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To the shallow water theory foundation

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THE PAPER is devoted to shallow water model foundation in the class of analytical functions for plane unsteady motions of an incompressible liquid in the field of gravity above a horizontal bottom in the case of periodic waves. The conformal mapping of the flow domain on a strip is used. For estimates, a special mechanism of the type of analytical majorants for nonlocal operators is applied. The basic result is obtained with the help of the author's abstract theorem of existence and uniqueness of the Cauchy problem solution in the scale of Banach spaces.

Praca poświęcona jest podstawom modelu wody płytkiej w klasie funkcji analitycznych dla płaskich, nieustalonych ruchów cieczy nieściśliwej w polu ciężkości przy założeniu płaskiego dna i fal okresowych. Zastosowano odwzorowanie konforemne obszaru przepływu na pasmo. Dla oszacowań wprowadzono specjalne narzędzie typu majoranty analitycznej dla operatorów nielokalnych. Podstawowy wynik uzyskano za pomocą wyprowadzonego przez autora abstrakcyjnego twierdzenia o istnieniu i jednoznaczności rozwiązania problemu Cauchy'ego w obszarze przestrzeni Banacha.

Работа посвящена обоснованию модели мелкой воды в классе аналитических функций для плоских, неустановившихся движений несжимаемой жидкости в поле тяжести, при предположении плоского дна и периодических волн. Применено конформное преобразование области течения в полосу. Для оценок использован специальный аппарат типа аналитических мажорант для нелокальных операторов. Основный результат получен при помощи ранее доказанной автором абстрактной теоремы существования и единственности решения задачи Коши в шкале банаховых пространств.

1. Introduction

THE SHALLOW water model as an asymptotic approximation in the problems of the wave motions on a liquid surface in the field of gravity with a small depth is widely known and is being applied in many of the modern investigations of both theoretical and applied character. The greatest known hitherto advance in this theory was obtained by FRIEDRICHS [1] who showed the way of producing the formal shallow water equations with the aid of the regular development with respect to the small parameter. The notable peculiarity of this approximation, in which the starting elliptic problem is replaced with the hyperbolic one, and the desirability of its precise derivation are mentioned in Stoker's monograph [2]. Nevertheless, the problem of mathematical foundation of the shallow water theory remained absolutely untouched up to now. It is not a surprise as for such a foundation one should have beforehand, as a minimum, a theorem of existence of the Cauchy-Poisson's problem solution in the exact statement. Such theorems have been obtained for plane problems only recently by NALIMOV [3] who based upon the author's paper [4]. The possibility was ascertained in these papers of the constructive investigations in the field of the problems with the free boundary in the classes of the analytical functions and thereby the way was found out by which one may try to ground various approximations. In particular, NALIMOV has recently fulfilled a strict foundation of the linear theory of the surface waves in a plane problem.

In the present paper, by the foundation of the shallow water theory is meant the demonstration of the two statements: (A) the exact solution of the Cauchy-Poisson's problem exists at any value of the small parameter generating the shallow water model and (B) this solution tends to the solution of the limit problem for the shallow water equations with the small parameter tending to zero.

Here such a foundation is given for plane potential motions of an incompressible liquid above the horizontal bottom in the class of periodic analytical functions. To this end, the auxiliary conformal mapping of the domain on a strip is used. For analysis of the equivalent system obtained, the technique of scales of Banach spaces of analytical functions, adapted to the concrete problem under consideration, is used. The results have been summarized in two theorems corresponding to the above mentioned points (A) and (B). Theorem 1 is the theorem of existence of the Cauchy-Poisson's problem solution with a uniform estimate with respect to the small parameter. Its demonstration is reduced to the verification of the condition of a general abstract author's theorem [5]. Theorem 2 establishes the uniform convergence of the exact Cauchy-Poisson's problem solution towards the solution of the shallow water equations in the scale of Banach space metric. In conclusion, some of the perspective directions associated with the subject of the given investigation are noted.

2. The plane Cauchy-Poisson's problem

The Cauchy-Poisson's problem of unsteady waves on the surface of an ideal incompressible liquid of the finite depth above the horizontal bottom in the case of two-dimensional potential motion is stated as follows. The initial form of free surface $y = \Gamma_0(x) > 0$ and the function $\varphi_0^*(x, y)$ harmonic in the domain $\Omega_0 = \{(x, y) | -\infty < x < +\infty, 0 < y < \Gamma_0(x)\}$ and representing the initial velocity potential field in Ω_0, provided that $\varphi_{0y}^*(x, 0) = 0$, are given. The functions $\Gamma = \Gamma(t, x)$ and $\varphi^* = \varphi^*(t, x, y)$ for $t \geqslant 0$ are to be determined so that, with each fixed t, the function φ^* be harmonic in the domain

$$\Omega_t = \{(x, y) | -\infty < x < +\infty,\ 0 < y < \Gamma(t, x)\},$$

the relations being performed on the line $y = \Gamma(t, x)$ (the force of gravity acceleration is assumed to be equal to one):

$$\Gamma_t + \Gamma_x \varphi_x^* = \varphi_y^*, \quad \varphi_t^* + \frac{1}{2}(\varphi_x^{*2} + \varphi_y^{*2}) + \Gamma = 0, \tag{2.1}$$

on the line $y = 0$:

$$\varphi_y^* = 0 \tag{2.2}$$

and, with $t = 0$, the given initial values were accepted

$$\Gamma(0, x) = \Gamma_0(x), \quad \varphi^*(0, x, y) = \varphi_0^*(x, y). \tag{2.3}$$

Let ψ^* be a harmonic function conjugate with φ^* and satisfying at $y = 0$ the condition

(2.4) $$\psi^* = 0.$$

These two functions are connected by the equations

(2.5) $$\varphi^*_x - \psi^*_y = 0, \quad \varphi^*_y + \psi^*_x = 0.$$

Being the values of functions φ^*, ψ^* brought into consideration on the line $y = \Gamma$ with the formulae

(2.6) $$\varphi^*(t, x, \Gamma(t, x)) = \Phi(t, x), \quad \psi^*(t, x, \Gamma(t, x)) = \Psi(t, x),$$

the basic equations (2.1) are reduced to the following ones

(2.7) $$\Gamma_t + \Psi_x = 0, \quad \Phi_t + \frac{1}{2}\,\frac{\Phi_x^2 + 2\Gamma_x\Phi_x\Psi_x - \Psi_x^2}{1+\Gamma_x^2} + \Gamma = 0.$$

Here Ψ is to be considered as image of Φ at the linear mapping operating by the following rule: with given Φ by means of Dirichlet problem solution at the complementary condition (2.2) in the domain Ω_t, the harmonic function φ^* is determined, according to it the function ψ^* and thereafter Ψ is being calculated in accordance with (2.6). The obtained mapping $\Phi \to \Psi$ depends on Γ and will be designated by a symbol $K(\Gamma)$, so that $\Psi = K(\Gamma)\Phi$. The natural initial conditions following from (2.3) are being added to equations (2.7):

(2.8) $$\Gamma(0, x) = \Gamma_0(x), \quad \Phi(0, x) = \varphi_0^*(x, \Gamma_0(x)).$$

Thus, the starting problem is reduced to the Cauchy problem for a system (2.7), (2.8). The main difficulty in investigating this problem is the realization and estimation of the operator $K(\Gamma)$ owing to its complicated non-linear dependency on the function Γ.

3. Shallow water theory

Let the initial data (2.8) be small in the sense that with the small parameter $\varepsilon > 0$ representation holds

(3.1) $$\Gamma_0(x) = \varepsilon\Gamma'_0(x, \varepsilon), \quad \Phi(0, x) = \varepsilon^{\frac{1}{2}}\Phi'_0(x, \varepsilon),$$

in which functions Γ'_0 and Φ'_0 remain finite at $\varepsilon \to 0$.

The approximate shallow water equations are being obtained in the modelling process connected with introduction of the small parameter ε into the Eqs. (2.7), (2.8) and with the limiting transition if $\varepsilon \to 0$. The starting point of this modelling is the representation

(3.2) $$x = x', \quad y = \varepsilon y', \quad t = \varepsilon^{-\frac{1}{2}}t', \quad \varphi^* = \varepsilon^{\frac{1}{2}}\varphi', \quad \psi^* = \varepsilon^{\frac{3}{2}}\psi'$$

and the assumption that the primed values (and their derivatives as well) remain finite at $\varepsilon \to 0$. According to (3.2), for functions (2.6) will also hold

$$\Gamma = \varepsilon\Gamma^1, \quad \Phi = \varepsilon^{\frac{1}{2}}\Phi^1, \quad \Psi = \varepsilon^{\frac{3}{2}}\Psi^1.$$

The substitution of these expressions in (2.7) and $\varepsilon \to 0$ transition give the equations

$$(3.3) \qquad \Gamma^1_{t'} + \Psi^1_x = 0, \quad \Phi^1_{t'} + \frac{1}{2}(\Phi^1_x)^2 + \Gamma^1 = 0.$$

For an approximate representation of the operator $K(\Gamma)$, it should be noted that the system (2.5) in this modelling process assumes the shape

$$\varphi'_x - \psi'_{y'} = 0, \quad \varphi'_{y'} = 0$$

and under conditions (2.2), (2.4) has a single solution

$$\varphi' = \Phi^1(t', x), \quad \psi' = y'\Phi^1_x(t', x).$$

Therefore, the operator $K(\Gamma)$ with $\varepsilon \to 0$ is being realized with the formula

$$(3.4) \qquad \Psi^1 = \Gamma^1 \Phi^1_x.$$

Finally, the shallow water theory equations take the form

$$(3.5) \qquad \Gamma^1_{t'} + (\Gamma^1\Phi^1_x)_x = 0, \quad \Phi^1_{t'} + \frac{1}{2}(\Phi^1_x)^2 + \Gamma^1 = 0$$

with the initial conditions

$$(3.6) \qquad \Gamma^1(0, x) = \Gamma'_0(x, 0), \quad \Phi^1(0, x) = \Phi'_0(x, 0).$$

If one assumes that in the initial data (3.1) functions $\Gamma'_0(x, \varepsilon)$, $\Phi'_0(x, \varepsilon)$ are infinitely differentiable with respect to x and are the analytical functions of the parameter ε with a regular point $\varepsilon = 0$, the said modelling process then can be obviously continued for obtaining the approximations of the highest orders. As a result, a recurrent system of equations can be drawn which allows to determine step by step all the members of the development into the formal series of the (2.7), (2.8) problem solution with the initial data of the form (3.1), namely

$$\Gamma(t, x, \varepsilon) = \varepsilon \sum_{n=1}^{\infty} \Gamma^n(\varepsilon^{\frac{1}{2}} t, x)\varepsilon^{n-1}, \quad \Phi(t, x, \varepsilon) = \varepsilon^{\frac{1}{2}} \sum_{n=1}^{\infty} \Phi^n(\varepsilon^{\frac{1}{2}} t, x)\varepsilon^{n-1}.$$

However, for shallow water theory foundation there is no necessity in such an assumption. One may consider this theory to be founded (in a class S of functions) if two statements were proved:

A. A solution $\Gamma(t, x, \varepsilon)$, $\Phi(t, x, \varepsilon)$ of the problem (2.7), (2.8) with the initial data (3.1) exists for any $\varepsilon > 0$.

B. Functions $\varepsilon^{-1}\Gamma(\varepsilon^{-\frac{1}{2}}t', x, \varepsilon)$, $\varepsilon^{-\frac{1}{2}}\Phi(\varepsilon^{-\frac{1}{2}}t', x, \varepsilon)$ have as a limit with $\varepsilon \to 0$ the solution of the problem (3.5), (3.6).

4. Conformal mapping on a strip

To overcome one of the difficulties arising in the proving of solution existence connected with the dependence of the domain Ω_t on time t, it is convenient to use an auxiliary conformal mapping Ω_t onto the fixed (independent of t) strip.

Let $z = x+iy$, $\zeta = \xi+i\eta$ and let function $z = g(t, \zeta)$ give such conformal mapping of the strip $\Pi_\delta = \{\zeta | 0 < \eta < \delta\}$ onto the domain Ω_t that the axis $y = 0$ mapped onto the axis $\eta = 0$. Assuming

$$f(t, \zeta) = (\varphi^* + i\psi^*)\big(t, g(t, \zeta)\big) = \varphi(t, \zeta) + i\psi(t, \zeta), \tag{4.1}$$

basic equations (2.1) or equivalent to them (2.7) may be transformed into the following at $\eta = \delta$:

$$\operatorname{Im}\frac{g_t}{g_\zeta} = -\operatorname{Im}\frac{f_\zeta}{|g_\zeta|^2}, \qquad \operatorname{Re}\left(f_t - f_\zeta\frac{g_t}{g_\zeta}\right) = -\frac{1}{2}\left|\frac{f_\zeta}{g_\zeta}\right|^2 - y. \tag{4.2}$$

Since functions g and f are analytical in Π_δ, left parts of these equations are the values of harmonic functions which, moreover, satisfy the following conditions at $\eta = 0$:

$$\operatorname{Im}\frac{g_t}{g_\zeta} = 0, \qquad \frac{\partial}{\partial\eta}\operatorname{Re}\left(f_t - f_\zeta\frac{g_t}{g_\zeta}\right) = 0. \tag{4.3}$$

Therefore analytical in Π_δ functions g_t/g_ζ and $f_t - f_\zeta g_t/g_\zeta$ may be restored by their values at the boundary $\eta = \delta$. This may be done by means of the well known Schwartz formula for a strip; however it is more convenient for the future not to write this formula actually but to use the inscription with the help of operators A_δ and B_δ introduced below.

Let Q_s denotes a set of functions $p(\xi)$, infinitely differentiable on the whole axis and satisfying inequalities of the form

$$|p(\xi)| < N(1+|\xi|)^s, \qquad |p_\xi(\xi)| < N, \qquad N = \text{const}.$$

Let class Q be the set of functions $u+iv$, continuous in $\overline{\Pi}_\delta$, analytical in Π_δ, symmetrical in the sense that $v(\xi, 0) = 0$ and satisfying the condition

$$u(\xi, \delta) \in Q_1, \qquad v(\xi, \delta) \in Q_0. \tag{4.4}$$

As in the class Q setting of the real part $u(\xi, \delta)$ uniquely determines analytical in Π_δ function $u+iv$, the operator $A_\delta: Q_1 \to Q_0$ is determined, acting in accordance with formula $v(\xi, \delta) = A_\delta u(\xi, \delta)$. The operator A_δ has the one-dimensional kernel consisting of constants. Therefore its inverted B_δ is defined as the mapping into the set of classes of equivalent functions in Q_1 with respect to the equivalence relation $u_1 \backsim u_2 \Leftrightarrow u_1 - u_2 = \text{const}$. In this paragraph, $B_\delta v$ will denote that representative for which $B_\delta v(0) = 0$.

By means of operators A_δ and B_δ from relations (4.2), (4.3), the representations of analytical functions on $\eta = \delta$ follow:

$$\begin{aligned} \frac{g_t}{g_\xi} &= -B_\delta\left(\frac{\psi_\xi}{|g_\xi|^2}\right) - i\frac{\psi_\xi}{|g_\xi|^2}, \\ f_t - f_\xi\frac{g_t}{g_\xi} &= -\frac{1}{2}\left|\frac{f_\xi}{g_\xi}\right|^2 - y + iA_\delta\left(\frac{1}{2}\left|\frac{f_\xi}{g_\xi}\right|^2 + y\right). \end{aligned} \tag{4.5}$$

Besides the representations (4.5), the following ones are valid (at $\eta = \delta$):

$$y = A_\delta x, \qquad \psi = A_\delta\varphi \tag{4.6}$$

and their analogues for derivatives on ξ.

The separation in (4.5) of real and imaginary parts and using of formulae (4.6) lead to the following system for two functions, $x = x(t, \xi, \delta)$ and $\varphi = \varphi(t, \xi, \delta)$:

$$
\begin{aligned}
x_t &= \frac{A_\delta x_\xi \cdot A_\delta \varphi_\xi}{x_\xi^2 + (A_\delta x_\xi)^2} - x_\xi B_\delta\left(\frac{A_\delta \varphi_\xi}{x_\xi^2+(A_\delta x_\xi)^2}\right),\\
\varphi_t &= -A_\delta x + \frac{1}{2}\frac{(A_\delta\varphi_\xi)^2-\varphi_\xi^2}{x_\xi^2+(A_\delta x_\xi)^2} - \varphi_\xi B_\delta\left(\frac{A_\delta \varphi_\xi}{x_\xi^2+(A_\delta x_\xi)^2}\right),
\end{aligned}
\tag{4.7}
$$

with the initial conditions

$$
x(0, \xi, \delta) = x_0(\xi, \delta), \quad \varphi(0, \xi, \delta) = \varphi_0(\xi, \delta), \tag{4.8}
$$

where functions $x_0(\xi, \delta) = \mathrm{Re}\, g(0, \xi + i\delta)$ and $\varphi_0(\xi, \delta) = \varphi^*(0, g(0, \xi+i\delta))$ are completely determined by the initial data (2.3).

The following lemma formulates the main asymptotic property of operators A_δ, B_δ.

LEMMA 1. *If functions $u(\xi)$, $v(\xi)$ satisfy condition* (4.4), *then uniformly on the whole axis*

$$
\lim_{\delta\to 0}\frac{1}{\delta}A_\delta u = u_\xi, \quad \lim_{\delta\to 0}\delta B_\delta v = \int_0^\xi v(s)ds. \tag{4.9}
$$

Now, it is possible to apply to the system (4.6)–(4.8) formal modelling process described in paragraph 4 with the small parameter δ and to show that it leads precisely to the shallow water Eqs. (3.5). The ground of this modelling is the suggestion of smallness of function $\varphi_0(\xi, \delta)$, concerning which it is supposed that $\varphi_0(\xi, \delta) = \delta^{\frac{1}{2}}\varphi_0'(\xi, \delta)$, when $\delta \to 0$ functions $x_0(\xi, \delta)$ and $\varphi_0'(\xi, \delta)$ are bounded. By analogy to (3.2), it is assumed that the substitution as follows:

$$
x = x', \quad y = \delta y', \quad t = \delta^{-\frac{1}{2}}\tau, \quad \varphi = \delta^{\frac{1}{2}}\varphi', \tag{4.10}
$$

is made into the system (4.6), (4.7) and the limit process is performed when $\delta \to 0$, using relations (4.9). In the limit for functions

$$
x'(\tau, \xi, 0) = X(\tau, \xi), \quad y'(\tau, \xi, 0) = Y(\tau, \xi), \quad \varphi'(\tau, \xi, 0) = \Phi(\tau, \xi),
$$

the following system of equations is obtained:

$$
\begin{aligned}
Y &= X_\xi,\\
X_\tau &= -X_\xi \cdot \int_0^\xi \frac{\Phi_{\xi\xi}}{X_\xi^2}d\xi,\\
\Phi_\tau &= -X_\xi - \frac{1}{2}\frac{\Phi_\xi^2}{X_\xi^2} - \Phi_\xi \int_0^\xi \frac{\Phi_{\xi\xi}}{X_\xi^2}d\xi.
\end{aligned}
\tag{4.11}
$$

It is easy to verify that for functions $\Gamma^1(\tau, x)$ and $\Phi^1(\tau, x)$, defined by the equalities

$$
\Gamma^1(\tau, X(\tau, \xi)) = Y(\tau, \xi), \quad \Phi^1(\tau, X(\tau, \xi)) = \Phi(\tau, \xi),
$$

equations (3.5) follow from Eqs. (4.11). For the inverse transition it is sufficient to introduce an auxiliary variable ξ with the first of equations (4.11) and then all equations (4.11) will follow from (3.5). Therefore, in order to find the shallow water theory, it is enough to prove statements A and B (see paragraph 3) for the system (4.7).

5. Periodic solutions

The further results will concern solutions periodic in ξ. The function $x(t, \xi)$ being not periodic (it contains an additive term of $a\xi$ type), its derivative x_ξ may already be considered as the periodic one. That's why such solutions of the system (5.3) will be called 2π-periodic ones in which functions φ_ξ and x_ξ possess the period 2π on ξ.

It is useful to introduce new unknown functions

$$x_\xi = u, \quad \varphi_\xi = v \tag{5.1}$$

and to proceed from Eqs. (4.7), (4.8) to the same but differentiated on ξ. If the substitution is made in the obtained equations analogous to (4.10) and the primes are omitted, one obtains the following Cauchy problem for functions $u(\tau, \xi, \delta)$, $v(\tau, \xi, \delta)$:

$$\begin{aligned} u_\tau &= \frac{\partial}{\partial\xi}\left[\frac{A_\delta u A_\delta v}{u^2+(A_\delta u)^2} - uB'_\delta\left(\frac{A_\delta v}{u^2+(A_\delta u)^2}\right)\right], \\ v_\tau &= \frac{\partial}{\partial\xi}\left[\frac{1}{2}\,\frac{(A_\delta v)^2-v^2}{u^2+(A_\delta u)^2} - vB'_\delta\left(\frac{A_\delta v}{u^2+(A_\delta u)^2}\right)\right] - \frac{1}{\delta}A_\delta u, \\ u(0, \xi, \delta) &= u_0(\xi, \delta), \quad v(0, \xi, \delta) = v_0(\xi, \delta). \end{aligned} \tag{5.2}$$

Here functions $u_0(\xi, \delta)$ and $v_0(\xi, \delta)$ are assumed to be 2π-periodic in ξ, continuous and possessing uniform limits $\mathring{u}(\xi)$, $\mathring{v}(\xi)$ when $\delta \to 0$.

For the future it is essential to pay attention to the fact that operator B_δ transforms constants into functions linear with respect to ξ, namely, $B_\delta(1) = \xi/\delta$. Therefore, if 2π-periodic function $w \in Q_0$ has mean over period value $w_0 = \frac{1}{2\pi}\int_0^{2\pi} w(\xi)\,d\xi$, unequal to zero, then the image $B_\delta w$ will contain an additive term $\xi w_0/\delta$ which will disturb smoothness. This unpleasant situation could as well arise in the system (2.2), where operator B_δ acts on the function which has a form $\psi_\xi/|g_\xi|^2$ in the initial variables. Now it will be shown that the mean value of that function is actually equal to zero.

It appears that this fact is closely connected with period of solutions on the plane (x, y). It is not obvious *a priori* that this period is independent on time t. For each t and η this period is equal to the increment of coordinate $x(t, \xi, \eta)$ on any interval of the length 2π and will be denoted as $[x]_{2\pi}$. Of course, it is independent of η and may be calculated at $\eta = \delta$.

LEMMA 2. *The period* $[x]_{2\pi}$ *is independent of* t.

For proof, it is enough to note that from first Eq. (4.5) follows the relation

$$x_t y_\xi - x_\xi y_t = \psi_\xi, \tag{5.3}$$

from which, due to 2π-periodicity of functions y_ξ, x_ξ, y_t and ψ_ξ, equality $[x_t]_{2\pi}\cdot y_\xi = 0$ follows. Only the uniform flow exists in the case $y_\xi \equiv 0$; the general case, therefore, ought to be $[x_t]_{2\pi} = \frac{\partial}{\partial t}[x]_{2\pi} = 0$.

LEMMA 3. *The meun value of* $\psi_\xi/|g_\xi|^2$ *is equal to zero.*

For proof, the analytical function g_t/g_ζ is considered in rectangle $0 < \xi < 2\pi$, $0 < \eta < \delta$ and it is used that its integral over the contour of this rectangle is equal to zero. Equation (5.3) leads to

$$\int_0^{2\pi} \frac{\psi_\xi}{|g_\xi|^2}\bigg|_{\eta=\delta} d\xi + [x_t]_{2\pi} \int_0^{\delta} \frac{x_\xi}{|g_\xi|^2}\bigg|_{\xi=0} d\eta = 0$$

and the result follows from Lemma 2.

Therefore in the solutions of the system (5.2), the presence of operator B_δ does not violate smoothness of its right side. For this property conservation it is expedient to introduce the operator B'_δ instead of B_δ [what has already been done in (5.2)] acting on an arbitrary 2π-periodic function $u(\xi)$ according to the formula

$$B'_\delta u = B_\delta(u - u_0), \quad u_0 = \frac{1}{2\pi}\int_0^{2\pi} u(s)\,ds. \tag{5.4}$$

Moreover, in the class of 2π-periodic functions it is convenient to change the convention adopted in paragraph 4 on the selection of representative $B'_\delta u$ with the convention that $\int_0^{2\pi} B'_\delta u(\xi)d\xi = 0$.

To each 2π-periodic smooth enough function $u(\xi)$ corresponds the sequence of its Fourier coefficients

$$u_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\xi}u(\xi)d\xi, \quad n = 0, \pm 1, \pm 2, \ldots$$

and the representation holds

$$u(\xi) = \sum_n u_n e^{in\xi} \tag{5.5}$$

(here and further the symbol $\sum_n$ denotes the sum spreading over all whole numbers n).

For further consideration, the explicit form of the results of operators A_δ and B'_δ action upon any harmonic is essential; the corresponding formulae are the following:

$$A_\delta(e^{in\xi}) = ie^{in\xi}\mathrm{th}(n\delta), \quad B'_\delta(e^{in\xi}) = -ie^{in\xi}\mathrm{cth}(n\delta), \quad n \neq 0. \tag{5.6}$$

With the aid of these formulae, the validity of lemma 1 is easily obtained in the class of smooth enough 2π-periodic functions.

6. Scale of Banach spaces

Further limitations of the class of initial data and the formation of the functional class in which the solution is sought, are connected with introduction of the special norms. To every function $u(\xi)$ of the type (5.5) and number $\varrho > 0$ is put into correspondence a number (presumably equal to $+\infty$) called the norm of function $u(\xi)$ with index ϱ:

$$||u||_\varrho = \sum_n e^{|n|\varrho}|u_n|. \tag{6.1}$$

The totality of those $u(\xi)$ for which $||u||_\varrho < \infty$ forms, as can easily be tested, a Banach space, designated E_ϱ further on. Since for $\sigma < \varrho$ always $||u||_\sigma \leqslant ||u||_\varrho$, then in this case $E_\sigma \supset E_\varrho$ due to which the unification $S = \bigcup_{0<\varrho} E_\sigma$ forms the scale of Banach spaces.

It is obvious that functions $u \in S$ are analytical ones and if $u(\xi) \in E_\varrho$, the function $u(\xi+i\eta)$ will evidently be analytical in strip $|\eta| < \varrho$. Inversely, every function $u(\xi)$ admitting an analytical continuation of the form $u(\xi+i\eta)$ on a strip $|\eta| < \varrho$, belongs to any E_σ at $\sigma < \varrho$. Thus, the scale of Banach spaces as a set consists of all the 2π-periodic functions $u(\xi)$ every one of which is uniformly analytical on the whole axis.

The fact is of principal significance that in the scale S the following two properties of the norm $||\cdot||_\varrho$ are fulfilled (which are however characteristic for most of analytical functions scales of Banach spaces):

1. If $u \in E_\varrho$, for any $\sigma < \varrho$, norm $||u||$ is continuously differentiable with respect to variable σ and is a convex function of σ (actually, here the norm $||u||_\sigma$ is analytical with respect to σ, all its derivatives being non-negative).

2. A triangle inequality can be term by term differentiated with respect to parameter ϱ: if $u, v \in E_\varrho$, then

$$\frac{\partial}{\partial\varrho}||u+v||_\varrho \leqslant \frac{\partial}{\partial\varrho}||u||_\varrho + \frac{\partial}{\partial\varrho}||v||_\varrho.$$

Other properties are constructive; they are associated with a concrete realization of norm (6.1). Most important are the following:

$$u, v \in E_\varrho \Rightarrow ||uv||_\varrho \leqslant ||u||_\varrho \cdot ||v||_\varrho; \tag{6.2}$$

$$u \in E_\varrho \Rightarrow ||u_\xi||_\varrho = \frac{\partial}{\partial\varrho}||u||_\varrho. \tag{6.3}$$

7. Estimates

For the objectives of this paper it is necessary to perform an estimate of som functions and operators which have been met in the right parts of system (6.2).

It follows from Lemma 1 that $A_\delta u \to 0$ uniformly as $\delta \to 0$. As a matter of fact it is possible to give more precise estimate showing that for each δ the operator $\delta^{-1}A_\delta$ is analogous to the differential one.

LEMMA 4. *If $u \in E_\varrho$, then $A_\delta u \in E_\varrho$ and*

$$||A_\delta u||_\varrho \leqslant ||u||_\varrho, \qquad \left\|\frac{1}{\delta}A_\delta u\right\|_\varrho \leqslant \frac{\partial}{\partial\varrho}||u||_\varrho. \tag{7.1}$$

These facts follow directly from formulae (5.6) and from inequalities $|\mathrm{th}\,x| < 1$, $\mathrm{th}\,x/x \leqslant 1$.

The estimation of operation $B'_\delta(wA_\delta v)$ presented in (5.2) with $w = \left(u^2+(A_\delta u)^2\right)^{-1}$ proves to be more difficult. To this end is needed previous estimate of the function

$$M(k, l, \varrho, \delta) = e^{(|k+l|-|k|-|l|)\varrho}\,\mathrm{cth}\,|k+l|\delta\,\mathrm{th}\,|l|\delta$$

with arbitrary whole numbers k, l, so that $k+l \neq 0$, and real positive ϱ, δ.

LEMMA 5. *The estimate holds*

$$M(k, l, \varrho, \delta) \leqslant \max\left\{1, \frac{1}{2\varrho}\right\}.$$

The deduction of this estimate is connected with consideration of function M in different domains of the plane (k, l) obtained by its division with straight lines $k = 0$, $k+l = 0$, $k+2l = 0$.

Now it is possible to obtain the main estimate.

LEMMA 6. *If* $w, v \in E_\varrho$, *then* $B'_\delta(wA_\delta v) \in E_\varrho$ *and*

$$\|B'_\delta(wA_\delta v)\|_\varrho \leqslant \|w\|_\varrho \cdot \|v\|_\varrho \cdot \max\left\{1, \frac{1}{2\varrho}\right\}. \tag{7.2}$$

Proof. In virtue of (5.6) and the convention in paragraph 6 about operator B'_δ, the estimated function has the following Fourier coefficients

$$\big(B'(wA_\delta v)\big)_m = \operatorname{cth} m\delta \sum_k w_k v_{m-k} \operatorname{th}(m-k)\delta.$$

Therefore (prime in symbol $\sum\limits_m{}'$ denotes the blank of value $m = 0$)

$$\|B'_\delta(wA_\delta v)\|_\varrho = \sum_m{}' e^{|m|\varrho} \operatorname{cth}|m|\delta \Big|\sum_k w_k v_{m-k} \operatorname{th}(m-k)\delta\Big|$$

$$\leqslant \sum_m{}' \sum_k e^{|m|\varrho} \operatorname{cth}|\ldots|\delta \operatorname{th}|m-k|\delta |w_k|\,|v_{m-k}| = \sum_{\substack{k\\k+l\neq 0}} \sum_l e^{|k+l|\varrho} \operatorname{cth}|k+l|\delta \operatorname{th}|l|\delta |w_k|\,|v_l|$$

$$= \sum_{\substack{k\\k+l\neq 0}} \sum_l e^{|k|\varrho}|w_k| e^{|l|\varrho}|v_l| M(k, l, \varrho, \delta) \leqslant \|w\|_\varrho \|v\|_\varrho \max M(k, l, \varrho, \delta)$$

and the result (7.2) follows from Lemma 5.

The consequence of estimate (7.2) is the following one, which will actually be used

$$\frac{1}{2} \leqslant \varrho \Rightarrow \|B_\delta(wA_\delta v)\|_\varrho \leqslant \|w\|_\varrho \cdot \|v\|_\varrho. \tag{7.3}$$

The last of required estimates is the below estimate of function $u = x_\xi$. For its obtaining one notes that the coordinate $x \in Q_1$ and may be represented in the form

$$x(\tau, \xi, \delta) = \gamma(\tau, \delta)\xi + X'(\tau, \xi, \delta), \quad X' \in Q'_0,$$

where Q'_0 denotes the class of 2π-periodic functions with the mean over period value being equal to zero. Therefore function $u = x_\xi$ has the representation

$$u(\tau, \xi, \delta) = \gamma(\tau, \delta) + u'(\tau, \xi, \delta), \quad u' \in Q'_0,$$

where $u' = X'_\xi$. Substitution of this expression into the first of equations (5.2) and note that its right part is a function from Q'_0, leads to equality $\gamma_\tau(\tau, \delta) = 0$. So γ is independent of τ and is completely determined by the initial data. It is useful to note that the value γ determines the period of solution in the (x, y)-plane:

$$[x]_{2\pi} = 2\pi\gamma. \tag{7.4}$$

Later on for simplicity it will be supposed that γ is independent of δ too. In this case function $u(\tau, \xi, \delta)$ has the representation

$$u(\tau, \xi, \delta) = \gamma + u'(\tau, \xi, \delta), \quad u' \in Q_0' \tag{7.5}$$

with the given positive constant γ.

The a priori restriction connected with γ is to be laid on a solution. For example, it may be assumed that

$$||u'||_\varrho < \frac{1}{4}\gamma. \tag{7.6}$$

LEMMA 7. *With the assumptions made the estimate is valid*

$$\left\|\frac{1}{u^2+(A_\delta u)^2}\right\|_\varrho < \frac{8}{3\gamma^2}. \tag{7.7}$$

This estimate follows from the representation (7.5), the first of inequalities (7.1) and the property (7.2).

Remark. Some of inequalities obtained for the norm may be differentiated with respect to ϱ. Particularly for estimation of right parts of system (5.2) there are useful following inequalities which may easily be proved by means of the property (6.3):

$$\begin{gathered}
\frac{\partial}{\partial\varrho}||uv||_\varrho \leqslant \frac{\partial}{\partial\varrho}(||u||_\varrho||v||_\varrho),\\
\frac{\partial}{\partial\varrho}\left\|\frac{1}{u}\right\|_\varrho \leqslant \left\|\frac{1}{u}\right\|_\varrho^2 \frac{\partial}{\partial\varrho}||u||_\varrho, \quad u>0,\\
\frac{\partial}{\partial\varrho}||A_\delta u||_\varrho \leqslant \frac{\partial}{\partial\varrho}||u||_\varrho,\\
\frac{\partial}{\partial\varrho}||B_\delta'(wA_\delta v)||_\varrho \leqslant \frac{\partial}{\partial\varrho}(||w||_\varrho||v||_\varrho), \quad \varrho \geqslant \frac{1}{2}.
\end{gathered} \tag{7.8}$$

The significant peculiarity of the estimates received above is that they are valid independently of the value $\delta > 0$ and are also right for $\delta = 0$.

8. The existence theorem

It will be shown below how it is possible to establish the existence and uniqueness of the solution of the problem (5.2) in the scale S for sufficiently small values of time τ. For this approach, the problem (5.2) is transformed into another Cauchy problem with zero data and fulfillment of conditions of a general abstract author's theorem [5] is verified.

In the set of pairs (u_1, u_2), where $u_1, u_2 \in E_\varrho$ which is denoted below as $S^2 = \bigcup_{0<\varrho} E_\varrho \times E_\varrho$, the norm $||(u_1, u_2)||_\varrho = ||u_1||_\varrho + ||u_2||_\varrho$ is introduced converting S^2 into the scale of Banach spaces. In the scale S^2 are valid fundamental properties 1° and 2°, as well as the properties (6.2), (6.3).

System (5.2) is considered as an equation in S^2 for the pair

$$q = (u-u_0,\ v-v_0)$$

and is rewritten in the form

$$q_\tau = f(q,\delta), \quad q|_{\tau=0} = 0, \tag{8.1}$$

where f is the pair consisting of right parts of (5.2) after the substitution of expressions

$$u = u_0+P_1q, \quad v = v_0+P_2q \tag{8.2}$$

with projectors P_1 and P_2 on the first and the second multipliers, correspondingly.

In order to guarantee the estimate (7.6), it is sufficient to suppose that with some $\varrho_0 > 1/2$ the inequalities are valid

$$||u_0'||_{\varrho_0} < r, \quad ||v_0||_{\varrho_0} < r, \quad ||q||_{\varrho_0} < r, \qquad r = \frac{1}{8}\gamma. \tag{8.3}$$

Then in virtue of estimates from paragraph 8 the operation f for each $\delta \geqslant 0$ will be the mapping $f\colon O(r,\varrho_0) \to S^2$, where

$$O(r,\varrho_0) = \bigcup_{\frac{1}{2}\leqslant\varrho\leqslant\varrho_0} \{q|q\in S^2,\ ||q||_\varrho < r\}.$$

After slightly tiresome but quite elementary calculations one establishes the following decisive estimate.

LEMMA 8. *There exists such independent of δ constant C that for each $\varrho\in\left[\frac{1}{2},\varrho_0\right]$ and for every $q_i\in E_\varrho\times E_\varrho$, $||q_i||_\varrho < r$ $(i = 1, 2)$, the inequality is valid*

$$||f(q_1,\delta)-f(q_2,\delta)||_\varrho \leqslant C\left(1+\frac{\partial}{\partial\varrho}\right)[(1+||q_1||_\varrho+||q_2||_\varrho)||q_1-q_2||_\varrho]. \tag{8.4}$$

The inequality (8.4) shows that mapping f is quasi-differential uniformly relative to $\delta \geqslant 0$. Now, it is clear that all conditions of the fundamental theorem from [5] (where it is called the Theorem 1) will be valid if additionally the existence of such a constant C_1 is assumed that

$$||u_{0\xi}'||_{\varrho_0} < C_1, \quad ||v_{0\xi}||_{\varrho_0} < C_1. \tag{8.5}$$

Therefore application of the above mentioned theorem to the problem (1) gives the following result:

THEOREM 1. *If the initial data of the problem* (5.2) *satisfy conditions* (8.3) *and* (8.5), *there exists such independent of $\delta \geqslant 0$ number $k > 0$ that problem* (5.2) *has in S^2 the unique solution $(u(\tau), v(\tau))$ satisfying inequality $||u(\tau)-u_0||_\varrho+||v(\tau)-v_0||_\varrho \leqslant r$ for all (ϱ, r) from the domain*

$$\tau \geqslant 0, \quad \frac{1}{2} \leqslant \varrho, \quad \varrho+k\tau \leqslant \varrho_0. \tag{8.6}$$

Here it is essential that neither constant k nor estimate of the solution norm are independent of parameter δ including the value $\delta = 0$. Therefore according to system (5.2), in the scale of Banach spaces S is proved the statement A from paragraph 3.

The process of proving Theorem 1 (see [5]) also gives the following solution estimate in domain (8.6) in terms of the initial data,

$$(8.7) \qquad \|u-u_0\|_\varrho+\|v-v_0\|_\varrho \leqslant \frac{1}{k}\int\limits_{\varrho}^{\varrho+k\tau} e^{\sigma}\|f(0,\delta)\|_\sigma d\sigma,$$

where the right part may be estimated, in its turn, independently of δ.

9. The shallow water theory foundation in the case of periodic waves

Equations of the shallow water theory are obtained from system (5.2) by formal limit transition as $\delta \to 0$. Taking into account the result of Lemma 1 and singularities of operator B'_δ mentioned in paragraph 6, the limit problem for limit functions $\mathring{u}(\tau, \xi)$, $\mathring{v}(\tau, \xi)$ has the form

$$(9.1) \qquad \begin{aligned} \mathring{u}_\tau &= -\frac{\partial}{\partial\xi}\left[\mathring{u}B_0\left(\frac{\mathring{v}_\xi}{\mathring{u}^2}\right)\right],\\ \mathring{v}_\tau &= -\frac{\partial}{\partial\xi}\left[\mathring{u}+\frac{\mathring{v}}{2\mathring{u}^2}+\mathring{v}B_0\left(\frac{\mathring{v}_\xi}{\mathring{u}^2}\right)\right],\\ \mathring{u}(0,\xi) &= u_0(\xi,0), \quad \mathring{v}(0,\xi) = v_0(\xi,0), \end{aligned}$$

where operator B_0 acts according to formula

$$(9.2) \qquad B_0\Big(\sum_n e^{in\xi}w_n\Big) = \sum_n{}' e^{in\xi}\frac{1}{n}w_n.$$

The existence of the solution in problem (9.1) is guaranteed by Theorem 1. For this solution, estimates (8.6) and (8.7) are valid.

Let $u(\tau, \delta)$, $v(\tau, \delta)$ be the solution of (5.2) and $\mathring{u}(\tau)$, $\mathring{v}(\tau)$ — the solution of (9.1). Let, moreover, the initial data of the problem (5.2) converge in S^2 as $\delta \to 0$ to the initial data of the problem (9.1) in the sense that

$$(9.3) \qquad \lim_{\delta\to 0}(\|u_0(\delta)-\mathring{u}\|_\varrho+\|v_0(\delta)-\mathring{v}\|_\varrho) = 0$$

uniformly relative to ϱ on the segment $\left[\frac{1}{2}, \varrho_0\right]$.

THEOREM 2. *The condition* (9.3) *being valid, the number* $k > 0$ *in the inequality* (8.6) *may be chosen as such that in domain* (8.6) *uniformly*

$$(9.4) \qquad \lim_{\delta\to 0}(\|u(\tau,\delta)-\mathring{u}(\tau)\|_\varrho+v(\tau,\delta)-\mathring{v}(\tau)\|_\varrho) = 0.$$

Proof. If we put $u = \mathring{u}+\tilde{u}$, $v = \mathring{v}+\tilde{v}$ and fulfil the substitution in (5.2) taking into account (9.1) then for the pair $\tilde{q} = (\tilde{u}, \tilde{v})$ arises the Cauchy problem of the form

$$(9.5) \qquad \begin{aligned} \tilde{q}_\tau &= L_\delta\tilde{q}+h_\delta,\\ \tilde{q}(0) &= \tilde{q}_0 = \big(u_0(\delta)-\mathring{u}, v_0(\delta)-\mathring{v}\big), \end{aligned}$$

where L_δ is linear operator and element $h_\delta \in S^2$ is determined by the solution $(\mathring{u}, \mathring{v})$ only and has a form of linear combination of terms each of which is proportional to a value of the type

(9.6) $$A_\delta w, \quad \left(\frac{1}{\delta} A_\delta - \frac{\partial}{\partial \xi}\right) w, \quad (\partial B_\delta' - B_0) w$$

with different functions w, with coefficients which are uniformly bounded in the domain (8.6) together with their derivatives. Operator L_δ is quasi-differential and admits the estimate

(9.7) $$\|L_\delta q\|_\varrho \leqslant C_2 \left(1 + \frac{\partial}{\partial \varrho}\right) \|q\|_\varrho,$$

uniform in domain (8.6) with the constant C_2 independent of δ.

The estimates for values (9.6) are valid with some positive constants α, β

(9.8) $$\begin{gathered} \|A_\delta w\|_\varrho \leqslant \delta \frac{\partial}{\partial \varrho} \|w\|_\varrho, \\ \left\| \left(\frac{1}{\delta} A_\delta - \frac{\partial}{\partial \xi}\right) w \right\|_\varrho \leqslant \delta \cdot \alpha \frac{\partial^2}{\partial \varrho^2} \|w\|_\varrho, \\ \|(\delta B_\delta' - B_0) w\|_\varrho \leqslant \delta^2 \cdot \beta \frac{\partial}{\partial \varrho} \|w\|_\varrho . \end{gathered}$$

It is possible to interprete the problem (9.5) as a particular case of already mentioned general abstract theorem from [5], for what it is sufficient to subtract the initial data from the solution. If we put $\tilde{q} = \tilde{q}_0 + q^*$, the problem for q^* will be of the form

(9.9) $$q_\tau^* = L_\delta q^* + L_\delta \tilde{q}_0 + h_\delta, \quad q^*(0) = 0.$$

Here the sum $L_\delta \tilde{q}_0 + h_\delta$ plays a part of $f(0)$. Therefore estimate of the form (8.7) is valid for the solution of the problem (9.9) with perhaps larger than in (8.7) value of the constant $k > 0$ [the latter depends on the constant C_2 in the estimate (9.7) only]. Namely, in the domain (8.6)

(9.10) $$\|q^*(\tau)\|_\varrho \leqslant \frac{1}{k} \int\limits_\varrho^{\varrho + k\tau} e^\sigma \|L_\delta \tilde{q}_0 + h_\delta\|_\sigma d\sigma .$$

Due to the representation of the function h_δ, estimates (9.7) and (9.8) and the condition (9.3), it follows from estimate (9.10) that $\|q^*\|_\varrho \to 0$, consequently $\|\tilde{q}\|_\varrho \to 0$ as $\delta \to 0$ uniformly in domain (8.6).

Herewith the proof of the statement B from paragraph 3 is given and according to the collorary in the end of paragraph 4, the shallow water theory is founded for periodic analytical solutions.

It is useful to point out that between parameters ε (paragraph 3) and δ (paragraph 4) the connection may be established showing that these parameters have the same order of smallness. Let free boundary at the initial moment $t = 0$ be given by the equation $y = \varepsilon \Gamma_0'(x, \varepsilon)$. If function $g_0 = x_0 + iy_0$ realizes the conformal mapping of the strip Π_δ onto the domain Ω_0, then the same boundary will be given parametrically: $x = x_0(\xi, \delta)$,

$y = y_0(\xi, \delta)$. Hence $y_0(\xi, \delta) = \varepsilon\Gamma_0'(x_0(\xi, \delta), \varepsilon)$ and since, according to (4.6), $y_0 = A_\delta(x_0)$, there arises the equation

$$\frac{1}{\delta}A_\delta(x_0(\xi, \delta)) = \frac{\varepsilon}{\delta}\Gamma_0'(x_0(\xi, \delta), \varepsilon). \tag{9.11}$$

Here function $\Gamma_0'(x, \varepsilon)$ is bounded from above and it may be supposed that $\Gamma_0'(x, \varepsilon) > \mu > 0$ (the bottom does not dry). Further, in virtue of (7.5) the representation $x_0 = \gamma\xi + X_0'(\xi, \delta)$ is valid, thereafter $\frac{1}{\delta}A_\delta(x_0) = \gamma + \frac{1}{\delta}A_\delta(X_0')$ with $\gamma > 0$ and the second term being bounded and small due to the assumption (8.3). Therefore ε/δ is bounded from above and below by positive numbers. In order to establish the concrete connection $\delta = \delta(\varepsilon)$, it is sufficient to take (9.11) at one value of ξ. It is interesting that the limit value of the ratio ε/δ may be calculated directly from the given boundary on the physical plane:

$$\lim_{\delta\to 0}\frac{\varepsilon}{\delta} = \frac{1}{2\pi}\int_0^{2\pi\gamma}\frac{dx}{\Gamma_0'(x, 0)}.$$

10. Concluding remarks

The suggested in this paper shallow water theory foundation by no means exhaust the whole problem. A series of directions can be pointed out in which a certain amount of work should be done,.e.g.:

(a) non-periodic solutions, in particular, with the initial data damping at infinity;

(b) approximations of highest orders and finding out the possibility of the solution development into power series with respect to the small parameter;

(c) shallow water theory foundation in classes of functions with finite smoothness;

(d) results transfer onto the three-dimensional motions;

(e) generalizations for the case of non-horizontal bottom.

The direction (a) seems quite accessible though the fundamental estimate of the form (7.3) has not yet been obtained. In the (b) direction, the highest order approximations foundation is likely to comprise no great difficulty. However, the opinion was expressed that the development into the convergent power series did not exist. As far as the perspectives of obtaining results in the (c) direction are concerned, the author thinks them to be fantastic in difficulty; it is enough to pay more attention to limitation $\varrho \geqslant 1/2$ (or, generally, $\varrho \geqslant \varrho_1$ with a certain constant $\varrho_1 > 0$) in estimate (7.3) to understand how great is the distance to the finite smoothness functions. However, it is quite possible that this limitation is only a shortcoming of the applied method, improvement or replacement of which will permit to obtain the desired result. As far as the author knows, there are still not any exact results in the (d) direction, even the existence theorems of solution of the Cauchy-Poisson problem in the analytical case. For the case of a non-horizontal bottom (e), it would be interesting to give the foundation of the known solutions obtained in the shallow water approximation in the problems of waves flow near the slope beach.

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