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Low-Reynolds-number bubbles in fluidised beds

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THIS PAPER is concerned with the shape and speed of rise of a bubble of clear fluid in a fluidised bed of identical solid particles, given that the Reynolds numbers of the motions associated with a single particle and with a rising bubble are both small compared with unity. Under these conditions the mixture of fluid and particles outside a bubble can be regarded as a continuum with certain equivalent properties. It is found that a spherical bubble remains spherical, and the associated distributions of velocity of fluid and particles are obtained analytically. Spherical bubbles can also exist in a fluidised bed of particles of different sizes.

Doświadczenie wykazuje, że kształt pęcherzy cieczy podnoszących się pod wpływem sił ciężkości w ośrodku fluidyzowanym jest w analogicznych warunkach (tj. tej samej liczby Reynoldsa i pomijalności wpływu napięcia powierzchniowego) podobny do kształtu pęcherzy gazu, podnoszących się w cieczy. Wykorzystanie tej analogii pozwala na zrozumienie mechanizmu ruchu pęcherzy. Uzyskane rozwiązanie ważne jest dla małych liczb Re, lecz sugeruje przybliżony sposób opisu ruchu pęcherza w przypadku, gdy to założenie nie jest spełnione.

Как показывает опыт, существует аналогия между формами пузырей жидкости, уносящимися под влиянием сил тяжести во флюидизированной среде, и формами пузырей газа, уносящимися в жидкости (при таких же числах Рейнольдса и возможности пренебрежения влияния поверхностного натяжения). Использование этой аналогии позволяет понять механизм движения пузыря. Полученное решение важно для малых чисел Re, но это описание движения пузыря можно рассматривать как приближение в случае, когда это предложение не осуществляется.

1. Introduction

SOLID particles at rest and packed under their own weight in a vertical cylinder form a porous medium through which fluid may be forced. If the rate at which fluid flows upward through the bed is increased above a certain critical value, the particles are lifted and separated a little and become mobile. The bed then has the general properties of a rather viscous fluid and is said to be "fluidised", and in that state allows a higher rate of transfer of heat or mass between the particles and the fluid than when the bed is "packed". It is this higher transfer rate that makes fluidised beds so important in chemical engineering, although here I shall be concerned only with the dynamics of the particles and the ambient fluid. A general account of the mechanics of fluidised beds is available in the book by DAVIDSON & HARRISON (1963), and recent developments are described in a collective work edited by these authors (DAVIDSON & HARRISON, 1971).

At flow rates above the critical value, it is observed that "bubbles" of clear fluid may form near the base of the bed and rise through it. Bubble formation is common in a gas-fluidised bed, and may also occur in a liquid-fluidised bed when the ratio of the particle density to the liquid density is large. The number of these bubbles present at any instant appears to be such that the volume fraction of the fluid in the remainder of the bed is reduced to a value near the minimum for which the particles are mobile. There is evidently some dynamical process, some instability in the supply of fluid at the base of the bed, that makes the fluid prefer to pass up through the bed partly in particle-free bubbles, leaving only the minimum flow rate needed for fluidisation in the part of the bed containing the particles.

These rising bubbles in a fluidised bed can be photographed, either by X-rays in the interior of the bed or optically in the case of a bubble near a transparent vertical wall. Observation shows that various bubble shapes occur and that the shapes are generally similar to those of gas bubbles rising through pure liquid. The shape of a gas bubble in liquid which is too large for effects of surface tension to be important depends only on the Reynolds number of the bubble motion, and it is known that there is a transition in shape from a sphere at Reynolds numbers below unity to a spherical cap at large Reynolds

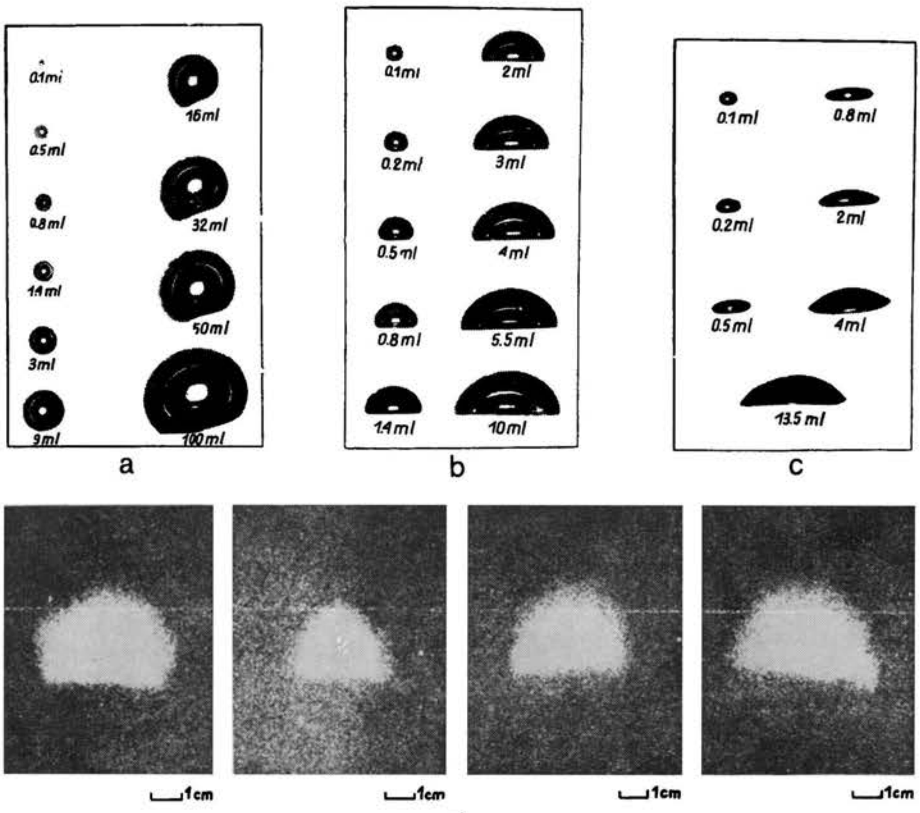

FIG. 1. Bubble shapes under different conditions. Air bubbles of different sizes rising (a) in glycerol of viscosity 8.3 poise, (b) in liquid paraffin of viscosity 2.93 poise, and (c) in sugar solution of viscosity 0.67 poise (from JONES, 1965). The effect of surface tension on the shape of the bubbles of volume greater than 1 cm^3 is probably small. (d) X-ray photographs of air bubbles in a fluidized bed of sand particles (from ROWE and PARTRIDGE, 1965).

numbers compared with unity; see Fig. 1. Figure 1 also shows photographs of bubbles in fluidised beds which resemble gas bubbles in liquid at intermediate values of the Reynolds number; so do most of the other available photographs of bubbles in a fluidised bed. Too little information about the effective viscosity of the fluidised beds is available for it to be possible to ascribe a Reynolds number to the bubbles in fluidised beds shown in Fig. 1.

The separate motions of particles and fluid near a bubble in a fluidised bed, and the way in which they combine to produce a steadily-moving boundary of a region of clear fluid, are now understood in general terms (see the two books cited), although for some years there was debate about the merits of rival theories. There appears to be widespread agreement in particular that particles are prevented from falling through the top of the bubble by a continual flux of fluid through the bubble surface, outwards over the upper part of the bubble and inwards over the lower part, and that particle-particle contact has only a secondary role and does not lead to the formation of "arches". The fluidised bed is treated as a two-phase flow system, with the primary contribution to the force of interaction between the two phases being made by fluid stresses at the particle surfaces. So far as I know, no complete and exact solution for the flow field associated with a single rising bubble in a fluidised bed is available, although there are some partial analytical descriptions based on simplifying hypotheses or models.

The starting-point for the two-phase flow solution to be described below is the fact that a spherical bubble of one fluid rising steadily through a second (and immiscible) fluid under gravity at small Reynolds number remains spherical, even in the absence of surface tension, and that an exact solution of the governing equations for this flow system is available (BATCHELOR, 1967; § 4.9). It seemed to me to be likely that the additional process present in the case of a bubble in a fluidised bed, viz. the movement of each particle relative to the surrounding fluid, could be accommodated within the mathematical form of this exact solution; and I shall show this to be so.

The two main assumptions concerning the conditions under which this solution for a bubble in a fluidised bed are to be sought are as follows:

(a) The Reynolds number of the flow due to a particle moving through the surrounding fluid is small.

(b) The Reynolds number of the mean flow due to the bubble rising through the fluidised bed is small. One consequence of this is that the acceleration of an element of the mixture near a rising bubble is small compared with g (provided we may anticipate that the speed of rise of a bubble in a fluidised bed is comparable with the speed of rise of a drop of fluid of the same size in a second fluid having properties equivalent to those of the fluidised bed) so that the motion of a particle relative to the surrounding fluid is due wholly to the force of gravity.

The Reynolds numbers of the flow due to bubbles occurring in fluidised beds are usually larger than unity (although they may sometimes be less than 10), so the solution to be described is perhaps not of direct practical value. However, it shows clearly the mechanism of this intriguing kind of two-phase flow system, and it suggests some general properties of bubbles in fluidised beds in more complex circumstances, such as when the particles are not uniform in size.

2. The governing equations and their solution

The fluid in which the fluidised particles are immersed is assumed to be Newtonian, with viscosity μ and density ϱ. A small-Reynolds-number motion of this fluid is governed by the equations

$$\nabla p = \varrho \mathbf{g} + \mu \nabla^2 \mathbf{u}, \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.2}$$

where $\mathbf{u}$ and p denote the velocity and (absolute) pressure at a point in the fluid. Equations (2.1) and (2.2) apply everywhere inside a bubble of clear fluid.

The part of the fluidised bed containing particles will be assumed to be statistically homogeneous in composition, with a particle spacing which is small compared with the bubble dimensions. Now the particles are falling under gravity relative to the surrounding fluid, but since the relative motion of the two phases is constant it is not necessary to consider the motions of the two phases separately. We shall regard the two phases as components of a mixture whose statistical properties are like those of an equivalent homogeneous medium, and, more specifically, like those of a Newtonian fluid with viscosity μ_1 and density ϱ_1. This latter requirement is unlikely to be satisfied exactly, because it is known that even for neutrally-buoyant spherical particles in a dilute suspension the effect of hydrodynamic interaction between particles is to make the stress system depend (weakly) on the local flow field and to differ in form in the two cases of pure straining and simple shearing motions (BATCHELOR & GREEN, 1972). However, in the present state of ignorance of the mean stress system in a concentrated suspension there is little else that one can do; and it is plausible to suppose that in a suspension of particles without strong directional properties the main effect of the presence of the particles is represented by an increased effective viscosity of the medium.

The governing equations for the motion of the mixture of particles and fluid are now the same as (2.1) and (2.2), with ϱ_1 and μ_1 replacing ϱ and μ. The velocity $\mathbf{u}$ in these equations is to be interpreted here as the local mean velocity of the mixture, taking account of the different velocities of the fluid and solid constituents, and p as the local mean value of $(-1/3)$-times the sum of the diagonal elements of the stress tensor.

The solution to be considered concerns the flow due to a bubble rising vertically under buoyancy forces through a mixture whose mean velocity is uniform at large distances from the bubble; and we choose axes of reference such that this uniform velocity at infinity is zero. The bubble boundary will be assumed to be spherical, and so involves no directional parameters. The only vector parameter on which the solution of the governing equations can depend, other than the position vector $\mathbf{x}$ (with origin at the instantaneous position of the centre of the spherical bubble), is the gravitational acceleration $\mathbf{g}$. Moreover, it is evident that the velocity and pressure (relative to some reference value) at any point in the flow field must be linear functions of $\mathbf{g}$. It follows that p, which is a harmonic function, must be of the form

$$\frac{p - p_0 - \varrho \mathbf{g} \cdot \mathbf{x}}{\mu} = \mathbf{g} \cdot \mathbf{x} \left(\frac{C}{r^3} + D \right), \tag{2.3}$$

in a region between two spherical surfaces occupied by a medium with density ϱ and viscosity μ, where p_0, C and D are constants and $r = |\mathbf{x}|$. The corresponding expression for a solenoidal velocity may then be found from (2.1) to be

$$\mathbf{u} = \mathbf{g}\frac{f'}{r}+\mathbf{x}\frac{\mathbf{x}\cdot\mathbf{g}}{r^2}\left(\frac{2f}{r^2}-\frac{f'}{r}\right), \tag{2.4}$$

where

$$f(r) = \frac{1}{2}Cr+\frac{1}{20}Dr^4+\frac{L}{r}+Mr^2 \tag{2.5}$$

and $f' = df/dr$.

We note also for future use that the stress on a spherical surface at a place where the unit normal vector is $\mathbf{n}(= \mathbf{x}/r)$ is

$$\begin{aligned}\tau_{ij}n_j &= -pn_i+\mu n_j\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) \\ &= -pn_i+\mu n_i\frac{\mathbf{g}\cdot\mathbf{x}}{r^2}\left(-f''+\frac{6f'}{r}-\frac{10f}{r^2}\right)+\mu g_i\left(f''-\frac{2f'}{r}+\frac{2f}{r^2}\right) \\ &= -(p_0+\varrho\mathbf{g}\cdot\mathbf{x})n_i+\mu n_i\frac{\mathbf{g}\cdot\mathbf{x}}{r^2}\left(-\frac{3C}{r}+\frac{9}{10}Dr^2-\frac{18L}{r^3}\right)+\mu g_i\left(\frac{3}{10}Dr^2+\frac{6L}{r^3}\right).\end{aligned} \tag{2.6}$$

The total force exerted on a spherical surface of radius r by stresses in the fluid on the outer side of this surface is thus

$$\int_{r\,\mathrm{const}} \tau_{ij}n_j dA = -4\pi\mu g_i C.$$

3. Matching conditions at the bubble surface

We now suppose that the bubble of clear fluid which is rising with velocity $\mathbf{U}$ through the fluidised bed has a spherical boundary, with radius R. The test of whether this is a possible permanent shape for the bubble lies in being able to satisfy all the boundary conditions, at the bubble surface and at infinity, by appropriate choice of the constants occuring in the expressions for $\mathbf{u}$ and p.

In each of the two regions $r > R$ and $r < R$ the pressure and velocity are given by expressions of the form (2.3) and (2.4). The four scalar constants appropriate to the flow in the outer region $r > R$, which is occupied by the fluid-particle mixture with density ϱ_1 and viscosity μ_1, will be designated as C_1, D_1, L_1, M_1, and those for the inner region $r < R$, which is occupied by clear fluid with density ϱ and viscosity μ, as C_2, D_2, L_2, M_2. The requirement of zero velocity at infinity gives

$$D_1 = 0, \quad M_1 = 0,$$

and the requirement that the pressure and velocity be non-singular at the centre of the bubble gives

$$C_2 = 0, \quad L_2 = 0.$$

The remaining boundary conditions apply to the bubble surface $r = R$. There is first the obvious condition, from conservation of material volume, that the normal component of $\mathbf{u}$ be continuous across the boundary, which gives

$$\frac{1}{2}C_1 R + \frac{L_1}{R} = \frac{1}{20}D_2 R^4 + M_2 R^2. \tag{3.1}$$

The condition that should be imposed on the tangential component of $\mathbf{u}$ is less evident. On one side of the boundary there is pure fluid, and on the other side there is a mixture of particles and fluid with the particles moving relative to the fluid. I think the appropriate condition is continuity of the tangential component of the mean velocity across the boundary, but there is need for separate investigation of the question. (It will be noted that, if the mean velocity is continuous, the mean velocity of the *fluid* on the mixture side of the boundary is not equal to the fluid velocity on the clear-fluid side.) On this basis we have

$$\frac{1}{2}C_1 R - \frac{L_1}{R} = \frac{1}{5}D_2 R^4 + 2M_2 R^2. \tag{3.2}$$

We have already assumed that the mixture of particles and fluid is equivalent to a Newtonian fluid of viscosity μ_1, and, if we regard this as holding right up to the bubble surface (another point which perhaps needs further investigation), the dynamical condition of continuity, across the boundary, of the tangential component of the stress on an element of a spherical surface is seen from (2.6) to require

$$\mu_1 \frac{6L_1}{R^3} = \mu \frac{3}{10} D_2 R^2. \tag{3.3}$$

The normal component of the stress on an element of a spherical surface must also be continuous across the bubble surface, and since there is no surface tension at the bubble boundary this requires [see (2.6) again]

$$-\varrho_1 R\mathbf{g}\cdot\mathbf{n} + \mu_1 \mathbf{g}\cdot\mathbf{n}\left(-\frac{3C_1}{R^2} - \frac{18L_1}{R^4}\right) = -\varrho R\mathbf{g}\cdot\mathbf{n} + \mu\mathbf{g}\cdot\mathbf{n}\left(-\frac{9}{10}D_2 R^2\right).$$

This relation holds for all $\mathbf{n}$, and so, with the help of (3.3), we have

$$\mu_1 C_1 = \frac{1}{3}R^3(\varrho - \varrho_1). \tag{3.4}$$

We now have four matching conditions for the determination of the four constants C_1, L_1, D_2, M_2, and we find

$$L_1 = -C_1 R^2 \frac{\frac{1}{2}\mu}{3\mu + 2\mu_1}, \quad D_2 = -\frac{C_1}{R^3}\frac{10\mu_1}{3\mu+2\mu_1}, \quad M_2 = C_1 R \frac{\mu + \frac{3}{2}\mu_1}{3\mu + 2\mu_1}, \tag{3.5}$$

with C_1 being given by (3.4). It appears from (2.3), (2.4), (2.5), (3.4) and (3.5), that the velocity and pressure distributions inside and outside the bubble are fully determined and are identical with those for a spherical bubble of radius R containing fluid of density ϱ and viscosity μ moving under gravity at low Reynolds number through fluid of density ϱ_1 and viscosity μ_1.

There is of course a further boundary condition which specifies the flux of material volume across the bubble surface, and this enables the bubble velocity $\mathbf{U}$ to be determined in terms of R and the properties of the two media. Relative to axes moving with the bubble, the normal component of the volume-flux velocity at a point on the bubble surface is

$$\mathbf{n}\cdot(\mathbf{u}-\mathbf{U})_{r=R} = \mathbf{n}\cdot\left(\mathbf{g}\frac{2f(R)}{R^2}-\mathbf{U}\right). \tag{3.6}$$

A significant aspect of (3.6) is that it is the normal component of a vertical velocity which is uniform over the spherical bubble surface and whose value depends on $\mathbf{U}$. By choosing $\mathbf{U}$ appropriately we can make this vertical velocity zero and so demonstrate that a spherical bubble of one fluid rising in another remains spherical. And by a different choice of $\mathbf{U}$ we shall be able to allow for relative motion of the particles and fluid in the case of a bubble in a fluidised bed.

The average velocity of a particle relative to the surrounding medium (i.e. relative to local zero-volume-flux axes) in the interior of the mixture is a vertical velocity, to be denoted by $\mathbf{V}$, which depends on the particle size, shape and density and on the concentration of the particles. And for a particle at a (plane) boundary between clear fluid and the mixture, the component of its mean velocity, again relative to zero-volume-flux axes, normal to the boundary is equal to $\mathbf{n}\cdot\mathbf{W}$, say, where $\mathbf{W}$ is also a vertical velocity and the magnitude W is presumably greater than V but less than the falling speed of a single particle in fluid at rest at infinity(1). Now, if the bubble surface is to be a permanent boundary between the inner clear fluid and the outer uniform mixture, there must be a flux of clear fluid across the bubble surface which makes the particles move along the bubble surface and not across it. This is achieved by equating the expression (3.6) to $-\mathbf{n}\cdot\mathbf{W}$. Note that $\mathbf{W}$ is a statistical quality, and that as a consequence of chance arrangements of neighbouring particles there are always some particles with speeds of fall relative to the surrounding medium which are less than or greater than W; consequently, there will in practice be some particles which cross the bubble surface and fall through it.

The condition that the bubble surface is not crossed by particles is thus

$$-\mathbf{W} = \mathbf{g}\frac{2f(R)}{R^2}-\mathbf{U} = \mathbf{g}\left(\frac{C_1}{R}+\frac{2L_1}{R^3}\right)-\mathbf{U} = \mathbf{g}\frac{C_1}{R}\,\frac{2\mu+2\mu_1}{3\mu+2\mu_1}-\mathbf{U}. \tag{3.7}$$

After elimination of C_1 with the help of (3.4), this relation becomes

$$\begin{aligned}\mathbf{U} &= \mathbf{W}+\frac{\frac{1}{3}R^2(\varrho-\varrho_1)\mathbf{g}}{\mu_1}\,\frac{2\mu+2\mu_1}{3\mu+2\mu_1}, \\ &= \mathbf{W}+\mathbf{U}_0, \text{ say,}\end{aligned} \tag{3.8}$$

(1) The assumption made earlier, that the mean number density of particles in the mixture is uniform, is evidently incompatible with a significant difference between $\mathbf{V}$ and $\mathbf{W}$.

where $\mathbf{U}_0$ is the velocity of a spherical bubble of radius R containing fluid of density ϱ and viscosity μ moving under gravity at low Reynolds number through fluid of density ϱ_1 and viscosity μ_1.

The whole flow field due to a low-Reynolds-number bubble in a fluidised bed is thus a very simple modification of that due to a bubble of the same size and constitution rising through fluid having the same density and viscosity as the particle-fluid mixture. The instantaneous pressure and velocity distributions are identical, provided "velocity" in the case of the particle-fluid mixture is defined as the local average velocity over the two components. The only consequence of the motion of the particles relative to the mixture in which they are immersed is to give the bubble boundary a corresponding kinematical downward velocity, thereby reducing the apparent upward rate of rise of the bubble. $\mathbf{U}_0$ is the dynamic velocity of the bubble resulting from the balance between buoyancy and viscous forces, $\mathbf{W}$ is the kinematical contribution due to the motion of particles relative to the particle-fluid mixture, and $\mathbf{U}_0+\mathbf{W}$ is the observable bubble velocity.

4. Features of the solution

If we take it for granted that the particle density ϱ_p is greater than the fluid density ϱ, then the velocity $\mathbf{U}_0$ is necessarily vertically upward and $\mathbf{W}$ is vertically downward. Consequently, $\mathbf{U}$ is a smaller upward velocity than $\mathbf{U}_0$. If $W > U_0$, where W and U_0 are magnitudes of the velocities $\mathbf{W}$ and $\mathbf{U}_0$, the direction of $\mathbf{U}$ is downward. We get some idea of the circumstances in which this happens by taking the particles to be rigid spheres of radius a which occupy a fraction c of the volume of the particle-fluid mixture everywhere outside the bubble. Then we may put

$$\mathbf{W} = \frac{2a^2(\varrho_p-\varrho)\mathbf{g}}{9\mu P(c)}, \qquad \frac{\mu_1}{\mu} = Q(c), \tag{4.1}$$

where P and Q are functions of the particle concentration about which little reliable information is available except for $c \ll 1$. In the limit $c \to 0$ both P and Q approach unity by definition. At any non-zero c, P and Q are both greater than unity; and Q takes very large values when c approaches the maximum value for which the mixture is mobile. We also have

$$\varrho_1 = c\varrho_p+(1-c)\varrho. \tag{4.2}$$

It follows that

$$\frac{U_0}{W} = c\frac{R^2}{a^2}\frac{P}{Q}\frac{1+Q}{1+\frac{2}{3}Q}. \tag{4.3}$$

The factor R/a is usually large (and must of course be so for our representation of the mixture outside the bubble as a continuum to be valid); and Q is also usually large in fluidised beds. The additional power of R/a probably makes that the dominant factor, in which case U_0/W will usually be greater than unity and the observed bubble velocity will be upward.

In the limiting case $U_0/W \to 0$, which can be thought of as being achieved by letting $c \to 0$ although this is unlikely to be realizable in practice, the bubble velocity is $\mathbf{W}$, the fluid inside the bubble is stationary, and the mixture outside the bubble has zero mean velocity everywhere. The case $U_0/W = 1$ is intriguing because here the dynamical and kinematical contributions to the bubble velocity balance exactly and the bubble appears to be stationary even though the velocity distribution in the mixture is the same as for a bubble rising through a fluid with the equivalent properties ϱ_1 and μ_1.

All the above remarks and analysis refer to velocities relative to axes such that the mean velocity of the mixture far from the bubble is zero. In an ordinary fluidised bed a particle has zero mean velocity relative to an observer, and in order to convert the above results to this observer's frame of reference it is necessary to add $-\mathbf{V}$ to all velocities.

Although the velocity distribution in the mixture outside the bubble is identical with that for a bubble of density ϱ and viscosity μ rising through fluid of density ϱ_1 and viscosity μ_1, the pattern of streamlines of the motion in the mixture *relative to the bubble* looks quite different as a consequence of the different bubble speeds in the two cases. One striking difference is revealed by the expression for the normal component of the mean velocity, relative to axes moving with the bubble, at a point on a spherical surface in the particle-fluid mixture, viz.

(4.4)
$$\mathbf{n}\cdot(\mathbf{u}-\mathbf{U}) = \mathbf{n}\left\{\mathbf{g}\left(\frac{C_1}{r}+\frac{2L_1}{r^3}\right)-\mathbf{U}\right\} = \mathbf{n}\cdot\left\{\mathbf{U}_0\frac{\frac{3}{2}\mu+\mu_1}{\mu+\mu_1}\left(\frac{R}{r}-\frac{R^3}{r^3}\frac{\mu}{3\mu+2\mu_1}\right)-\mathbf{U}_0-\mathbf{W}\right\}.$$

If $\mathbf{W}$ were zero, this expression would be zero at the bubble surface $r = R$; and if $\mathbf{W}$ is a non-zero downward velocity with $U_0 > W$, it is zero at a value of r/R which is greater than unity and given by a root of the equation

$$\frac{R^3}{r^3}\left(\frac{\mu}{3\mu+2\mu_1}\right)-\frac{R}{r}+\frac{U_0-W}{U_0}\left(\frac{\mu+\mu_1}{\frac{3}{2}\mu+\mu_1}\right) = 0. \tag{4.5}$$

When $\mu_1/\mu \gg 1$, an approximate value of the relevant root of (4.5) is simply $R/r = (U_0-W)/U_0 = U/U_0$.

Thus the mixture streamlines relative to axes moving with the bubble are divided into an outer set, which come from and go to infinity, and an inner set which are confined within the sphere whose radius is given by (4.5), as sketched in Fig. 2.

Since the mean velocity of a particle in the interior of the mixture, and relative to local zero-volume-flux axes, is $\mathbf{V}$, the corresponding mean velocity of the fluid at any point, relative to the same zero-volume-flux axes, is

$$-\frac{c}{1-c}\mathbf{V}.$$

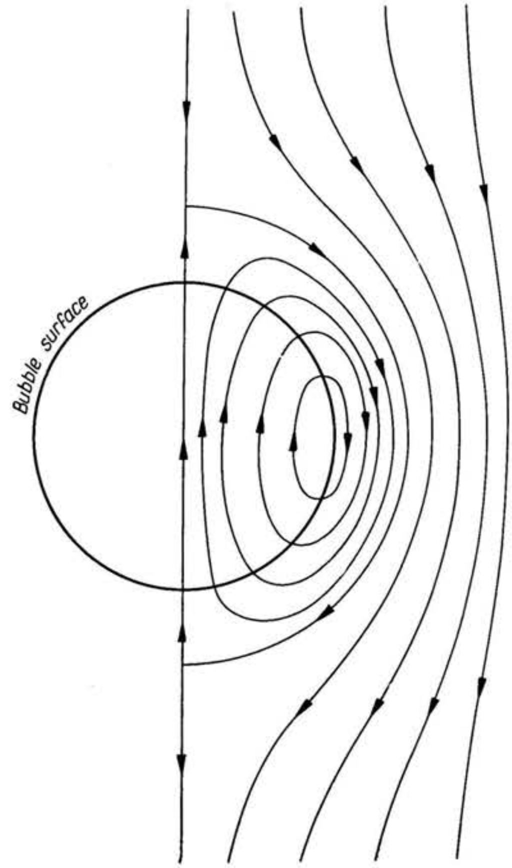

FIG. 2. Sketch of the streamlines of the flow relative to a low-Reynolds-number bubble in a fluidised bed. The bubble is spherical and the streamlines shown are those in a plane through the vertical axis of symmetry. The velocity on which the streamlines are based is the local mean velocity over all constituents of the medium. The streamlines based on the mean velocity of the fluid alone may be obtained by adding a uniform upward velocity to the mean velocity of the particle-fluid mixture; likewise the paths of the particles may be obtained by adding a uniform downward velocity to the velocity of the mixture.

Hence there is also a spherical surface on which the normal component of the mean velocity of the fluid, relative to axes moving with the bubble, is zero, and its radius satisfies the relation

$$\mathbf{n}\cdot(\mathbf{u}-\mathbf{U}) = \frac{c}{1-c}\,\mathbf{n}\cdot\mathbf{V},$$

that is

$$\frac{R^3}{r^3}\left(\frac{\mu}{3\mu+2\mu_1}\right)-\frac{R}{r}+\left(\frac{U_0-W}{U_0}-\frac{c}{1-c}\,\frac{V}{U_0}\right)\frac{\mu+\mu_1}{\frac{3}{2}\mu+\mu_1}=0. \tag{4.6}$$

If $U_0-W = U > \dfrac{c}{1-c}V$, one of the three real roots of equation (4.6) lies in the region of interest, viz. $R/r < 1$, and is given approximately, provided $\mu_1/\mu \gg 1$, by

$$\frac{R}{r} = \frac{U}{U_0}-\frac{c}{1-c}\,\frac{V}{U_0}.$$

In that case fluid that passes into the bubble across the lower half of its surface and out of it across the upper half thus moves on closed paths, relative to the bubble, and travels upward with the bubble. This is a feature of real bubbles in fluidised beds, and not only of those that are nearly spherical(²) and it is also predicted by some of the theoretical models. But if

$$U < \frac{c}{1-c} V,$$

there is no real root of (4.6) in the range $0 < R/r < 1$, and the streamlines of the fluid part of the mixture have a quite different form, essentially because the fluid far from the bubble is now moving upwards faster than the bubble itself. The important dependence of the flow pattern on whether the ratio of U and $\frac{c}{1-c} V$ is greater or less than one is also a familiar aspect of the theoretical models of bubbles in fluidised beds.

At the beginning of this section it was supposed that the particle density is greater than that of the fluid. If $\varrho_p < \varrho$, all velocities in the above solution are simply reversed in direction. A spherical bubble of clear fluid falling through a uniform mixture of light (rising) particles and fluid is thus dynamically possible. I do not know whether such bubbles have been observed; perhaps experiments with water containing many small gas bubbles might show them.

5. A low-Reynolds-number bubble with two spherical interfaces

The solution described above is applicable when the particles in the fluidised bed are uniform. If the particles vary in size, and so also in their speed of fall relative to the mixture, it is not possible to find a single value of $\mathbf{U}$ such that all the different particles move along the bubble boundary and do not cross it. However, if there is a finite number of types of particle, it is possible to imagine a bubble with several concentric spherical interfaces, of which the outermost one is a barrier for the type of particle with the smallest speed of fall, the next one excludes the particles with the next larger speed of fall, and so on. Of course, we cannot choose different values of the bubble velocity $\mathbf{U}$ to suit the kinematical condition at the different interfaces, but we are free to choose the radii of the various spherical interfaces in such a way that the kinematical condition is satisfied at each interface with a common value of $\mathbf{U}$.

This type of multiple-interface solution will be illustrated by a description of a low-Reynolds-number bubble of permanent form rising through a fluidised bed containing just two different types of particle, uniformly mixed. A particle of type 1 has a mean velocity, relative to local zero-volume-flux axes, which is $\mathbf{V}_1$, say, when the particle is in the interior of the mixture and $\mathbf{W}_1$ when it is at a horizontal plane boundary separating the mixture of fluid and particles of types 1 and 2 from a mixture of fluid and particles of type 2. And a particle of type 2 has a mean velocity, relative to local zero-volume-flux axes, which is $\mathbf{V}_2$ ($V_2 > V_1$) when the particle is in the interior of the fluidised bed and $\mathbf{W}_2$ when it is at a horizontal plane boundary separating the mixture of fluid and particles of type 2 from pure fluid.

(²) See DAVIDSON and HARRISON, 1963, and the photograph of a two-dimensional bubble in Fig. 4.43 of the article by P. N. ROWE in Fluidization, edited by DAVIDSON and HARRISON, 1971.

In the outer shell ($\infty > r > R$), occupied by the mixture of fluid and both types of particle, the constants C, D, L, M in the general solution (2.3), (2.4) and (2.5) have the values C_1, 0, L_1, 0, respectively; in the middle shell ($R > r > S$), occupied by a mixture of fluid and particles of type 2 only, the constants are C_2, D_2, L_2, M_2; and in the inner shell ($S > r > 0$), occupied by pure fluid, the constants are 0, D_3, 0, M_3. The mean density and effective viscosity of the mixture in the outer shell are ϱ_1 and μ_1, those for the

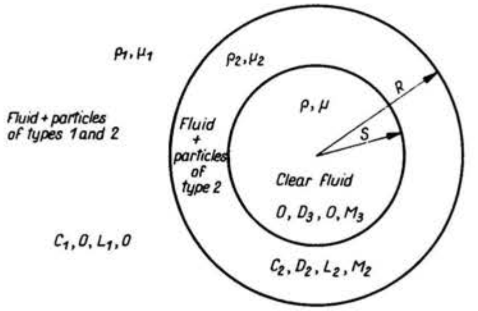

FIG. 3. Definition sketch for a low-Reynolds-number bubble with two spherical interfaces in a fluidised bed of non-uniform particles.

middle shell are ϱ_2 and μ_2, and for the pure fluid in the inner shell the values are ϱ and μ as before. Figure 3 shows the notation schematically.

At each of the two interfaces $r = R$ and $r = S$ there are four matching conditions like (3.1), (3.2), (3.3) and (3.4), and all eight non-zero constants (C_1, L_1, C_2, D_2, L_2, M_2, D_3, M_3) may be determined without difficulty. There is in addition a kinematical condition like (3.7) to be satisfied at each of the two interfaces. The condition that the interface $r = R$ is not crossed by particles of type 1 is

$$(5.1) \qquad -\mathbf{W}_1 = \mathbf{g}\left(\frac{C_1}{R}+\frac{2L_1}{R^3}\right)-\mathbf{U},$$

and the condition that the interface $r = S$ is not crossed by particles of type 2 is

$$(5.2) \qquad -\mathbf{W}_2 = \mathbf{g}\left(\frac{1}{10}D_3S^2+2M_3\right)-\mathbf{U}.$$

After the expressions for the constants C_1, L_1, etc. have been determined and substituted in (5.1) and (5.2), the scalar equivalents of (5.1) and of the difference between (5.1) and (5.2) are found to be

$$(5.3) \qquad (U+W_1)\mu_1 R\{(2\mu_1+3\mu_2)(2\mu_2+3\mu_3)+6\alpha^5(\mu_1-\mu_2)(\mu_2-\mu_3)\}$$
$$= \frac{1}{3}g(\varrho_1-\varrho_2)R^3\{2(\mu_1+\mu_2)(2\mu_2+3\mu_3)+2\alpha^5(3\mu_1-2\mu_2)(\mu_2-\mu_3)\}$$
$$+\frac{1}{3}g(\varrho_2-\varrho_3)R^3\alpha^3\{(3\mu_1+2\mu_2)(2\mu_2+3\mu_3)+4\alpha^5(\mu_1-\mu_2)(\mu_2-\mu_3)-5\alpha^2\mu_1\mu_3\}$$

and

$$(5.4)\quad (W_2-W_1)\mu_2 R\{(2\mu_1+3\mu_2)(2\mu_2+3\mu_3)+6\alpha^5(\mu_1-\mu_2)(\mu_2-\mu_3)\}$$
$$= \frac{1}{3}g(\varrho_1-\varrho_2)R^3\mu_2\{(2\mu_2+3\mu_3)-5\alpha^3\mu_3-2(\mu_2-\mu_3)\alpha^5\}$$
$$+\frac{1}{3}g(\varrho_2-\varrho_3)R^3\alpha^2\{2(2\mu_1+3\mu_2)(\mu_2+\mu_3)-\alpha(2\mu_2+3\mu_3)(3\mu_1+2\mu_2)$$
$$+5\alpha^3\mu_3(2\mu_1-\mu_2)+3\alpha^5(\mu_1-\mu_2)(2\mu_2-3\mu_3)-4\alpha^6(\mu_1-\mu_2)(\mu_2-\mu_3)\},$$

where $\alpha = S/R$. These two equations determine the bubble speed U and the radius ratio α in terms of R and the properties of the fluid and of the particles. They appear to be too complicated to allow general conclusions to be drawn. If we choose $\alpha = 1$ (coalescence of the two interfaces), (5.4) gives $W_2 = W_1$ and (5.3) gives the same expression for U as (3.8), as expected.

The conditions for generation of these bubbles with two interfaces in a fluidised bed containing two types of particle should be generally similar to those for generation of single-interface bubbles in a bed containing uniform particles. As a result of some kind of instability in the supply of clear fluid through the horizontal base of the bed, a finite volume of clear fluid is released at some point and rises through the bed. The faster-falling particles penetrate further into the top of the rising bubble and in this way a separation of the two types of particle occurs and a flow system of the above nature may be established. The conditions of the initial release of the volume of clear fluid at the base determine R, and all other features of the two-interface bubble are determined by the equations of motion and the properties of the particles and the fluid.

6. Higher-Reynolds-number bubbles in a fluidised bed

The simplicity of the relation between a low-Reynolds-number bubble in a fluidised bed and a bubble of the same size and shape rising through a uniform medium with properties equivalent to those of the fluid-particle mixture invites speculation about the relation between bubbles in fluidised beds and those in ordinary fluids in other circumstances. The first of the two main assumptions used in the preceding sections was that the Reynolds number of the flow due to a particle moving through the surrounding fluid is small and that, as a consequence, the relative motion of a particle and the surrounding fluid is determined by the instantaneous force on a particle. This is often true in practice, and is necessary if we are to be able to treat the mixture of particles and fluid as dynamically equivalent to a new uniform fluid with appropriate values of the density and viscosity, so we shall retain it. The second main assumption was that the Reynolds number of the flow due to the bubble rising through the fluidised bed is small. The purpose of this assumption was to allow exploitation of the known exact solution for a spherical bubble of one fluid rising under gravity through a second fluid at low-Reynolds-number. We may abandon this assumption and instead try to make use of empirical information about the shape of a bubble of one fluid rising steadily through a second fluid at Reynolds numbers which are not small. However, it soon becomes clear that there are difficulties.

The ratio of the accelerations produced in the surrounding medium by a bubble with representative dimension R rising with speed U to the gravitational acceleration (i.e. U^2/Rg) is given, in order of magnitude, by the Reynolds number of the flow due to the moving bubble when that Reynolds number is small [see (3.8)]. On the other hand, when the Reynolds number is large, it is known that U is of order $\{gR(\varrho_1-\varrho)/\varrho_1\}^{1/2}$, and so the acceleration ratio is of order unity. Consequently, when the bubble Reynolds number is not small, the elements of the mixture in a fluidised bed are subject to non-negligible accelerations and the motion of a particle relative to local zero-volume-flux axes is not simply a vertical fall due to gravity. If this acceleration vector is not solenoidal, a non-uniform number density of particles will be produced near the bubble and this in turn will lead to new gravitational influences on the motion.

Let us ignore these effects, which probably will be small if the bubble Reynolds number is not more than order unity, and let us continue to suppose that the mixture of particles and fluid outside a bubble in a fluidised bed is equivalent to a uniform Newtonian fluid of density ϱ_1 and viscosity μ_1. The motion of the clear fluid inside a rising bubble and of the equivalent fluid outside the bubble are then governed by the Navier-Stokes equation, with the following boundary conditions (all as in § 3):

medium at rest at infinity,

normal and tangential components of $\mathbf{u}$ continuous across bubble surface,

normal and tangential components of stress continuous across bubble surface.

Also, if we again designate the mean velocity of a particle, relative to local zero-volume-flux axes, at an interface between clear fluid and the particle-fluid mixture by $\mathbf{W}$, the remaining condition to be satisfied at the surface of a bubble of constant shape rising with observable velocity $\mathbf{U}$ in a fluidised bed is

$$\mathbf{n}\cdot(\mathbf{u}-\mathbf{U}) = -\mathbf{n}\cdot\mathbf{W}. \tag{6.1}$$

All these equations and boundary conditions are identical with those that describe the flow due to a bubble of fluid of density ϱ and viscosity μ rising with constant shape and velocity $\mathbf{U}-\mathbf{W}$ through fluid of density ϱ_1 and viscosity μ_1, both the bubble shape and the value of $\mathbf{U}-\mathbf{W}$ being determined by the equations and boundary conditions. No analytical solution for this latter problem is available, except that already described for the case of small Reynolds number when the bubble surface is spherical. But we know from observation of ordinary fluids that rising bubbles with steady shapes and orientations do exist at non-small Reynolds numbers, being characterized by a progressive flattening of the underside as the Reynolds number is increased through unity (Fig. 1). Does it follow then that to every bubble rising steadily with velocity $\mathbf{U}_0$ through ordinary fluid of viscosity μ_1 there corresponds a bubble of clear fluid of the same size and shape which will rise steadily with observable velocity $\mathbf{U}_0+\mathbf{W}$ through a fluidised bed, with the velocities in the medium outside the bubble being the same in the two cases? It appears not, for the reason that, relative to axes moving with the bubble, the velocity distribution in the medium outside the bubble can be steady for only one value of the velocity of the inner boundary — except at low Reynolds number when the term $\partial\mathbf{u}/\partial t$ in the equation of motion is negligible anyway. The bubble of one fluid rising with velocity $\mathbf{U}_0$ through another fluid is a flow system for which the condition

$$\mathbf{n}\cdot(\mathbf{u}-\mathbf{U}_0) = 0$$

for which is satisfied at the bubble surface and, the term $\partial\mathbf{u}/\partial t$ in the equation of motion is equal, with axes such that the fluid at infinity is at rest, to $-\mathbf{U}_0\cdot\nabla\mathbf{u}$ (for the flow would not otherwise be steady when referred to axes moving with the bubble); whereas the bubble in a fluidised bed is a flow system for which (6.1) is satisfied at the bubble surface and the term $\partial\mathbf{u}/\partial t$ in the equation of motion is equal to $-\mathbf{U}\cdot\nabla\mathbf{u}$. The two flow systems thus cannot be identical, and we cannot conclude that the steady shape of a bubble in a fluidised bed is identical with that of a bubble of the same size rising at the same Reynolds number through an ordinary fluid, except when the Reynolds number is small.

Unless our assumption of small Reynolds number of the flow associated with a single particle is invalid, we may conclude that the observed general similarity between the shapes of the two kinds of bubble is evidently approximate only, cxcept at small Reynolds number of the flow due to a bubble.

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