Archives of Mechanics • Archiwum Mechaniki Stosowanej • 26, 4, pp. 665–674, Warszawa 1974

Dynamic incompatibility problem

E. KOSSECKA (WARSZAWA)

THE MEDIUM with incompatibilities depending on time is considered. With the help of the idea of plastic deformations, the constraint equations for the elastic strain and velocity fields are derived. Constraints equations lead to definitions of the incompatibility tensor $\boldsymbol{\eta}$ and the new quantity—the incompatibility current tensor $\mathbf{F}$. Next, the dynamic problem for the anisotropic, infinite, linearly elastic medium is examined. The strain field $\mathbf{e}$ and velocity $\mathbf{v}$ are obtained in terms of $\boldsymbol{\eta}$ and $\mathbf{F}$. In the expression for $\mathbf{e}$ the dynamic Green potential $\mathbf{K} = -\Delta^{-1}\mathbf{G}$ appears. The explicit expression for $\mathbf{K}$ for the isotropic medium is found.

Rozważa się ośrodek z niezgodnościami zależnymi od czasu. Posługując się pojęciem deformacji plastycznej wyprowadza się równania więzów dla sprężystego pola odkształcenia i prędkości. Równania te prowadzą do definicji tensora niezgodności $\boldsymbol{\eta}$ i nowej wielkości, tensora prądu niezgodności $\mathbf{F}$. Następnie rozwiązuje się problem dynamiczny dla anizotropowego, nieskończonego ośrodka liniowo sprężystego — znajduje się pole odkształcenia $\mathbf{e}$ i prędkości $\mathbf{v}$ wyrażone przez niezgodności $\boldsymbol{\eta}$ i $\mathbf{F}$. W wyrażeniu dla $\mathbf{e}$ występuje dynamiczny potencjał Greena $\mathbf{K} = -\Delta^{-1}\mathbf{G}$. Znalezione jest jawne wyrażenie dla $\mathbf{K}$ dla ośrodka izotropowego.

Обсуждается среда с несовместностями зависящими от времени. Пользуясь понятием пластической деформации выводятся уравнения связей для упругого поля деформаций и скоростей. Эти уравнения приводят к определению тензора несовместности $\boldsymbol{\eta}$ и новой величины тензора тока несовместности $\mathbf{F}$. Затем решается динамическая задача для анизотропной, бесконечной, линейно-упругой среды — вычисляются поля деформаций $\mathbf{e}$ и скоростей $\mathbf{v}$, выраженные через несовместности $\boldsymbol{\eta}$ и $\mathbf{F}$. В выражении для $\mathbf{e}$ выступает динамический потенциал Грина $\mathbf{K} = -\Delta^{-1}\mathbf{G}$. Вычислено явное выражение для $\mathbf{K}$ для изотропной среды.

1. Introduction

THIS article treats the problem, which appears to be a generalization of the static incompatibility problem formulated by KRÖNER in [1]. We consider linearly elastic, infinitely extended, homogeneous medium. The sources of incompatibility in the medium may be arbitrary defects distributions. To them correspond plastic deformations, which produce incompatibilities. To a discrete defect corresponds singular plastic deformation, having character of a delta function (see [2, 3]).

The state of the medium is represented by the elastic strain field $\mathbf{e}$ and the elastic velocity field $\mathbf{v}$. They have the good physical interpretation, irrespective of kind of defects to be found in the medium.

The constraints equations are developed, being the relations between the elastic and plastic strain and velocity fields. These equations lead to the definitions of the two source quantities: incompatibility tensor $\boldsymbol{\eta}$, which describes geometric incompatibility and is the basic source quantity in the static theory, and the new quantity—incompatibility current tensor, which describes kinematic incompatibility.

We solve the equilibrium equation for the dynamic incompatibility problem for the case of general anisotropy using the Green function technique. The strain field is expressed by the Green potential **K**, to find which the Green tensor **G** of the dynamic Lamé equation is necessary. For the isotropic case, the **K** tensor is calculated explicitly; for anisotropy, the difficulties are of the same kind as for the tensor **G**.

The static incompatibility problem was discussed in details in the paper [5] of Simmons and Bullough, see also [6, 7].

2. Defect kinematics

Before we proceed to the general formulation of the dynamic incompatibility problem—few remarks about defects theory.

When constructing a defects theory, although in principle we tend to deal with "physical" quantities as the fields of stresses, strains and velocities, and the sources in the form of densities of some defects, we often introduce some subsidiary quantities. The quantity of this kind is the displacement field; it is specially convenient when describing a single defect.

In the displacement description, the surface defect has the clear geometric interpretation. It is the surface S, on which the displacement field $\mathbf{u}$ suffers the discontinuity $\mathbf{U}$. We express this fact:

$$|[\mathbf{u}(\boldsymbol{\zeta}, t)]| = \mathbf{U}(\boldsymbol{\zeta}, t); \quad \boldsymbol{\zeta} \in S. \tag{2.1}$$

The double bracket denotes here the discontinuity of the function $\mathbf{u}(\mathbf{x})$ at the point $\boldsymbol{\zeta}$ of S at the instant t.

We assume then that the surface can move in an arbitrary way. For the purpose of the theory of linear defects—dislocations and disclinations—this model is slightly too general. It is sufficient to assume that the open "defect surface" was formed and changes in time only through the motion of its boundary. But for the purpose of the theory of point defects, which we obtain by the limitary transition to the infinitesimal closed surfaces, this generality is necessary (see [4]).

To the discontinuity $\mathbf{U}$ of the displacement field $\mathbf{u}(\mathbf{x}, t)$ are associated (see [3]) the singularities of its derivatives with respect to space variables and time. We denote them appropriately $\mathring{\boldsymbol{\beta}}$ and $\mathring{\mathbf{v}}$. The fields $u_{i,k}$ and $\dot{u}_i$ we write in the form:

$$u_{i,k} = \beta_{ik} + \mathring{\beta}_{ik}, \tag{2.2$_1$}$$

$$\dot{u}_i = v_i + \mathring{v}_i. \tag{2.2$_2$}$$

For the strains, we introduce the denotations:

$$u_{<i,k>} = e_{ik} + \mathring{e}_{ik}. \tag{2.3}$$

$\mathring{\boldsymbol{\beta}}$ and $\mathring{\mathbf{v}}$ are given by the formulae:

$$\mathring{\beta}_{ik} = \int_S ds_k\, U_i\, \delta_3(\mathbf{x} - \boldsymbol{\zeta}); \quad \boldsymbol{\zeta} \in S, \tag{2.4$_1$}$$

(2.4)$_2$ $$\overset{\circ}{v}_i = -\int_S ds_k \dot{\zeta}_k U_i \delta_3(\mathbf{x}-\boldsymbol{\zeta}).$$

For the case, when the normal velocity of the points of the surface S is equal to zero, e.g. when it is the real surface outlined by the dislocation line, $\overset{\circ}{\mathbf{v}} = 0$.

The state of the medium in the defect's surrounding is represented by the fields $\mathbf{e}$ and $\mathbf{v}$; we call them elastic. We also call $\boldsymbol{\beta}$ the elastic distortion, however, one should point out that, in the presence of disclinations, $\boldsymbol{\beta}$ is not uniquely defined, and plays the subsidiary role, like the $\mathbf{u}$ field [7]. The fields $\overset{\circ}{\boldsymbol{\beta}}$, $\overset{\circ}{\mathbf{e}}$ and $\overset{\circ}{\mathbf{v}}$ we call plastic (or initial). They describe the constrained deformation, leading from the ideal medium to the medium with incompatibilities. The name "plastic" is in some sense conventional because the constrained deformation needs not be the real deformation. The quantities $\mathbf{e}$ and $\mathbf{v}$, which we shall use further on, can be obviously generalized to the case of continuous distribution of defects.

3. Incompatibility tensor and incompatibility current tensor

We now proceed to the general formulation of the incompatibility problem, e.g. to the derivation of the formulae, being the relationships between the elastic fields $\mathbf{e}$ and $\mathbf{v}$ and the plastic fields $\overset{\circ}{\mathbf{e}}$ and $\overset{\circ}{\mathbf{v}}$. We make use again of the subsidiary quantity, the total displacement field $\mathbf{u}$, in the ideal medium subjected to the elastic and plastic deformation. The total strain is then the sum of the elastic and plastic strain, the same applies to velocities:

(3.1)$_1$ $$u_{<i,k>} = e_{ik} + \overset{\circ}{e}_{ik},$$

(3.1)$_2$ $$\dot{u}_i = v_i + \overset{\circ}{v}_i.$$

Let us eliminate from the system of Eqs. (3.1) the field $\mathbf{u}$. Second derivatives of $\mathbf{u}$ can be represented in the following way:

(3.2) $$u_{i,km} = \frac{1}{2}[u_{i,km} + u_{k,im}] + \frac{1}{2}[u_{i,mk} + u_{m,ik}] - \frac{1}{2}[u_{k,mi} + u_{m,ki}].$$

Thus, from $(3.1)_1$, we obtain:

(3.3) $$u_{i,km} = e_{ik,m} + e_{im,k} - e_{km,i} + \overset{\circ}{e}_{ik,m} + \overset{\circ}{e}_{im,k} - \overset{\circ}{e}_{km,i}.$$

Differentiating once more, we obtain:

(3.4) $$u_{i,kml} = e_{ik,ml} + e_{im,kl} - e_{km,il} + \overset{\circ}{e}_{ik,ml} + \overset{\circ}{e}_{im,kl} - \overset{\circ}{e}_{km,il}.$$

Simultaneously:

(3.5) $$u_{i,klm} = e_{ik,lm} + e_{il,km} - e_{kl,im} + \overset{\circ}{e}_{ik,lm} + \overset{\circ}{e}_{il,km} - \overset{\circ}{e}_{kl,im}.$$

Subtracting (3.5) from (3.4), we obtain:

(3.6) $$e_{im,kl} + e_{kl,im} - e_{km,il} - e_{il,km} = -[\overset{\circ}{e}_{im,kl} + \overset{\circ}{e}_{kl,im} - \overset{\circ}{e}_{km,il} - \overset{\circ}{e}_{il,km}].$$

In view of the antisymmetry in the indices (i, k), (l, m), it is convenient to write (3.6) in the form:

(3.7) $$-\varepsilon_{rik}\varepsilon_{plm} e_{km,il} = \varepsilon_{rik}\varepsilon_{plm}\overset{\circ}{e}_{km,il}.$$

This is the basic constraints equation for the $\mathbf{e}$ field in the static theory. When there are only elastic deformations in the medium ($\overset{\circ}{e} = 0$), (3.6) is the classical de Saint Venant compatibility equation for the strain field. However, in the dynamic case, the second equation appears, being the relation between the time derivative of the strain and the symmetric part of gradient of the velocity. Differentiating with respect to time equation $(3.1)_1$, we obtain:

$$\dot{u}_{<i,k>} = \dot{e}_{ik} + \dot{\overset{\circ}{e}}_{ik}. \tag{3.8}$$

Differentiating Eq. $(3.1)_2$ with respect to x_k and symmetrising the result, we obtain:

$$\dot{u}_{<i,k>} = v_{<i,k>} + \overset{\circ}{v}_{<i,k>}. \tag{3.9}$$

If we subtract now (3.9) from (3.8), we come to:

$$\dot{e}_{ik} - v_{<i,k>} = -[\dot{\overset{\circ}{e}}_{ik} - \overset{\circ}{v}_{<i,k>}]. \tag{3.10}$$

Equations (3.7) and (3.10) lead us to the definitions of the two quantities: the incompatibility tensor $\boldsymbol{\eta}$ and the incompatibility current tensor $\mathbf{F}$:

$$\eta_{ij} = \varepsilon_{ikl}\varepsilon_{jmn}\overset{\circ}{e}_{ln,km}, \tag{3.11$_1$}$$

$$F_{ik} = -[\dot{\overset{\circ}{e}}_{ik} - \overset{\circ}{v}_{<i,k>}]. \tag{3.11$_2$}$$

$\boldsymbol{\eta}$ and $\mathbf{F}$ will be the sources of the elastic deformation.

From the definition $(3.11)_1$ results the symmetry and vanishing of the divergence of the tensor $\boldsymbol{\eta}$.

Differentiating twice the Eq. $(3.11)_2$, we obtain:

$$-\left[\dot{\overset{\circ}{e}}_{ik,lm} - \frac{1}{2}(\overset{\circ}{v}_{i,klm} + \overset{\circ}{v}_{k,ilm})\right] = F_{ik,lm}. \tag{3.12}$$

Contracting (3.12) with $\varepsilon_{ali}\varepsilon_{bmk}$ gives:

$$-\dot{\eta}_{ab} = \varepsilon_{ali}\varepsilon_{bmk}F_{ik,lm}. \tag{3.13}$$

The system of equations:

$$\eta_{ab,b} = 0, \tag{3.14$_1$}$$

$$\dot{\eta}_{ab} = -\varepsilon_{ali}\varepsilon_{bmk}F_{ik,lm}, \tag{3.14$_2$}$$

is the system of compatibility equations for the tensors $\boldsymbol{\eta}$ and $\mathbf{F}$.

4. Incompatibilities as the sources of deformation in the linearly elastic medium

We calculate now the strain and velocity fields, produced by incompatibilities in the infinite, linearly elastic medium. In his basic work [1], KRÖNER gave the exact solution of the static problem for the isotropic medium. SIMMONS and BULLOUGH have discussed the case of the anisotropic medium in [5]; the incompatibility source tensor which they defined is given in terms of the Green tensor of the Lamé equation.

Here, we consider the general anisotropic medium; the elastic strain and velocity fields $\mathbf{e}$ and $\mathbf{v}$ satisfy the equilibrium equation:

(4.1) $$\varrho \frac{\partial}{\partial t} v_j - c_{jklm} e_{lm,k} = 0.$$

$\mathbf{c}$ is the tensor of elastic moduli of the medium.

At the same time the constraints equations are to be satisfied:

$(4.2)_1$ $$\varepsilon_{ikl}\varepsilon_{jmn} e_{ln,km} = -\eta_{ij},$$

$(4.2)_2$ $$\dot{e}_{ik} - v_{<i,k>} = F_{ik}.$$

The above three equations we replace by two equations, from which the strain field $\mathbf{e}$ and velocity $\mathbf{v}$ are to be determined. We are not making use any more of the ideas of the $\mathbf{u}$, $\mathring{\mathbf{e}}$ and $\mathring{\mathbf{v}}$ fields.

The Eq. (4.1) will be submitted to the following subsequent transformations. First, we differentiate it:

(4.3) $$\varrho \frac{\partial}{\partial t} v_{j,s} - c_{jklm} e_{lm,ks} = 0.$$

Then we add and subtract appropriate terms:

(4.4) $$\varrho \frac{\partial^2}{\partial t^2} e_{js} - c_{jklm} e_{ls,km} = c_{jklm}[e_{lm,ks} - e_{ls\,km}] + \varrho \frac{\partial}{\partial t} [\dot{e}_{js} - v_{j,s}].$$

Next, we act with the Laplace operator:

(4.5) $$\varrho \frac{\partial^2}{\partial t^2} \Delta e_{js} - c_{jklm} \Delta e_{ls,km} = c_{jklm} \nabla_k \nabla_a [e_{lm,sa} - e_{ls,ma}] + \varrho \frac{\partial}{\partial t} \nabla_a [\dot{e}_{js,a} - v_{j,sa}]$$

and complete the equation to the form:

(4.6) $$\varrho \frac{\partial^2}{\partial t^2} [\Delta e_{js} + e_{ja,as} - e_{sa,aj}] - c_{jklm} \nabla_k \nabla_m \Delta e_{ls} + c_{jklm} \nabla_k \nabla_a [e_{sa,lm} - e_{ma,ls}]$$
$$= c_{jklm} \nabla_k \nabla_a [e_{lm,sa} + e_{sa,lm} - e_{ls,ma} - e_{ma,ls}] + \varrho \frac{\partial}{\partial t} \nabla_a [\dot{e}_{js,a} + \dot{e}_{ja,s} - \dot{e}_{sa,j} - v_{j,sa}].$$

From $(4.2)_{1,2}$ follows:

$(4.7)_1$ $$e_{lm,sa} + e_{sa,lm} - e_{ls,ma} - e_{ma,ls} = \varepsilon_{smr}\varepsilon_{alp}\varepsilon_{rbc}\varepsilon_{pdg} e_{bd,gc} = -\varepsilon_{smr}\varepsilon_{alp}\eta_{rp},$$

$(4.7)_2$ $$\dot{e}_{js,a} + \dot{e}_{ja,s} - \dot{e}_{sa,j} - v_{j,sa} = \dot{e}_{js,a} + \dot{e}_{ja,s} - \dot{e}_{sa,j} - v_{<j,s>a} - v_{<j,a>s} + v_{<s,a>j}$$
$$= F_{js,a} + F_{ja,s} - F_{sa,j}.$$

If we take into account $(4.7)_{1,2}$, (4.6) takes the form:

(4.8) $$\varrho \frac{\partial^2}{\partial t^2} [\Delta e_{js} + e_{ja,as} - e_{sa,aj}] - c_{jklm} \nabla_k \nabla_m [\Delta e_{ls} + e_{la,as} - e_{sa,al}]$$
$$= -c_{jklm} \nabla_k \nabla_a \varepsilon_{smr}\varepsilon_{alp}\eta_{rp} + \varrho \frac{\partial}{\partial t} \nabla_a [F_{js,a} + F_{ja,s} - F_{sa,j}].$$

Denoting by $\check{\mathbf{L}}$ the Lamé operator

(4.9) $$\check{L}_{jl} = \varrho \delta_{jl} \frac{\partial^2}{\partial t^2} - c_{jklm} \nabla_k \nabla_m,$$

we write (4.8) in the form:

$$\check{L}_{ji}[\Delta e_{ls}+e_{la,as}-e_{sa,al}] = -c_{jklm}\nabla_k\nabla_a\varepsilon_{smr}\varepsilon_{alp}\eta_{rp}+\varrho\frac{\partial}{\partial t}\nabla_a[F_{js,a}+F_{ja,s}-F_{sa,j}]. \tag{4.10}$$

We are considering the infinite medium, so we are interested in the particular solutions of the above equation only. This is the equation of the Lamé type, and its solution, given in terms of the dynamic Green tensor **G**, has the form:

$$\Delta e_{is}+e_{ia,as}-e_{sa,ai} = -G_{ij}*\left\{c_{jklm}\nabla_k\nabla_a\varepsilon_{smr}\varepsilon_{alp}\eta_{rp}-\varrho\frac{\partial}{\partial t}\nabla_a[F_{js,a}+F_{ja,s}-F_{sa,j}]\right\}. \tag{4.11}$$

Here, the star denotes the convolution with respect to four variables x_1, x_2, x_3, t; **G** satisfies the equation:

$$\check{L}_{jl}G_{ln} = \delta_{jn}\delta(t)\delta_3(\mathbf{x}). \tag{4.12}$$

Symmetrising (4.11) with respect to indices i, s, we obtain:

$$\Delta e_{is} = -G_{ij}*\left\{c_{jklm}\nabla_k\nabla_a\varepsilon_{smr}\varepsilon_{alp}\eta_{rp}-\varrho\frac{\partial}{\partial t}\nabla_a[F_{js,a}+F_{ja,s}-F_{sa,j}]\right\}_{<is>}, \tag{4.13$_1$}$$

$$e_{is} = -\Delta^{-1}G_{ij}*\left\{c_{jklm}\nabla_k\nabla_a\varepsilon_{smr}\varepsilon_{alp}\eta_{rp}-\varrho\frac{\partial}{\partial t}\nabla_a[F_{js,a}+F_{ja,s}-F_{sa,j}]\right\}_{<is>}. \tag{4.13$_2$}$$

We introduce the tensor **K**, which may be called the Green potential, satisfying the Poisson equation:

$$\Delta K_{ij} = -G_{ij}, \tag{4.14$_1$}$$

$$K_{ij} = -\Delta^{-1}G_{ij}. \tag{4.14$_2$}$$

At the same time it satisfies the Lamé equation of the form:

$$\check{L}_{ji}K_{ln} = \delta_{jn}\delta(t)\frac{1}{4\pi r}. \tag{4.15}$$

The expression for **e**, in terms of **K**, is given as:

$$e_{is} = K_{ij,a}*\left\{c_{jklm}\nabla_k\varepsilon_{smr}\varepsilon_{alp}\eta_{rp}-\varrho\frac{\partial}{\partial t}[F_{js,a}+F_{ja,s}-F_{sa,j}]\right\}_{<is>}. \tag{4.16}$$

If we are to calculate the **v** field, we differentiate (4.1) with respect to time:

$$\varrho\frac{\partial^2}{\partial t^2}v_j-c_{jklm}\nabla_k\dot{e}_{lm} = 0, \tag{4.17}$$

and complete to the form:

$$\varrho\frac{\partial^2}{\partial t^2}v_j-c_{jklm}\nabla_k\nabla_m v_l = c_{jklm}\nabla_k[\dot{e}_{lm}-v_{<l,m>}] = c_{jklm}\nabla_k F_{lm}. \tag{4.18}$$

Hence

$$v_l = G_{lj}*c_{jklm}\nabla_k F_{lm}. \tag{4.19}$$

For **F** equal to zero, **v** is equal to zero, and **e** is given by the static expression discussed in [5].

5. The dynamic Green potential for the isotropic medium

We calculate now the dynamic Green potential **K** for the isotropic medium. For completeness, we calculate the dynamic Green tensor **G**. For the isotropic medium, the tensor of elastic moduli has the form:

(5.1)
$$c_{iklm} = \lambda\delta_{ik}\delta_{lm} + \mu[\delta_{il}\delta_{km} + \delta_{im}\delta_{kl}].$$

The Lamé equation for the **u** field produced by the force distribution **X** is as follows:

(5.2)
$$\varrho\ddot{u}_i - (\lambda+\mu)u_{k,ki} - \mu u_{i,kk} = X_i.$$

We introduce wave velocities:

(5.3)
$$c_1^2 = \frac{\lambda+2\mu}{\varrho}, \quad c_2^2 = \frac{\mu}{\varrho},$$

then

(5.4)
$$(c_1^2 - c_2^2)u_{k,ki} + c_2^2 u_{i,kk} - \ddot{u}_i = -\frac{1}{\varrho}X_i.$$

We denote by $\square_{(1)}$, $\square_{(2)}$ d'Alembert operators:

(5.5)
$$\square_{(1)} = \Delta - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}, \quad \square_{(2)} = \Delta - \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}.$$

Equation (5.4) yields:

$(5.6)_1$
$$\square_{(2)}u_i = \frac{1}{c_1^2}\left\{-\frac{1}{\varrho}X_i - (c_1^2 - c_2^2)u_{k,ki}\right\},$$

$(5.6)_2$
$$u_i = \frac{1}{c_2^2}\square_{(2)}^{-1}\left\{-\frac{1}{\varrho}X_i - (c_1^2 - c_2^2)u_{k,ki}\right\}.$$

Taking the divergence of (5.4), we obtain:

$(5.7)_1$
$$\square_{(1)}u_{k,k} = -\frac{1}{c_1^2}\frac{1}{\varrho}X_{k,k},$$

$(5.7)_2$
$$u_{k,k} = \frac{1}{c_1^2}\square_{(1)}^{-1}\left\{-\frac{1}{\varrho}X_{k,k}\right\}.$$

Hence

(5.8)
$$u_i = \frac{1}{\varrho}\left\{-\frac{1}{c_2^2}\square_{(2)}^{-1}\delta_{ik} + \frac{c_1^2 - c_2^2}{c_1^2 c_2^2}\square_{(1)}^{-1}\square_{(2)}^{-1}\nabla_i\nabla_k\right\}X_k.$$

Because

(5.9)
$$u_i = G_{ik} * X_k,$$

G is equal to:

(5.10)
$$G_{ik} = \frac{1}{\varrho}\left\{-\frac{1}{c_2^2}\square_{(2)}^{-1}\delta_{ik} + \frac{c_1^2 - c_2^2}{c_1^2 c_2^2}\square_{(1)}^{-1}\square_{(2)}^{-1}\nabla_i\nabla_k\right\}\delta(t)\,\delta_3(\mathbf{x}).$$

It is of importance to have (5.10) in a slightly different form; we make use of the following formal transformation:

$$\frac{c_1^2}{\Box_{(2)}}-\frac{c_2^2}{\Box_{(1)}}=(c_1^2-c_2^2)\frac{\Delta}{\Box_{(1)}\Box_{(2)}}. \tag{5.11}$$

Hence

$$G_{ik}=\frac{1}{\varrho}\left\{-\frac{1}{c_2^2}\Box_{(2)}^{-1}\delta_{ik}+\nabla_i\nabla_k\Delta^{-1}\left[\frac{1}{c_2^2}\Box_{(2)}^{-1}-\frac{1}{c_1^2}\Box_{(1)}^{-1}\right]\right\}\delta(t)\,\delta_3(\mathbf{x}). \tag{5.12}$$

Here, we deal with the expressions $\Box^{-1}\delta(t)\,\delta_3(\mathbf{x})$; these are the Green functions of the wave equations. Because in what follows we are going to calculate the retarded Green tensor of the Lamé equation, we make use here of the retarded singular solutions of the wave equations. So we take:

$$\Box^{-1}\delta(t)\,\delta_3(\mathbf{x})=-\frac{1}{4\pi}\frac{\delta\left(t-\frac{r}{c}\right)}{r}, \tag{5.13$_1$}$$

$$\overset{\text{ret}}{G}_{ik}=\frac{1}{4\pi\varrho}\left\{\frac{1}{c_2^2}\delta_{ik}\frac{\delta\left(t-\frac{r}{c}\right)}{r}+\nabla_i\nabla_k\Delta^{-1}\left[\frac{1}{c_1^2}\frac{\delta\left(t-\frac{r}{c_1}\right)}{r}-\frac{1}{c_2^2}\frac{\delta\left(t-\frac{r}{c_2}\right)}{r}\right]\right\}. \tag{5.13$_2$}$$

The retarded Green tensor has its support in the region $t\geqslant 0$; in what follows, we shall always assume $t\geqslant 0$ without pointing out this fact explicitly.

The Laplace operator acts on the function depending on r only in the following way:

$$\Delta f(r)=\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}rf(r)\right); \tag{5.14}$$

so its reverse is:

$$\Delta^{-1}f(r)=\frac{1}{r}\int\limits_0^r dr'\int\limits_0^{r'}dr''r''f(r''). \tag{5.15}$$

It acts on the generalized function $\dfrac{\delta\left(t-\frac{r}{c}\right)}{c^2r}$ in the following way:

$$\Delta^{-1}\frac{\delta\left(t-\frac{r}{c}\right)}{c^2r}=\frac{1}{rc^2}\int\limits_0^r dr'\int\limits_0^{r'}dr''\delta\left(t-\frac{r''}{c}\right)=\frac{1}{rc}\int\limits_0^{r'}dr''\theta(r'-ct) \tag{5.16}$$

$$=\frac{1}{rc}\int\limits_{ct}^{r}dr'\theta(r-ct)=\left(\frac{1}{c}-\frac{t}{r}\right)\theta(r-ct).$$

θ is the Heaviside function:

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0, \end{cases} \tag{5.17$_1$}$$

$$\theta(x)+\theta(-x) = 1, \tag{5.17$_2$}$$

$$\theta(cx) = \theta(x) \quad \text{for} \quad c \geqslant 0. \tag{5.17$_3$}$$

From $(5.13)_2$, (5.16) and $(5.17)_{1,2}$, we obtain (see also [8, 9]):

$$\overset{\text{ret}}{G}_{ik} = \frac{1}{4\pi\varrho}\left\{\frac{\delta_{ik}}{c_2^2}\frac{\delta\left(t-\dfrac{r}{c_2}\right)}{r} + \nabla_i\nabla_k\left[-\frac{1}{c_1}\theta\left(t-\frac{r}{c_1}\right)+\frac{t}{r}\theta\left(t-\frac{r}{c_1}\right)\right.\right.$$
$$\left.\left.+\frac{1}{c_2}\theta\left(t-\frac{r}{c_2}\right)-\frac{t}{r}\theta\left(t-\frac{r}{c_2}\right)\right]\right\}. \tag{5.18}$$

We perform now calculations important for the evaluation of the **K** tensor.

$$\Delta^{-1}\frac{1}{c}\theta(ct-r) = \frac{1}{rc}\int_0^r dr'\int_0^{r'} dr''r''\theta(ct-r'') = \frac{1}{rc}\int_0^r dr'\left\{\frac{r''^2}{2}\bigg|_0^{ct}\theta(r'-ct)\right. \tag{5.19}$$
$$\left.+\frac{r''^2}{2}\bigg|_0^{r'}\theta(ct-r')\right\} = \frac{1}{rc}\int_0^r dr\left\{\frac{c^2t^2}{2}\theta(r'-ct)+\frac{r'^2}{2}\theta(ct-r')\right\}$$
$$= \frac{1}{rc}\left\{r'\bigg|_{ct}^{r}\frac{c^2t^2}{2}\theta(r-ct)+\frac{r'^3}{6}\bigg|_0^{ct}\theta(r-ct)+\frac{r'^3}{6}\bigg|_0^{r}\theta(ct-r)\right\}$$
$$= \frac{1}{rc}\left\{\left[\frac{rc^2t^2}{2}-\frac{c^3t^3}{3}\right]\theta(r-ct)+\frac{r^3}{6}\theta(ct-r)\right\}.$$

Proceeding in the same way, we obtain also:

$$\Delta^{-1}\frac{t}{r}\theta(ct-r) = \frac{1}{rc}\left\{\left[rc^2t^2-\frac{c^3t^3}{2}\right]\theta(r-ct)+\frac{r^2ct}{2}\theta(ct-r)\right\}. \tag{5.20}$$

Hence:

$$\Delta^{-1}\left[-\frac{1}{c_1}\theta(c_1t-r)+\frac{t}{r}\theta(c_1t-r)\right]-\Delta^{-1}\left[-\frac{1}{c_2}\theta(c_2t-r)+\frac{t}{r}\theta(c_2t-r)\right] \tag{5.21}$$
$$= \frac{1}{6r}\left\{\frac{1}{c_1}(r-c_1t)^3\theta(r-c_1t)-\frac{1}{c_2}(r-c_2t)^3\theta(r-c_2t)\right\}+\frac{1}{6}\left(\frac{1}{c_1}-\frac{1}{c_2}\right)r^2.$$

So, up to a constant, the tensor **K** is equal:

$$\overset{\text{ret}}{K}_{ij} = -\frac{1}{4\pi\varrho}\left\{\frac{\delta_{ik}}{c_2}\frac{1}{r}(r-c_2t)\theta(r-c_2t)\right. \tag{5.22}$$
$$\left.+\frac{1}{6}\nabla_i\nabla_k\frac{1}{r}\left[\frac{1}{c_1}(r-c_1t)^3\theta(r-c_1t)-\frac{1}{c_2}(r-c_2t)^3\theta(r-c_2t)\right]\right\}, \quad t \geqslant 0.$$

References

1. E. KRÖNER, *Kontinuumstheorie der Versetzungen und Eigenspannungen*, Ergeb. angew. Math., **5**, 1958,
2. E. KOSSECKA, *The theory of dislocation lines in a continuous medium*, Arch. Mech. Stos., **21**, 2, 167 1969.
3. E. KOSSECKA, *Defects as initial deformations*, Arch. Mech., **25**, 1, 3–11, 1973.
4. E. KOSSECKA, *Point defects in motion*, in preparation.
5. J. A. SIMMONS, R. BULLOUGH, *Internal stress and the incompatibility problem in infinite anisotropic elasticity*, in Fundamental aspects of dislocation theory, Eds. J. A. SIMMONS, R. de WIT, and R. BULLOUGH, Nat. Bur. Stand. (U.S.), Spec. Publ. 317, Vol. I, 1970.
6. I. A. KUNIN, *Supplement to Russian edition of Tensor Analysis for Physicists by* J. A. SCHOUTEN, Moscow 1965.
7. R. de WIT, *The relation between continuous and discrete disclinations*, Arch. of Mech., **24**, 3, 499–510, 1972.
8. A. E. H. LOVE, *A treatise on the mathematical theory of elasticity*, Ed. 4, Dover Publ., 1944.
9. W. NOWACKI, *Theory of elasticity* [in Polish], PWN, Warszawa 1970.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received October 12, 1973.