теплопрводящей, сжимаемой жидкости при характеристическом числе Маха стремящимся к нулю. Обычно в аэродинамике пренебрегут силой тяжести, но если принять ее во внимание, картина течения принципиально отличается от классической. Представлена в этой работе асимптотическая теория позволяет не только получить классические уравнения Буссинеска но и указать границы применимости приближений нужных для их получения.

1. Introduction

Usually in aerodynamics, gravity is neglected. Hence we study flows at small Mach number in the stationary case by means of a classical perturbation method, known as the Janzen-Rayleigh method, and in the non-stationary case by means of the method of matched asymptotic expansions (Viviand, 1970).

But, if we take gravity into account, the study of the flow is fundamentally different from the classical study, because there appear two new non-dimensional parameters related respectively to the variations of the basic temperature $T_\infty(z)$ and pressure $p_\infty(z)$, as functions of the altitude z. Then, when $\mathsf{M}_\infty^0 \Rightarrow 0$, it will be necessary to define the manner in which these parameters vary with M_∞^0. This is done by writing similarity relations which depend upon constant parameters called *similarity parameters* (of the limiting flow).

Once these equations are written in non-dimensional form, one seeks their solution as expansions in powers of M_∞^0.

The first terms of these expansions are, in first approximation, the unknown functions which satisfy the limiting system obtained when $M^0_\infty \Rightarrow 0$, and when similarity relations are verified; then, these functions depend only upon the reduced variables and similarity parameters.

It is necessary to notice that the method used here, permits one to determine not only the exact form of the expansion in powers of M^0_∞ of the different characteristic functions of the flow, but the similarity relations as well and also the domain (D) of validity of these expansions and relations.

One of the results of this work is the rigourous definition of the limits in which Boussinesq approximations are valid.

This validity has been the object, during the last years, of rigourous works like those of SPIEGEL and VERONIS (1960), MIHALJAN (1962), PHILIPS (1967). However, their approach is implicitly based on the physics of the problem which differs it from ours. Note also the works of DUTTON and FICHTL (1969), DRAZIN (1961) and ZEYTOUNIAN (1972a and b).

2. General equations

p, ϱ, T, a and s are respectively the pressure, density, temperature, sound speed and specific entropy; p_∞, ϱ_∞, T_∞, a_∞ and s_∞ denote the corresponding variables relative to the unperturbed flow. Perturbations of the non-stationary flow of heavy viscous compressible and thermally conducting fluid are to be considered as a non-linear phenomenon in the space-time E^4 of the four variables x, y, z, t, where x, y, z are defined with respect to a rectangular frame fixed to a portion of the unperturbed flow.

The Coriolis force is neglected, but the force due to gravity g in the fluid is taken into account.

In what follows, we do not take into account the causes of the perturbation, which depend upon the physical problem to be considered.

The unit vectors of the x, y, z axes are denoted $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the flow velocity $\mathbf{V} = u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$.

The heavy viscous fluid is treated as a perfect gas with c_p and c_v being constant $\left(\gamma = \frac{c_p}{c_v},\ R = c_p - c_v\right)$.

The thermal conductivity and the viscosity parameters λ_0 and μ_0 are constant, satisfying Stokes hypothesis:

$$3\lambda_0+2\mu_0 = 0; \tag{2.1}$$

$\mathbf{k}$ is the upward vertical and the Navier-Stokes equations can be written as:

$$\frac{\partial\varrho}{\partial t}+\mathbf{V}\cdot\nabla\varrho+\varrho(\nabla\cdot\mathbf{V}) = 0;$$

$$\varrho\left\{\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla\right\}\mathbf{V}+\nabla p+\varrho g k = \mu_0\left\{\nabla^2\mathbf{V}+\frac{1}{3}\nabla(\nabla\cdot\mathbf{V})\right\},$$

$$c_p\varrho\left\{\frac{\partial T}{\partial t}+\mathbf{V}\cdot\nabla T-\frac{\gamma-1}{\gamma}\frac{T}{p}\left(\frac{\partial p}{\partial t}+\mathbf{V}\cdot\nabla p\right)\right\} = k_0\nabla^2 T+\mu_0\left\{\phi-\frac{2}{3}(\nabla\cdot\mathbf{V})^2\right\}, \tag{2.2}$$

$$p = R\varrho T,$$

$$\phi = 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right\} + \frac{1}{2}\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right\},$$

where ϕ is the viscous dissipation.

In the problem which it suffices to suppose that p_∞, ϱ_∞, T_∞, a_∞ and s_∞ are functions of the altitude z only; then

$$\frac{dp_\infty}{dz} + \varrho_\infty g = 0; \quad p_\infty = R\varrho_\infty T_\infty; \tag{2.3}$$
$$a_\infty = (\gamma R T_\infty)^{1/2}, \quad s_\infty = c_v \mathrm{Log}\frac{P_\infty}{\varrho_\infty^\gamma}.$$

For the unperturbed flow, (2.3) verify the general equations; we shall suppose that:

$$\frac{d^2 T_\infty}{dz^2} \equiv 0 \Rightarrow -\frac{dT_\infty}{dz} = \Gamma_\infty^0 = \text{const}, \tag{2.4}$$

and

$$u_\infty = U_\infty^0 = \text{const}, \quad v_\infty = V_\infty^0 = \text{const}, \tag{2.5}$$
$$w_\infty \equiv 0.$$

In the perturbed flow, we write

$$p = p_\infty(1+\pi), \quad \varrho = \varrho_\infty(1+\omega), \quad T = T_\infty(1+\theta), \tag{2.6}$$

and if we take account of the relations:

$$\frac{1}{\varrho_\infty}\frac{d\varrho_\infty}{dz} = \{-g/R + \Gamma_\infty^0\}\frac{1}{T_\infty}; \tag{2.7}$$
$$\frac{1}{T_\infty}\frac{dT_\infty}{dz} - \frac{\gamma-1}{\gamma}\frac{1}{p_\infty}\frac{dp_\infty}{dz} = \left\{-\Gamma_\infty^0 + \frac{\gamma-1}{\gamma}g/R\right\}\frac{1}{T_\infty},$$

we obtain for $\mathbf{V}$, π, ω and θ the following system, consequence of (2.2):

$$\pi = \omega + \theta + \omega\theta; \tag{2.8$_1$}$$

$$(1+\omega)\left\{\nabla\cdot\mathbf{V} + [-g/R + \Gamma_\infty^0]\frac{w}{T_\infty}\right\} + \frac{\partial\omega}{\partial t} + \mathbf{V}\cdot\nabla\omega = 0; \tag{2.8$_2$}$$

$$(1+\omega)\left\{\frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} - g\theta\mathbf{k}\right\} = -RT_\infty\nabla\pi + \frac{\mu_0}{\varrho_\infty}\left\{\nabla^2\mathbf{V} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{V})\right\}; \tag{2.8$_3$}$$

$$(1+\omega)\left\{\frac{\partial\theta}{\partial t} + \mathbf{V}\cdot\nabla\theta\right\} - \frac{\gamma-1}{\gamma}\left(\frac{\partial\pi}{\partial t} + \mathbf{V}\cdot\nabla\pi\right) + (1+\pi)\left[-\Gamma_\infty^0 + \frac{\gamma-1}{\gamma}g/R\right]\frac{w}{T_\infty} \tag{2.8$_4$}$$
$$= \frac{1}{\mathrm{Pr}}\frac{\mu_0}{\varrho_\infty}\left\{\nabla^2\theta - 2\frac{\Gamma_\infty^0}{T_\infty}\frac{\partial\theta}{\partial z}\right\} + \frac{\mu_0}{c_p\varrho_\infty T_\infty}\left\{\phi - \frac{2}{3}(\nabla\cdot\mathbf{V})^2\right\},$$

where $\mathrm{Pr} = \dfrac{\mu_0}{k_0}c_p$ is the Prandtl number.

3. Reduced equations and dimensionless parameters

Let L and H be the horizontal (along x and y) and vertical (along z) characteristic dimensions of the domain (D); then we define reduced coordinates:

$$x_1 = \frac{x}{L}, \quad x_2 = \frac{y}{L}, \quad x_3 = \frac{z}{H}. \tag{3.1}$$

Also let t^0 be a characteristic time of the perturbed flow in (D), different from the characteristic times deduced from L, H and U_∞^0, W_∞^0; U_∞^0 and $W_\infty^0 = U_\infty^0 H/L$ being the order of magnitude of u, v and w in (D). Finally, we define

$$\tau = t/t^\circ, \quad v_1 = \frac{u}{U_\infty^0}, \quad v_2 = \frac{v}{U_\infty^0}, \quad v_3 = \frac{w}{W_\infty^0},$$
$$\bar{p} = p/p_\infty^0, \quad \bar{\varrho} = \varrho/\varrho_\infty^0, \quad \bar{T} = T/T_\infty^0, \tag{3.2}$$

where p_∞^0, ϱ_∞^0 and T_∞^0 are reference values for p_∞, ϱ_∞ and T_∞.

With the reduced variables τ, x_i, we obtain for the dimensionless functions v_i, π, ω and θ the following system:

$$\pi = \omega + \theta + \omega\theta; \tag{3.3$_1$}$$

$$(1+\omega)\left\{\frac{\partial v_i}{\partial x_i} + [-\delta_0 + \alpha_0]\frac{v_3}{\bar{T}_\infty}\right\} + \beta_0 \frac{\partial \omega}{\partial \tau} + v_i \frac{\partial \omega}{\partial x_i} = 0; \tag{3.3$_2$}$$

$$(1+\omega)\left\{\beta_0 \frac{\partial v_\alpha}{\partial \tau} + v_i \frac{\partial v_\alpha}{\partial x_i}\right\} = -\frac{\bar{T}}{\gamma M_\infty^{02}} \frac{\partial \pi}{\partial x_\alpha} + \frac{1}{\mathrm{Re}} \frac{1}{\bar{\varrho}_\infty}\left\{\Delta_{12} v_\alpha + \frac{1}{\varepsilon^2}\frac{\partial^2 v_\alpha}{\partial x_3^2} + \frac{1}{3}\frac{\partial}{\partial x_\alpha}\left(\frac{\partial v_i}{\partial x_i}\right)\right\}, \quad (\alpha = 1, 2); \tag{3.3$_3$}$$

$$(1+\omega)\left\{\beta_0 \frac{\partial v_3}{\partial \tau} + v_i \frac{\partial v_3}{\partial x_i} - \frac{1}{\varepsilon^2}\frac{\delta_0}{\gamma M_\infty^{02}}\theta\right\} = -\frac{1}{\varepsilon^2}\frac{\bar{T}_\infty}{\gamma M_\infty^{02}}\frac{\partial \pi}{\partial x_3} + \frac{1}{\mathrm{Re}}\frac{1}{\bar{\varrho}_\infty}\left\{\Delta_{12} v_3 + \frac{1}{\varepsilon^2}\frac{\partial^2 v_3}{\partial x_3^2} + \frac{1}{3}\frac{1}{\varepsilon^2}\frac{\bar{\partial}}{\partial x_3}\left(\frac{\partial v_i}{\partial x_i}\right)\right\}; \tag{3.3$_4$}$$

$$\begin{aligned}(1+\omega)\left(\beta_0 \frac{\partial \theta}{\partial \tau} + v_i \frac{\partial \theta}{\partial x_i}\right) - \frac{\gamma-1}{\gamma}\left(\beta_0 \frac{\partial \pi}{\partial \tau} + v_i \frac{\partial \pi}{\partial x_i}\right) \\ + (1+\pi)\left[-\alpha_0 + \frac{\gamma-1}{\gamma}\delta_0\right]\frac{v_3}{\bar{T}_\infty} = \frac{1}{\mathrm{Pr}}\frac{1}{\mathrm{Re}}\frac{1}{\bar{\varrho}_\infty}\left\{\Delta_{12}\theta + \frac{1}{\varepsilon^2}\frac{\partial^2\theta}{\partial x_3^2} - \frac{2}{\varepsilon^2}\frac{\alpha_0}{\bar{T}_\infty}\frac{\partial \theta}{\partial x_3}\right\} \\ + (\gamma-1)\mathrm{M}_\infty^{02}\frac{1}{\mathrm{Re}}\frac{1}{\bar{\varrho}_\infty \bar{T}_\infty}\left\{\bar{\phi} - \frac{2}{3}\left(\frac{\partial v_i}{\partial x_i}\right)^2\right\},\end{aligned} \tag{3.3$_5$}$$

where we have noted:

$$\Delta_{12} \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

and $\overline{\phi}$ the non-dimensional form of $\phi = (U^0_\infty/L)^2\overline{\phi}$. The following dimensionless parameters appear in the reduced system:

$$\begin{aligned}
&\delta_0 = H \Big/ \frac{RT^0_\infty}{g} \equiv \frac{H}{H^0_\infty}, \quad \alpha_0 = \Gamma^0_\infty \Big/ \frac{T^0_\infty}{H} \equiv \frac{H}{H_\infty},\\
&\beta_0 = L/U^0_\infty t^0 \qquad \text{(Strouhal number)},\\
&\varepsilon = H/L \qquad \text{(form parameter of } (D);\\
&\mathsf{M}^{02}_\infty = U^{02}_\infty/\gamma RT^0_\infty \qquad \text{(Mach number)},\\
&\mathrm{Re} = \frac{U^0_\infty L}{\mu_0/\varrho^0_\infty} \qquad \text{(Reynolds number)},\\
&\mathrm{Pr} = \frac{\mu_0}{k_0} C_p \qquad \text{(Prandtl number)}.
\end{aligned} \tag{3.4}$$

In writing $(3.3)_4$ we have taken into account the relation:

$$\frac{\gamma \mathsf{M}^{02}_\infty}{\mathrm{Fr}^2} = \delta_0, \tag{3.5}$$

where $\mathrm{Fr}^2 = U^{02}_\infty/gH$ is the Froude number.

We can also define the Scorer-Dorodnitsyn number:

$$\frac{\mathsf{M}^0_\infty}{\mathrm{Fr}^2} = \frac{\delta_0}{\gamma \mathsf{M}^0_\infty} \equiv D_\infty = gH/U^0_\infty(\gamma RT^0_\infty)^{1/2}; \tag{3.6}$$

δ_0 and α_0 are the two new parameters mentioned in the Introduction.

4. Fundamental hypothesis

In what follows we suppose an infinitely small Mach number

$$\mathsf{M}^0_\infty \ll 1 \Rightarrow U^0_\infty \ll (\gamma RT^0_\infty)^{1/2} \tag{4.1}$$

and we will obtain limiting forms of the reduced system (3.3) when $\mathsf{M}^0_\infty \Rightarrow 0$. This system contains the dimensionless parameters (3.4) and with M^0_∞:

$$\delta_0,\ \alpha_0,\ \beta_0,\ \varepsilon,\ \mathrm{Re},\ \mathrm{Pr} \text{ and } \gamma. \tag{4.2}$$

It is necessary to specify how the dimensionless parameters (4.2) depend on M^0_∞. We shall assume some relations of the form:

$$F_j\{\delta_0, \alpha_0, \beta_0, \varepsilon, \ldots, \mathsf{M}^0_\infty\} = S_j, \quad j = 1, 2, \ldots, \tag{4.3}$$

in which S_j are constants independent of the fact "$\mathsf{M}^0_\infty \Rightarrow 0$"; (4.3) are called "similarity relations" and the S_j similarity parameters (of the obtained limiting flow).

We postulate, for $\mathsf{M}^0_\infty \ll 1$, the existence of asymptotic expansions of the following form:

$$\begin{aligned}
v_i &= v^0_i + \mathsf{M}^{0^a}_\infty v^a_i + \ldots;\\
\pi &= \mathsf{M}^{0^b}_\infty \pi_b + \ldots;\\
\omega &= \mathsf{M}^{0^c}_\infty \omega_c + \ldots;\\
\theta &= \mathsf{M}^{0^d}_\infty \theta_d + \ldots;,
\end{aligned} \tag{4.4}$$

where a, b, c, d are positive numbers.

The functions v_i^0, π_b, ω_c and θ_d are, in first approximation, the reduced unknowns and depend only on x_i, τ and S_j.

In (4.4) we implicitly make the hypothesis that $\mathsf{M}_\infty^0 \ll 1$, the perturbations π, ω and θ being also small; in a precise manner, we shall assume that they are respectively of order M_∞^{0b}, M_∞^{0c} and M_∞^{0d}. When $\mathsf{M}_\infty^0 \Rightarrow 0$, we shall keep certain parameters (4.2) fixed and independent of the manner in which M_∞^0 tends to zero; in this case, the similarity relations relative to these parameters are simply identities.

5. Boussinesq equations

Let

$$\mathsf{M}_\infty^0 \Rightarrow 0, \quad \delta_0 \Rightarrow 0, \quad \alpha_0 \Rightarrow 0, \tag{5.1}$$

so that the similarity relations:

$$\frac{\delta_0}{\mathsf{M}_\infty^0} = \gamma D_\infty = S_1, \quad \frac{\alpha_0}{\delta_0} = S_2, \tag{5.2}$$

are satisfied.

In obtaining the solution of the reduced Eqs. (3.3) in the form (4.4) with the hypothesis (5.1) and taking into account the relations (5.2), one sees that a minimum singularity of Eqs. (3.3) occurs for:

$$a = 1, \quad b = 2, \quad c = d = 1. \tag{5.3}$$

That is to say, one may seek a solution of the reduced equations (3.3) in the form:

$$\begin{aligned} v_i &= v_i^0 + \mathsf{M}_\infty^0 v_i^1 + \dots \\ \pi &= \mathsf{M}_\infty^{02}\pi_2 + \dots \\ \omega &= \mathsf{M}_\infty^0 \omega_1 + \dots \\ \theta &= \mathsf{M}_\infty^0 \theta_1 + \dots . \end{aligned} \tag{5.4}$$

The functions v_i^0, π_2, ω_1, θ_1 satisfy the following limiting equations:

$$\omega_1 = -\theta_1, \tag{5.5$_1$}$$

$$\frac{\partial v_i^0}{\partial x_i} = 0, \tag{5.5$_2$}$$

$$\beta_0 \frac{\partial v_\alpha^0}{\partial \tau} + v_i^0 \frac{\partial v_\alpha^0}{\partial x_i} = -\frac{1}{\gamma}\frac{\partial \pi_2}{\partial x_\alpha} + \frac{1}{\mathrm{Re}}\left\{\Delta_{12} v_\alpha^0 + \frac{1}{\varepsilon^2}\frac{\partial^2 v_\alpha^0}{\partial x_3^2}\right\}, \quad \alpha = 1, 2, \tag{5.5$_3$}$$

$$\beta_0 \frac{\partial v_3^0}{\partial \tau} + v_i^0 \frac{\partial v_3^0}{\partial x_i} = -\frac{1}{\varepsilon^2}\frac{1}{\gamma}\frac{\partial \pi_2}{\partial x_3} + \frac{1}{\varepsilon^2}\frac{1}{\gamma} S_1 \theta_1 + \frac{1}{\mathrm{Re}}\left\{\Delta_{12} v_3^0 + \frac{1}{\varepsilon^2}\frac{\partial^2 v_3^0}{\partial x_3^2}\right\}, \tag{5.5$_4$}$$

$$\beta_0 \frac{\partial \theta_1}{\partial \tau} + v_i^0 \frac{\partial \theta_1}{\partial x_i} + S_1\left[\frac{\gamma-1}{\gamma} - S_2\right] v_3^0 = \frac{1}{\mathrm{Pr}}\frac{1}{\mathrm{Re}}\left\{\Delta_{12}\theta_1 + \frac{1}{\varepsilon^2}\frac{\partial^2 \theta_1}{\partial x_3^2}\right\}. \tag{5.5$_5$}$$

The limiting system (5.5) is called the "Boussinesq equations" or "Shallow convection" equations. These equations contain the parameters β_0, γ, Re, ε and Pr, as well as the two similarity parameters:

$$S_1 = \gamma D_\infty, \quad S_2 = \frac{R}{g} \Gamma_\infty^0. \tag{5.6}$$

In $(5.5)_5$ the expression in brackets is the stability parameter of the unperturbed flow; three cases are possible:

$$\begin{aligned} \Gamma_\infty^0 &= \frac{g}{R}\frac{\gamma-1}{\gamma} \quad \text{(neutral)}, \\ \Gamma_\infty^0 &> \frac{g}{R}\frac{\gamma-1}{\gamma} \quad \text{(unstable)}, \\ \Gamma_\infty^0 &< \frac{g}{R}\frac{\gamma-1}{\gamma} \quad \text{(stable)}. \end{aligned} \tag{5.7}$$

If $S_1 = S_2 = 0$, (5.5) is the classical system for incompressible fluid flows.

If S_2 alone tends to zero, the parameter $D_\infty = S_1/\gamma$ is the unique fundamental one, depending on gravity, in the shallow convection equations; then, if $S_2 \ll 1$:

$$\frac{RT_\infty^0}{g} \ll \frac{T_\infty^0}{\Gamma_\infty^0} \Rightarrow H_\infty^0 \ll H_{T_\infty}, \tag{5.8}$$

i.e., the characteristic scale which is related to the variation of the basic temperature $T_\infty(z)$ with altitude z, is much greater than the altitude H_∞^0 of the homogeneous atmosphere corresponding to the reference temperature T_∞^0.

Note finally that, in obtaining (5.5), we have used the following fact: when $\alpha_0 \Rightarrow 0$ then $\bar{\varrho}_\infty \Rightarrow 1$ and $\bar{T}_\infty \Rightarrow 1$; this follows from Eqs. (3.4), making use of (2.4)([1])

$$\bar{T}_\infty = 1 - \alpha_0 x_3, \quad \bar{\varrho}_\infty = \bar{T}_\infty^{1/S_2 - 1}. \tag{5.9}$$

In the course of the analysis by means of which (5.5) was found, four characteristic length scales are used: L and H are related to the characteristic dimensions of $(\hat{D})$, the domain of validity of the Boussinesq equations, and H_∞^0 and H_{T_∞} related, respectively, to the variations of the pressure $p_\infty(z)$ and temperature $T_\infty(z)$, as functions of the altitude z.

The fact that the parameters δ_0 and α_0 are infinitely small with the Mach number M_∞^0 imposes the following constraints on the domain $(\hat{D})$ of validity of the Boussinesq equations:

$$H_{T_\infty} \gg H \ll H_\infty^0. \tag{5.10}$$

We then conclude that the *vertical characteristic dimension of* $(\hat{D})$ *must be much less than*:

i) the altitude of the homogeneous atmosphere corresponding to the reference temperature T_∞^0;

([1]) One can easily convince oneself that, in the case where $S_2 \to 0$, $\alpha_0 \to 0$ implies that $\bar{\varrho}_\infty \to 1$. Moreover, in (5.9) we have assumed $T_\infty^0 \equiv T_\infty(0)$.

ii) the characteristic vertical scale related to the variation of the basic tempearture $T_\infty(z)$ as a function of the altitude z.

Moreover if one wishes, within the framework of the Boussinesq Eqs. (5.5), to differentiate a perfect fluid region and a boundary layer in the vicinity of a wall bounding the limit flow described by (5.5), the characteristic vertical dimension of $(\hat{D})$ must also satisfy the inequality:

$$H \gg \sqrt{\frac{\nu_\infty^0 L}{U_\infty^0}} \Rightarrow \varepsilon \gg \sqrt{1/\mathrm{Re}}, \tag{5.11}$$

where $H_{\nu_\infty^0} \equiv \sqrt{\frac{\nu_\infty^0 L}{U_\infty^0}}$ $(\nu_\infty^0 = \mu_0/\varrho_\infty^0)$ enters as a characteristic vertical scale related to viscosity: (5.11) gives a lower bound for H.

6. Deep convection equations

We shall obtain, in this section, the limiting equations (when $\mathsf{M}_\infty^0 \Rightarrow 0$) which describe the flow in a domain $(\bar{D})$ of characteristic vertical dimension of order H_∞^0; these equations become precisely the "Deep convection" equations.

Since we suppose H to be of the same order as $H_\infty^0 = RT_\infty^0/g$, we must assume, at first, that δ_0 is bounded when $\mathsf{M}_\infty^0 \Rightarrow 0$. We do not know, *a priori*, the manner in which α_0 is related to M_∞^0 and δ_0.

We shall consider the reduced system (3.3); equation $(3.3)_4$ implies, for δ_0 bounded, the necessity of imposing in the expansions (4.4):

$$a = 1 \quad \text{and} \quad b = d = 2, \tag{6.1}$$

and, $(3.3)_1$(²) implies also that:

$$c = 2. \tag{6.2}$$

We are then led to seek the solution of the reduced Eqs. (3.3) in the following form:

$$\begin{gathered} v_i = \bar{v}_i^0 + \mathsf{M}_\infty^0 \bar{v}_i^1 + \ldots; \\ \begin{bmatrix} \pi \\ \omega \\ \theta \end{bmatrix} = \mathsf{M}_\infty^{02} \begin{bmatrix} \bar{\pi}_2 \\ \bar{\omega}_2 \\ \bar{\theta}_2 \end{bmatrix} + \ldots; \end{gathered} \tag{6.3}$$

then, $(3.3)_5$ gives, when $\mathsf{M}_\infty^0 \Rightarrow 0$:

$$\begin{aligned} (6.4)_1 \quad & \beta_0 \frac{\partial \bar{\theta}_2}{\partial \tau} + \bar{v}_i^0 \frac{\partial \bar{\theta}_2}{\partial x_i} - \frac{\gamma-1}{\gamma}\left(\beta_0 \frac{\partial \bar{\pi}_2}{\partial \tau} + \bar{v}_i^0 \frac{\partial \bar{\pi}_2}{\partial x_i}\right) + \frac{-\alpha_0 + \frac{\gamma-1}{\gamma}\delta_0}{\mathsf{M}^{02}} \frac{\bar{v}_3^0}{\bar{T}_\infty} \\ & + \left(-\alpha_0 + \frac{\gamma-1}{\gamma}\delta_0\right)\bar{\pi}_2 \frac{\bar{v}_3^0}{\bar{T}_\infty} = \frac{1}{\mathrm{Pr}} \frac{1}{\mathrm{Re}} \frac{1}{\bar{\varrho}_\infty}\left\{\Delta_{12}\bar{\theta}_2 + \frac{1}{\varepsilon^2}\frac{\partial^2 \bar{\theta}_2}{\partial x_3^2} - \frac{2}{\varepsilon^2}\frac{\alpha_0}{\bar{T}_\infty}\frac{\partial \bar{\theta}_2}{\partial x_3} \right. \\ & \left. + \mathrm{Pr}\frac{\gamma-1}{\bar{T}_\infty}\bar{\phi}_0 - \frac{2}{3}(\delta_0-\alpha_0)^2\left(\frac{\bar{v}_3^0}{\bar{T}_\infty}\right)^2\right\}, \end{aligned}$$

(²) Indeed, if $c = 1$, the state equation $(3.3)_1$ gives $\omega_1 \equiv 0$ and, if $c > 2$, it is easy to verify (since $(3.3)_1$ implies $\pi_2 = \theta_2$) that the limiting system is over-determined; that is, there are more equations than unknowns.

where $\bar{\Phi}_0$ is the viscous dissipation written in a non-dimensional form by means of the velocity components $\bar{v}_i^0$. In equation (6.4), the following expression appears:

$$\left(\frac{\gamma-1}{\gamma}\delta_0-\alpha_0\right)/\mathsf{M}_\infty^{02}$$

and if we require it to be bounded (for $\mathsf{M}_\infty^0 \Rightarrow 0$), then

$$\frac{\gamma-1}{\gamma}\delta_0-\alpha_0 \Rightarrow 0 \quad \text{vhen} \quad \mathsf{M}_\infty^0 \Rightarrow 0 \tag{6.5}$$

so that

$$\frac{\dfrac{\gamma-1}{\gamma}\delta_0-\alpha_0}{\mathsf{M}_\infty^{02}} = S_3 \tag{6.6}$$

remains bounded; S_3 is a similarity parameter for the limiting flow so obtained.

Hence we shall impose the following condition on α_0:

$$\alpha_0 = \frac{\gamma-1}{\gamma}\delta_0-\mathsf{M}_\infty^{02}S_3, \tag{6.7}$$

in *order to obtain a compatible limiting system.*

Hence, when:

$$\mathsf{M}_\infty^0 \Rightarrow 0, \quad \text{with } \delta_0 \text{ bounded, } \alpha_0 = \frac{\gamma-1}{\gamma}\delta_0-\mathsf{M}_\infty^{02}S_3, \tag{6.8}$$

with S_3 bounded, we obtain for the reduced functions $\bar{v}_i^0$, $\bar{\pi}_2$, $\bar{\omega}_2$ and $\bar{\theta}_2$ in the expansions (6.3), the following limiting system which follows from (3.3)

$$\bar{\pi}_2 = \bar{\omega}_2+\bar{\theta}_2, \qquad \frac{\partial \bar{v}_i^0}{\partial x_i} = \frac{\delta_0}{\gamma \bar{T}_\infty}\bar{v}_3^0,$$

$$\beta_0\frac{\partial \bar{v}_\alpha^0}{\partial \tau}+\bar{v}_i^0\frac{\partial \bar{v}_\alpha^0}{\partial x_i} = -\frac{\bar{T}_\infty}{\gamma}\frac{\partial \bar{\pi}_2}{\partial x_\alpha}+\frac{1}{\mathrm{Re}\bar{\varrho}_\infty}\left\{\Delta_{12}\bar{v}_\alpha^0+\frac{1}{\varepsilon^2}\frac{\partial^2 \bar{v}_\alpha^0}{\partial x_3^2}+\frac{1}{3\gamma\bar{T}_\infty}\delta_0\frac{\partial \bar{v}_3^0}{\partial x_\alpha}\right\}, \quad \alpha = 1, 2,$$

$$\beta_0\frac{\partial \bar{v}_3^0}{\partial \tau}+\bar{v}_i^0\frac{\partial \bar{v}_3^0}{\partial x_i} = -\frac{\bar{T}_\infty}{\varepsilon^2\gamma}\frac{\partial \bar{\pi}_2}{\partial x_3}+\frac{\delta_0}{\varepsilon^2\gamma}\bar{\theta}_2+\frac{1}{\mathrm{Re}\bar{\varrho}_\infty}\left\{\Delta_{12}\bar{v}_3^0+\frac{1}{\varepsilon^2}\frac{\partial^2 \bar{v}_3^0}{\partial x_3^2}+\frac{\delta_0}{3\gamma\varepsilon^2}\frac{\partial}{\partial x_3}\left(\frac{\bar{v}_3^0}{\bar{T}_\infty}\right)\right\}, \tag{6.9}$$

$$\beta_0\frac{\partial \bar{\theta}_2}{\partial \tau}+\bar{v}_i^0\frac{\partial \bar{\theta}_2}{\partial x_i}-\frac{\gamma-1}{\gamma}\left(\beta_0\frac{\partial \bar{\pi}_2}{\partial \tau}+\bar{v}_i^0\frac{\partial \bar{\pi}_2}{\partial x_i}\right)+\frac{S_3}{\bar{T}_\infty}\bar{v}_3^0 = \frac{1}{\mathrm{PrRe}\bar{\varrho}_\infty}\left\{\Delta_{12}\bar{\theta}_2+\frac{1}{\varepsilon^2}\frac{\partial^2 \bar{\theta}_2}{\partial x_3^2}-2\frac{\gamma-1}{\gamma}\frac{\delta_0}{\varepsilon^2\bar{T}_\infty}\frac{\partial \bar{\theta}_2}{\partial x_3}+\mathrm{Pr}\frac{\gamma-1}{\bar{T}_\infty}\bar{\phi}_0-\frac{2}{3}\frac{\delta_0^2}{\gamma^2}\left(\frac{\bar{v}_3^0}{\bar{T}_\infty}\right)^2\right\}.$$

The limiting equations (6.9) are those of *deep convection* where δ_0/γ and S_3 are similarity parameters.

Naturally, it may occur that $S_3 \equiv 0$, i.e.,

$$\Gamma^0_\infty = \frac{\gamma-1}{\gamma} g/R. \tag{6.10}$$

In any case, relation (6.7) indicates that Γ^0_∞ must be very near the "dry adiabatic gradient" of temperature: $\Gamma_A \equiv \frac{\gamma-1}{\gamma}\frac{g}{R}$. In the deep convection equations (6.9), we have:

$$\bar{T}_\infty = 1 - \frac{\Gamma_A H}{T^0_\infty} x_3, \quad \bar{\varrho}_\infty = \bar{T}_\infty^{1/\gamma-1}. \tag{6.11}$$

7. Relation between shallow and deep convection equations

From Secs. 5 and 6 we have, in the case of shallow convection equations (5.5), the expansions:

$$\left\{ \begin{array}{l} v_i \sim v_i^0 + \dots \\ \pi \sim \mathsf{M}^{02}_\infty \pi_2 + \dots \\ \omega \sim \mathsf{M}^0_\infty \omega_1 + \dots \\ \theta \sim \mathsf{M}^0_\infty \theta_1 + \dots \\ \text{with the assumptions that} \\ \mathsf{M}^0_\infty \Rightarrow 0, \quad \delta_0 \Rightarrow 0, \quad \alpha_0 \Rightarrow 0, \\ \dfrac{\delta_0}{\mathsf{M}^0_\infty} = S_1, \quad \dfrac{\alpha_0}{\delta_0} = S_2; \quad S_1 \text{ et } S_2 \text{ bounded,} \end{array} \right. \tag{7.1}$$

and for the deep convection (6.9):

$$\left\{ \begin{array}{l} v_i \sim \bar{v}_i^0 + \dots \\ \pi \sim \mathsf{M}^{02}_\infty \bar{\pi}_2 + \dots \\ \omega \sim \mathsf{M}^{02}_\infty \bar{\omega}_2 + \dots \\ \theta \sim \mathsf{M}^{02}_\infty \bar{\theta}_2 + \dots \\ \text{with the assumptions that} \\ \mathsf{M}^0_\infty \Rightarrow 0, \quad \delta_0 \text{ bounded,} \\ \alpha_0 = \dfrac{\gamma-1}{\gamma}\delta_0 - \mathsf{M}^{02}_\infty S_3; \quad S_3 \text{ bounded.} \end{array} \right. \tag{7.2}$$

We shall prove that the deep convection equations (6.9) include the shallow convection ones (5.5) or, more precisely, that *the shallow convection equations are the limiting form of the deep convection equations for* $\delta_0 \Rightarrow 0$.

In fact, let us consider the relations:

$$\bar{v}_i^0 = v_i^0, \quad \bar{\pi}_2 = \pi_2, \quad \bar{\omega}_2 = \omega_1/\mathsf{M}^0_\infty, \quad \bar{\theta}_2 = \theta_1/\mathsf{M}^0_\infty \tag{7.3$_1$}$$

$$S_2 \equiv \frac{\alpha_0}{\delta_0} = \frac{\gamma-1}{\gamma} - \frac{S_3}{S_1}\mathsf{M}^0_\infty \Rightarrow S_3 = \left(\frac{\gamma-1}{\gamma} - S_2\right) \Big/ \frac{\mathsf{M}^0_\infty}{S_1}, \tag{7.3$_2$}$$

which follow by comparing (7.1) and (7.2). Actually, if we take into account the relations (7.3) in (6.9), the limiting system, which results from (6.9) when:

$$\delta_0 \Rightarrow 0 \Rightarrow \alpha_0 = S_2\delta_0 \Rightarrow 0,$$

and

$$\mathsf{M}_\infty^0 \Rightarrow 0 \Rightarrow \frac{\delta_0}{\mathsf{M}_\infty^0} = S_1,$$

is the system which governs the shallow convection [Eqs. (5.5) of BOUSSINESQ].

It is interesting to note that: we can write the "deep convection" equations (6.9) using instead of x_3 the outer variable (vertical coordinate):

$$\tilde{x}_3 = \frac{z}{\mathsf{M}_\infty^0} = \delta_0 x_3, \tag{7.4}$$

and instead of $\bar{v}_3^0$ the outer vertical velocity component:

$$\tilde{v}_3^0 = \delta_0 \bar{v}_3^0. \tag{7.5}$$

In terms of the variables x_1, x_2, $\tilde{x}_3$ and the unknown functions $\bar{v}_1^0$, $\bar{v}_2^0$, $\tilde{v}_3^0$, $\bar{\pi}_2$, $\bar{\omega}_2$ and $\bar{\theta}_2$, the deep convection equations (outer equations) bring in the parameters

$$\sigma_0 \equiv \frac{\delta_0}{\varepsilon} = \frac{L}{H_\infty^0} \text{ and } \frac{S_3}{\delta_0}. \tag{7.6}$$

For the shallow convection equations (5.5), we can introduce the inner variable (vertical coordinate):

$$\hat{x}_3 = S_1 x_3 = S_1\frac{\tilde{x}_3}{\delta_0} = \frac{\tilde{x}_3}{\mathsf{M}_\infty^0} \tag{7.7}$$

and the inner vertical velocity component:

$$\hat{v}_3^0 = \tilde{v}_3^0/\mathsf{M}_\infty^0. \tag{7.8}$$

With the variables x_1, x_2, $\hat{x}_3$ and the unknown functions v_1^0, v_2^0, $\hat{v}_3^0$, π_2, ω_1 and θ_1, the shallow convection equations (inner equations) depend upon the parameter

$$\frac{s_1}{\varepsilon} = \gamma\frac{gL}{U_\infty^0(\gamma R T_\infty^0)^{1/2}}. \tag{7.9}$$

Thus we can conclude that, in fact, it is possible to *consider the deep convection equations as outer ones*, in the sense of matching asymptotic expansion (VAN DYKE, 1964), and *the shallow convection equations* (BOUSSINESQ) *as inner equations*.

The outer equations contain entirely the inner ones and are therefore uniformly valid everywhere in the domain $(\bar{D})$.

8. Conclusion

We have seen that the limiting equations governing the viscous, compressible fluid flows with gravity at low Mach number [Eqs. (6.9) of "deep convection"] contain the Boussinesq equation (5.5); these latter equations can be considered as "inner equations" describing the limiting flow in the domains $(\hat{D})$ with vertical characteristic dimension much smaller than the altitude $H^0_\infty = RT^0_\infty/g$ of the homogeneous atmosphere.

The asymptotic theory presented there, then permits one, to obtain not only the classical Boussinesq equations but also to define the limits of validity of the approximations through which these equations are obtained.

Moreover this theory is a rational one; that is to say, it is possible if necessary to go beyond the limiting case.

It is also clear that we could have taken into account, the complementary accelaration (Coriolis force) in equations (2.2); in this case and in the limiting equations (5.5) and (6.9) a new parameters is introduced: the Rossby number: $R_0 = U^0_\infty/2\Omega_0 L$, where Ω_0 is the constant angular speed of the frame $R(x, y, z)$, moving about some axis which can be chosen as the z axis. Then, it is easily conceivable that in flows at small Mach and Rossby numbers the ratio R_0/M^0_∞ will be essential, and analogously for flows with small Mach and Reynolds numbers it is the ratio Re/M^0_∞ which will be important. In these two cases, it is necessary to re-examine the reduced equations (3.3) and to look for the new corresponding limiting forms. We hope to do this in a forthcoming work.

Finally, notice that, every change other than (7.1) and (7.2) gives limiting systems "more degenerate" than those obtained in sections 5 and 6.

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