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BRIEF NOTES

On secondary flow phenomena in viscoelastic fluids near free boundaries

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WHAT happens near the free boundary of a "simple fluid" between two walls rotating about the common vertical axis of rotation with small angular velocities? A first approximation is the creeping flow of a Newtonian fluid which is characterised by circular stream lines. This "primary" flow induces a field of centrifugal forces and of normal stress differences, which produce a "secondary" flow and a secondary pressure field. Since the free surface must be free of stress, it cannot be plane. These secondary flow phenomena can be analysed analytically by a regular perturbation procedure. Two material constants characterising a second order fluid come into the theory. In some special cases, explicit formulas for the shape of the free surface are found. Thus, by measuring the shape of the surface, the material constants can be determined.

THE PURPOSE of this paper is to draw the reader's attention to certain secondary flow phenomena which are produced by both inertial forces and normal stress differences in the fluid. Figure 1 shows the flow situation under consideration. The gap between two

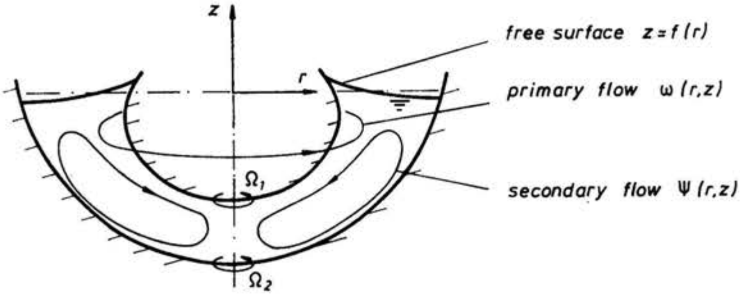

FIG. 1. Flow situation and notations.

rigid surfaces of revolution with a common vertical axis of symmetry is filled with an incompressible fluid up to the height $z = 0$, where the fluid is bounded by a gas of constant pressure. The fluid is assumed to be what is called a "simple fluid"—that is a memory fluid which is characterized by the fact that the Cauchy stresses in every material point depend on the history of the first deformation gradient of that material point only (cf. e.g. [3]). The steady fluid motion which arises if the two walls rotate about the common vertical z-axis with different small angular velocities, Ω_1 and Ω_2, will be described. In extremely slow motion, simple fluids behave like Newtonian fluids. Therefore, in a first approximation, the motion is the creeping flow of a Newtonian fluid. This first approximation is called "primary flow". It is characterized by circular stream lines perpendicular to the axis of symmetry. Each fluid particle moves with a certain angular velocity along a circle the center of which is situated on the axis. Disregarding surface tension, the surface of the fluid is plane in first approximation. Of course, the primary flow is not the exact solution of the problem, since it induces a field of centrifugal forces and a field of normal stresses which produce an additional motion of the fluid, called "secondary flow". The centrifugal forces

$$\varrho r \omega^2 \mathbf{e}_r$$

are proportional to the density ϱ, to the distance from the axis r, and to the square of the local angular velocity ω. In order to find the additional stresses induced by the primary flow, it must be borne in mind that the primary flow is "viscometric" in the sense of COLEMAN [2]. It is well known that in viscometric flows of simple fluids there appear one shear stress and two normal stress differences [4]. These are determined by one kinematic quantity, namely the shear rate $\varkappa$, which in the present case is essentially the absolute value of the gradient of the primary field, $\varkappa = r|\mathrm{grad}\,\omega|$. Therefore, in a viscometric flow, the connexion between the Cauchy stresses and the velocity field is given by three material functions $\tau(\varkappa)$, $\hat{\sigma}_1(\varkappa)$ and $\hat{\sigma}_2(\varkappa)$, for which can be substituted the first terms of their Taylor series for small $\varkappa$, in view of the slow flow assumption. Within a second-order theory, the shear stress function τ reduces to a linear function of ω with the viscosity η as proportional factor:

$$\tau \approx \eta \varkappa .$$

This linear Newtonian term describes the stresses which produce the primary flow. The normal stress functions $\hat{\sigma}_1$ and $\hat{\sigma}_2$ are proportional to the square of the shear rate:

$$\hat{\sigma}_1 \approx \nu_1 \varkappa^2, \quad \hat{\sigma}_2 \approx \nu_2 \varkappa^2,$$

ν_1 and ν_2 are constant material coefficients of second order. Thus, the primary flow induces two normal stress differences which are proportional to the square of the gradient of ω:

$$\nu_1 r^2 (\mathrm{grad}\,\omega)^2 \mathbf{e}_\phi \otimes \mathbf{e}_\phi + \nu_2 r^2 \mathrm{grad}\,\omega \otimes \mathrm{grad}\,\omega .$$

The symbol $\otimes$ signifies a dyadic product. These normal stresses, together with the centrifugal forces, produce the secondary flow. Only that part of the secondary flow is of interest which takes place in the meridian plane and can be described by a stream function ψ. Since the free surface must be free of stress, it cannot be plane in the second approximation. Under certain conditions, viscoelastic fluids climb on the inner rotating wall, as is assumed in Fig. 1. This is the well-known Weissenberg effect. The shape of the free surface may be described by the equation $z = f(r)$.

Now arises the question of how to analyse the secondary flow phenomena. Only the basic ideas are presented; for details see the author's article [8]. First, it is necessary to determine the primary motion. This is a classical Stokes flow problem and leads to a linear elliptic boundary-value problem of second order. In a second step, we have to solve a linear fourth-order boundary-value problem for the secondary stream function. The differential equation contains certain non-linear expressions of the primary solution ω as inhomogeneities which result from the induced centrifugal forces and the normal stresses induced. The boundary conditions are homogeneous. The boundary condition on the free surface can be linearized and fulfilled for $z = 0$ instead of the real surface. Finally, it is necessary to determine the shape of the free surface from the following ordinary first-order differential equation:

$$(1) \qquad \frac{df}{dr} = \frac{1}{\varrho g}\left[\varrho r\omega^2 - \nu_1 r\left(\frac{\partial\omega}{\partial r}\right)^2 + \nu_2 r^2\frac{\partial\omega}{\partial r}\frac{\partial^2\omega}{\partial r^2} - \frac{\eta}{r}\frac{\partial^3\psi}{\partial z^3} - 3\eta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2\psi}{\partial r\,\partial z}\right)\right]_{z=0},$$

which results from the dynamical boundary condition on the free surface after elimination of the pressure. It is clearly seen that, to determine f, both the primary flow field ω and the secondary stream function ψ must be known.

Without solving the boundary value problems, certain general properties of the solution can be stated concerning its dependence on the physical constants. The problem is described by the following constants: the specific gravity g, the density ϱ, the viscosity η, two normal stress coefficients ν_1 and ν_2, two angular velocities of the walls Ω_1 and Ω_2, and certain geometric parameters (G.P.) which characterize the geometrical properties of the gap. It is possible to write down exactly how the solution depends on the seven constants. The primary flow field ω results from a Stokes flow problem and therefore depends on the two characteristic angular velocities only, but neither on gravity nor on density, viscosity or the normal stress coefficients. Moreover, since the Stokes flow problem is a linear boundary-value problem, ω is a linear homogeneous function of the constants Ω_1 and Ω_2. If $\Omega_1 = \Omega_2$, ω reduces to a constant because, under that condition, the fluid rotates as a rigid body. Hence, the primary flow field can be written as

$$\omega(r, z) = \Omega_1 + (\Omega_2 - \Omega_1)\tilde{\omega}(r, z; \text{G.P.}).$$

The secondary stream function ψ can be written in the following form:

$$\psi(r, z) = \frac{\varrho}{\eta}(\Omega_2 - \Omega_1)[\Omega_1\psi_1(r, z; \text{G.P.}) + \Omega_2\psi_2(\ldots)] + \frac{\nu_1+\nu_2}{\eta}(\Omega_2 - \Omega_1)^2\psi_3(\ldots).$$

The formula shows that ψ is a quadratic function of the two angular velocities, which vanishes if Ω_1 and Ω_2 are equal—that is, in the state of rigid rotation. The terms proportional to ϱ/η describe the influence of the centrifugal forces, the last term shows the influence of the normal stress differences. Only the sum of the normal stress coefficients ν_1 and ν_2 comes into the result. ψ_1, ψ_2, ψ_3 and $\tilde{\omega}$ are functions of the variables r and z and, in addition, they depend on the geometric parameters of the problem, though not on the other physical constants. As regards these results, it is found from the Eq. (1) that the shape of the free surface can be represented in the following form:

$$f(r) = \frac{\Omega_1^2}{g}f_1(r; \text{G.P.}) + \frac{\Omega_2^2}{g}f_2(\ldots) + \frac{\Omega_1\Omega_2}{g}f_3(\ldots) + \frac{(\Omega_2-\Omega_1)^2}{\varrho g}[\nu_1 f_4(\ldots) + \nu_2 f_5(\ldots)],$$

which shows the explicit dependence on the physical constants of the problem. The first three terms result from the induced centrifugal forces, the last two terms describe the influence of the normal stresses in the fluid. Note that the elevation f is independent of the viscosity within a second-order approximation.

In the interest of simplicity, in what follows it is assumed that the outer wall does not rotate, which means $\Omega_2 = 0$. It is clear then that the general result for f reduces to three terms; the functions f_2 and f_3 need not be known. Two special flow situations are considered. The first example deals with a liquid filling the semi-infinite space outside a rotating cylindrical rod (Fig. 2). In this case, only one geometric parameter appears, namely the radius R of the cylinder. Therefore, the functions f_1, f_4 and f_5 depend on the variable r and the parameter R. Solving the boundary value problems, we find the following simple analytical solution which was given independently by JOSEPH *et al.* [6] and BÖHME [8](1):

$$f_1 = -\frac{R^4}{2r^2}, \quad f_4 = f_5/3 = \frac{R^4}{r^4}.$$

In particular, the dimensionless elevation of the fluid close to the cylinder is represented by

$$\frac{f(R)}{R} = \frac{R\Omega_1^2}{g}\left(-\frac{1}{2}+\frac{\nu_1+3\nu_2}{\varrho R^2}\right). \tag{2}$$

The second term in brackets dominates for sufficiently small radius R. Therefore, for sufficiently thin rod, the Weissenberg effect occurs if $\nu_1+3\nu_2$ is positive.

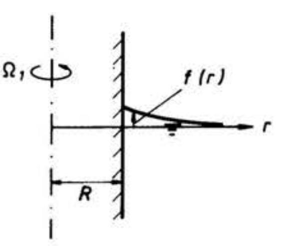

FIG. 2. One-cylinder configuration.

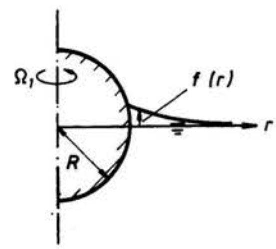

FIG. 3. One-sphere configuration.

As a second simple situation where the boundary-value problems can be solved analytically, the flow caused by a rotating sphere immersed in the fluid up to its equator is considered (Fig. 3). The radius R of the sphere is the only parameter that describes the geometry. As in the one-cylinder configuration, the analysis leads to simple expressions representing the shape of the free surface (for details see [8]):

$$f_1 = -\frac{3R^7}{4r^5}+\frac{R^6}{2r^4},$$

(1) The result agrees with that obtained by SERRIN [1] for Reiner-Rivlin fluids, which are special simple fluids without memory. However, Serrin's solution seems to be irrelevant, since, following ZIEGLER and YU [5], the normal stress differences in Reiner-Rivlin fluids should vanish in second approximation, i.e., $\nu_1 = \nu_2 = 0$.

$$f_4 = \frac{6R^6}{r^6} - \frac{9R^5}{2r^5},$$

$$f_5 = \frac{21R^6}{2r^6} - \frac{9R^5}{2r^5}.$$

Restricting the result to the point $r = R$, we obtain the following formula which gives the elevation of the fluid close to the sphere:

$$\frac{f(R)}{R} = \frac{R\Omega_1^2}{g}\left(-\frac{1}{4}+\frac{3}{2}\frac{\nu_1+4\nu_2}{\varrho R^2}\right). \tag{3}$$

The second term in brackets dominates for a sufficiently small sphere. Thus, the criterion for the Weissenberg effect is the positivity of the sum $\nu_1+4\nu_2$.

The results may serve as a basis for determining the second-order material coefficients ν_1 and ν_2 by means of suitable experiments. Measuring the Weissenberg effect on a rotating cylinder of known radius and known angular velocity and comparing the result with the Eq. (2), we find the material coefficients $(\nu_1+3\nu_2)/\varrho$. In fact, JOSEPH *et al.* [7] employed this method with good success. Nevertheless, the one-cylinder configuration does not make it possible to distinguish between the normal stress coefficients. Therefore, a corresponding procedure with a sphere is proposed, which leads to the coefficients $(\nu_1+4\nu_2)/\varrho$ (cf. Eq. (3)). Thus, the two experiments together should make it possible to estimate the two second-order normal stress coefficients. Note that the viscosity of the fluid need not be known. It should be mentioned that in order to compare the theory with the experiments, it is necessary to modify the theory by taking into account the effect of surface tension. But that can be done by numerical computation, as have shown JOSEPH *et al.* for the one-cylinder configuration; concerning this detail, the reader is referred to [6, 7].

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