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Construction of a flow of an ideal plastic material in a die, on the basis of the method of Riemann invariants

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THE PAPER contains analysis of the problem of flow of a perfectly plastic material inside an extruding die. It is concerned with an application of the method of Riemann invariants to this problem [1, 2]. When the above method is applied, certain difficulties are removed and mathematical precision may be improved.

Praca zawiera analizę przepływu idealnie plastycznego materiału wewnątrz narzędzia. Jest ona oparta na zastosowaniu metody inwariantów Riemanna do tego problemu [1, 2]. Stosowanie powyższej metody usuwa pewne trudności i pozwala osiągnąć większą precyzję matematyczną.

Работа содержит анализ задачи течения идеально пластического материала через матрицу. Опирается он на применении метода инвариантов Римана для этой задачи [1], [2]. Применение вышеупомянутого метода удаляет некоторые трудности и позволяет достигнуть большую математическую строгость.

1

IN THIS PAPER, we consider the system of partial differential equations

$$
\begin{aligned}
&\sigma_{,x}-2k(\theta_{,x}\cos 2\theta+\theta_{,y}\sin 2\theta) = 0,\\
&\sigma_{,y}-2k(\theta_{,x}\sin 2\theta-\theta_{,y}\cos 2\theta) = 0,\\
&(u_{,y}+v_{,x})\sin 2\theta+(u_{,x}-v_{,y})\cos 2\theta = 0,\\
&u_{,x}+v_{,y} = 0,
\end{aligned}
\tag{1.1}
$$

describing the plane flows of an ideal plastic material. The functions $u(x, y)$, $v(x, y)$ are the components of the velocity vector and $\sigma(x, y)$ and $\theta(x, y)$ define the components of the stress tensor

$$\sigma_{11} = \sigma-k\sin 2\theta, \quad \sigma_{22} = \sigma+k\sin 2\theta, \quad \sigma_{12} = k\cos 2\theta;$$

k is a material constant, and $k > 0$.

The following theory of plasticity problem is investigated (Fig. 1): it is required to determine the flow of an ideal plastic material in a die with friction — i.e., it is required to determine the plasticity region ALM and the solution of the system (1.1) defined in ALM and satisfying the following conditions:

1. on the contour of the die $\widetilde{LM}$

$$u\sin\varphi-v\cos\varphi = 0, \quad \varphi = \theta+\delta,$$

where φ is the angle between the tangent to the contour and the x-axis, and the angle δ is the friction angle, $0 < \delta \leqslant \pi/4$. The absence of friction means that $\delta = \pi/4$.

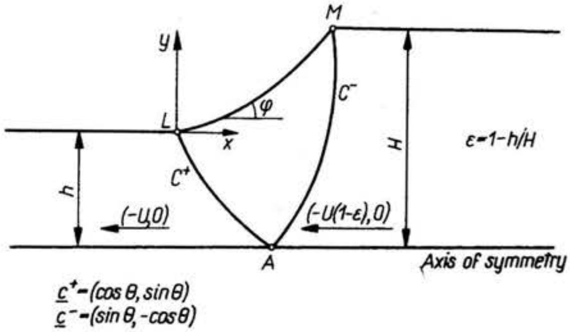

FIG. 1.

2. on the curve AL: $u\sin\theta - v\cos\theta = -U\sin\theta$, $U > 0$,

3. on the curve AM: $u\cos\theta + v\sin\theta = -U(1-\varepsilon)\cos\theta$, where ε is the reduction coefficient, $\varepsilon = 1-h/H$, and $(-U, 0)$, $(-U(1-\varepsilon), 0)$ are the velocity vectors before and after the plasticity region, respectively.

4. at point A: $\theta(A) = -\pi/4$. The curves AL and AM are the characteristic curves C^+ and C^-, respectively, tangent to the vectors $\mathbf{c}^+$ and $\mathbf{c}^-$; they are also the slip lines.

5. the solution should ensure a positive energy dissipation—i.e.,

$$\lambda(x, y) \equiv \frac{1}{2}(u_{,y}+v_{,x})/\cos 2\theta = v_{,y}/\sin 2\theta = -u_{,x}/\sin 2\theta \geqslant 0$$

in the region ALM.

Observe that if we confine ourselves to conditions 1–4, then we arrive at precisely the statement of Problem 1 in [2]. The physical meaning of conditions 1, 2, 3 and 5 is explained in, e.g., [4, 5, 6]. The problem stated above has already been examined many times, for example in [7–10]. However, problems such as the mathematical correctness of the methods of solution, existence and uniqueness of the solutions and their complete classification, are still not quite clear. Therefore, we present here a method of solution by means of Riemann invariants; this makes it possible to treat the above problems.

The aim of the paper is:

to present a method of constructing all solutions in the form of double waves by means of Riemann invariants [2, 1], both in the presence of friction on the die $\left(\delta \neq \frac{\pi}{4}\right)$, and in its absence $\left(\delta = \frac{\pi}{4}\right)$,

to carry out a complete classification of solutions satisfying the conditions 1–4, on the basis of the above method,

to prove that for a given die, for fixed ε, U, δ, there exists only one solution of the problem.

In the course of the solution, condition 5 is not taken into account, and in constructing solutions we confine ourselves to those corresponding to a profile of die with a constant sign of the curvature.

We do not consider in this paper the problem of appearence inside the plasticity region of what is known as the "gradient catastrophe" (derivatives of the solution become infinite, see [11]). We assume in deriving our solution that the boundary problems considered do not lead to the gradient catastrophe inside the plasticity region. The possible occurrence of the above catastrophe inside the plasticity region may be due only to the solution of the system (2.4), and this fact can be established in the course of a numerical construction of the solution. In the case of the method of characteristics, this consists more or less in the fact that in the region of the catastrophe the characteristic net becomes non-homogeneously denser. An example of the gradient catastrophe on the boundary of the plasticity region is provided by points from which there emanate characteristic fans.

Within the above limitation, the successively derived solutions are described in such a way that the analogy of the solution with friction $\delta \neq \pi/4$ and without friction $\delta = \pi/4$ is evident. We do not here consider these analogies, since they are readily observable in constructing the appropriate solutions.

2

The method presented below is based on the fact that the system (1.1) is simple hyperbolic [2]. Thus, we may apply the method of Riemann invariants. Then, instead of (1.1) we solve two systems of equations: one in the hodograph space H^4 of the variables u, v, σ, θ, and the second in the physical space E^2 of the variables x and y. Then it emerges that:

the equations in the space H^4 yield a general rule of determination of all possible non-uniquenesses connected with the change of variables (2.10);

there exists a simple criterion of selection of non-uniqueness in (2.10), leading to solutions satisfying 1–4;

it is easy to prove the uniqueness of the solution for a given tool U, ε, δ;

there is no need to solve the system in H^4 for every die.

We may confine ourselves to a few fixed and the characteristic net is obtained by solving the second system in E^2 with various boundary conditions.

All solutions of the system (1.1) can be derived by a successive solution of two systems of the form [2, 1]:

(2.2) $$u_{,\mu^1} = -\mathrm{tg}\theta v_{,\mu^1}, \quad u_{,\mu^2} = \mathrm{ctg}\theta v_{,\mu^2} \quad \text{[first system (2.2) and (2.3)]},$$

(2.3) $$\sigma_{,\mu^1} = k'\theta_{,\mu^1}, \quad \sigma_{,\mu^2} = -k'\theta_{,\mu^2}, \quad \text{where} \quad k' = 2k;$$

(2.4) $$\mu^1_{,x}\sin\theta - \mu^1_{,y}\cos\theta = 0, \quad \mu^2_{,x}\cos\theta + \mu^2_{,y}\sin\theta = 0 \quad \text{(second system (2.4))}.$$

In (2.4), $\theta = \theta(\mu^1(x, y), \mu^2(x, y))$.

We now proceed to an analysis of the above equations. Introducing new functions $H(\mu^1, \mu^2)$, $G(\mu^1, \mu^2)$ defined by the invertible formulae

(2.5) $$H = u\cos\theta + v\sin\theta, \quad G = v\cos\theta - u\sin\theta,$$

we reduce the Eqs. (2.2) to the form:

(2.6) $$H_{,\mu^1} - \theta_{,\mu^1} G = 0, \quad G_{,\mu^2} + \theta_{,\mu^2} H = 0.$$

If H and G are of class C^2, then differentiating (2.6) and taking into account (2.5) we obtain the system:

(2.7) $$H_{,\mu^1\mu^2} + \theta_{,\mu^1}\theta_{,\mu^2} H = 0, \quad G_{,\mu^1\mu^2} + \theta_{,\mu^1}\theta_{,\mu^2} G = 0.$$

Assume now that the arbitrary functions $g(\mu^1)$ and $f(\mu^2)$ yielding the general solution of the system of the Eqs. (2.3)

$$(2.8)\qquad \begin{aligned}\theta(\mu^1,\mu^2) &= \frac{1}{2}[g(\mu^1)-f(\mu^2)],\\ \sigma(\mu^1,\mu^2) &= \frac{k'}{2}[g(\mu^1)+f(\mu^2)]\end{aligned}$$

are such that

$$(2.9)\qquad \theta_{,\mu^1} \neq 0, \quad \theta_{,\mu^2} \neq 0$$

for $(\mu^1,\mu^2) \in \Omega$, where Ω is a domain in the plane of the variables μ^1 and μ^2. Then we can change the variables:

$$(2.10)\qquad (\mu^1,\mu^2) \to \big(\xi = g(\mu^1)-g(0),\ \eta = f(\mu^2)-f(0)\big).$$

Thus, the systems (2.6) and (2.7) take the form:

$$(2.11)\qquad H_{,\xi}-\frac{1}{2}G = 0, \quad G_{,\eta}-\frac{1}{2}H = 0;$$

$$(2.12)\qquad H_{,\xi\eta}-\frac{1}{4}H = 0, \quad G_{,\xi\eta}-\frac{1}{4}G = 0.$$

Consequently, we have

THEOREM 1. *If $u, v, \sigma, \theta \in C^2(\Omega)$ and the condition* (2.9) *is satisfied, then the system* (2.2) *is equivalent to* (2.12).

Consider now the boundary conditions corresponding to the Eqs. (2.2) and (2.3). Envisage a solution of the system (2.2), (2.3) constituting a two-dimensional variety $\mathcal{G}_2$ in the hodograph space of the variables u, v, σ, θ. If $U: E^2 \to \mathcal{G}_2 \subset H^4$ is the solution of the problem considered and constitutes a double wave, then the geometric interpretation of this solution [2, 1] implies that the mapping U maps the characteristics C^+ and C^- constituting the boundaries of the plasticity region, on to the characteristics Γ^+ and Γ^- in the space H^4:

$$U(C^+) \subset \Gamma^+ \subset \mathcal{G}_2, \quad U(C^-) \subset \Gamma^- \subset \mathcal{G}_2.$$

Since the characteristics Γ^+ and Γ^- are parametrized by the straight lines $\mu^2 = \text{const}$ and $\mu^1 = \text{const}$, respectively, it is evident that we may investigate the part of the variety $\mathcal{G}_2$ parametrized by the rectangle $[0, l] \times [0, m]$ on the plane of the variables μ^1 and μ^2, where $l > 0$ and $m > 0$. The conditions 2, 3 and 4 require that the solution (2.2), (2.3) in the rectangle $[0, l] \times [0, m]$ should satisfy the conditions

a) on the interval $[0, l]$ of the axis μ^1:

$$(2.13)\qquad u(\mu^1,0)\sin\theta(\mu^1,0)-v(\mu^1,0)\cos\theta(\mu^1,0) = -U\sin\theta(\mu^1,0)$$

b) on the interval $[0, m]$ of the μ^2-axis:

$$(2.14)\qquad u(0,\mu^2)\cos\theta(0,\mu^2)+v(0,\mu^2)\sin\theta(0,\mu^2) = -U(1-\varepsilon)\cos(0,\mu^2);$$

c)

$$(2.15)\qquad \theta(0,0) = -\frac{\pi}{4}.$$

Simple calculations prove that the solution of the system (2.2), (2.3) satisfying the conditions a, b and c, has the properties:

$$
\begin{aligned}
&\theta^+(\mu^1) \equiv \theta(\mu^1, 0) = \frac{1}{2}[g(\mu^1)-f(0)],\\
&v^+(\mu^1) \equiv v(\mu^1, 0) = C_2 \sin\theta^+(\mu^1),\\
&u^+(\mu^1) \equiv u(\mu^1, 0) = -U + C_2\cos\theta^+(\mu^1),\\
&\sigma^+(\mu^1) \equiv \sigma(\mu^1, 0) = k' \cdot [g(\mu^1)+f(0)]/2,\\
&\theta^-(\mu^2) \equiv \theta(0, \mu^2) = \frac{1}{2}[g(0)-f(\mu^2)],\\
&v^-(\mu^2) \equiv v(0, \mu^2) = C_1\cos\theta^-(\mu^2),\\
&u^-(\mu^2) \equiv u(0, \mu^2) = -U(1-\varepsilon) - C_1\sin\theta^-(\mu^2),\\
&\sigma^-(\mu^2) \equiv \sigma(0, \mu^2) = \frac{k'}{2}[g(0)+f(\mu^2)],\\
&\frac{1}{2}[g(0)-f(0)] = -\frac{\pi}{4},
\end{aligned}
\tag{2.16}
$$

where g and f are arbitrary functions of class C^1, and C_1 and C_2 are arbitrary constants, $0 \leqslant \mu^1 \leqslant l$, $0 \leqslant \mu^2 \leqslant m'$. We now require that $v^+(0) = v^-(0)$ and $u^+(0) = u^-(0)$; hence,

$$
-C_1 = C_2 = (U-V)\frac{\sqrt{2}}{2}, \quad V = (1-\varepsilon)U. \tag{2.17}
$$

The conditions (2.16) and (2.17) constitute the boundary conditions for the Eqs. (2.2), (2.3), for which there exists only one solution of class C^2 in the rectangle $[0, l] \times [0, m]$. Let $U: E^2 \to H^4$ be the solution of the problem considered in the general case $\delta \neq \pi/4$, which constitutes a regular double wave in the closure of the region ALM [2]. Then $U(\widetilde{LM})$ should be a curve on the manifold, connecting the points

$$
\big(u(l, 0), v(l, 0), \sigma(l, 0), \theta(l, 0)\big) \quad \text{and} \quad \big(u(0, m), v(0, m). \sigma(0, m), \theta(0, m)\big)
$$

along which the following condition, implied by 1, should be satisfied:

$$
u(\mu^1, \mu^2)\sin(\theta(\mu^1, \mu^2)+\delta) - v(\mu^1, \mu^2)\cos(\theta(\mu^1, \mu^2)+\delta) = 0;
$$

here, u, v, σ, θ are the solutions of (2.2), (2.3), (2.16), (2.17). However, the above equations define uniquely the variety $\mathscr{G}_2$ and the requirement of existence of this curve is an additional requirement, which for arbitrary functions f and g in (2.16) is not necessarily satisfied.

The above considerations prove that the determination of the solution of the problem stated in the introduction depends on the choice of the set $l_\delta = \mathscr{G}_2 \cap R_3^\delta$, where $\mathscr{G}_2$ is the variety defined by (2.2), (2.3), (2.16) and (2.17), and R_3^δ is a three-dimensional variety defined by the formula:

$$
R_3^\delta = \{(u, v, \sigma, \theta) \in H^4 : u\sin(\theta+\delta) - v\cos(\theta+\delta) = 0\}.
$$

The determination of the set $l_\delta = \mathcal{G}_2 \cap R_3^\delta$ is equivalent to determination of a set τ_δ defined as follows:

$$\tau_\delta = \{(\mu^1, \mu^2): (\mu^1, \mu^2) \in [0, l] \times [0, m] \quad \text{and} \quad (u(\mu^1, \mu^2), \ldots, \theta(\mu^1, \mu^2)) \in l_\delta\}. \tag{2.18}$$

Here, u, v, σ, θ are the solutions of (2.2), (2.3), (2.16), (2.17). To find the set τ_δ, we make use of Theorem 1. Making use of the new variables (2.10), we reduce the conditions (2.16) to the following:

$$\begin{aligned} H^+ &= H(\xi, 0) = C_2 - U\cos\left(\frac{1}{2}\xi - \frac{\pi}{4}\right), \\ H^- &= H(0, \eta) = -V\cos\left(\frac{1}{2}\eta + \frac{\pi}{4}\right), \\ G^+ &= G(\xi, 0) \doteq U\sin\left(\frac{1}{2}\xi - \frac{\pi}{4}\right), \\ G^- &= G(0, \eta) = C_1 - V\sin\left(\frac{1}{2}\eta + \frac{\pi}{4}\right), \end{aligned} \tag{2.19}$$

where ξ belongs to an interval with ends 0 and $f(l) - f(0)$, and η to an interval with ends 0 and $g(m) - g(0)$; the constants C_1 and C_2 satisfy (2.17) and $f_{,\mu^2} \neq 0$ for $0 \leqslant \mu^2 \leqslant m$; finally, $g_{,\mu^1} \neq 0$ for $0 \leqslant \mu^1 \leqslant l$.

The change (2.10) maps the rectangle $[0, l] \times [0, m]$ onto the rectangle

$$B = [0, f(l) - f(0)] \times [0, g(m) - g(0)]$$

on the plane of the variables ξ, η, and the set τ_δ onto the set

$$\tau'_\delta = \{(\xi, \eta): (\xi, \eta) \in B \quad \text{and} \quad H\sin\delta - G\cos\delta = 0\}. \tag{2.20}$$

Now, we are in a position to state

THEOREM 2. *The problem* (2.12), (2.16), (2.17) *has a unique solution in the rectangle* B *of class* C^2, *given by the formulae*:

$$\begin{aligned} H(\xi, \eta) = -J_0\left(\sqrt{-\xi\eta}\right)V\frac{\sqrt{2}}{2} + \frac{U}{2}\int_0^\xi J_0\left(\sqrt{-(\xi - t)\eta}\right) * \\ * \sin\left(\frac{1}{2}t - \frac{\pi}{4}\right)dt + \frac{V}{2}\int_0^\eta J_0\left(\sqrt{-\xi(\eta - t)}\right)\sin\left(\frac{1}{2}t + \frac{\pi}{4}\right)dt, \end{aligned} \tag{2.21}$$

$$\begin{aligned} G(\xi, \eta) = -J_0\left(\sqrt{-\xi\eta}\right)U\frac{\sqrt{2}}{2} + \frac{U}{2}\int_0^\xi J_0\left(\sqrt{-(\xi - t)\eta}\right) * \\ * \cos\left(\frac{1}{2}t - \frac{\pi}{4}\right)dt - \frac{V}{2}\int_0^\eta J_0\left(\sqrt{-\xi(\eta - t)}\right)\cos\left(\frac{1}{2}t + \frac{\pi}{4}\right)dt. \end{aligned} \tag{2.22}$$

If H and G are defined in B by (2.21) *and* (2.22), *then the set* τ'_δ *consists of* $(\xi, \eta) \in B$, *for which*

$$(2.23)\qquad F^\delta_\varepsilon(\xi,\eta) = 2\sin\big((\xi-\eta)/2+\delta-\pi/4\big) - \frac{\varepsilon}{2}\Big\{\sqrt{2}\sin\delta\, J_0\big(\sqrt{-\xi\eta}\big) - \int\limits_0^\eta J_0\big(\sqrt{-\xi(\eta-t)}\big)\cos\Big(\frac{t}{2}+\frac{\pi}{4}-\delta\Big)dt\Big\} = 0,$$

and of the point $(\xi,\eta) = (0,0)$. J_0 *is the Bessel function of zero order,* $J_0(0) = 1$.

For physical reasons (Fig. 1), *we have* $\varphi(L) \geqslant 0$ *and* $\varphi(M) \leqslant \pi/2$. *In other words,*

$$(2.24)\qquad 0 \leqslant \frac{1}{2}(\xi-\eta)-\frac{\pi}{4}+\delta \leqslant \frac{\pi}{2}.$$

The relation (2.23) was examined numerically by the author for various values of δ and various ε in the region (2.24); the numerical results are in part identical with numerical results obtained by other authors—e.g. [5]. They are represented in Figs. (2a–2h).

(2.23) implies that $F^\delta_\varepsilon(\xi, 0) = 0$ for $\xi = \xi^\delta_\varepsilon$, where

$$(2.25)\qquad \sin\Big(\xi/2+\delta-\frac{\pi}{4}\Big) = \varepsilon\sqrt{2}\sin\delta/2,$$

and $F^\delta_\varepsilon(0,\eta) = 0$ for $\eta = \eta^\delta_\varepsilon$, where

$$(2.26)\qquad \sin\Big(\eta^\delta_\varepsilon/2+\frac{\pi}{4}-\delta\Big) = \frac{\varepsilon}{\varepsilon-1}\Big(\frac{\sqrt{2}}{2}\sin\delta+\sin\Big(\frac{\pi}{4}-\delta\Big)\Big)$$

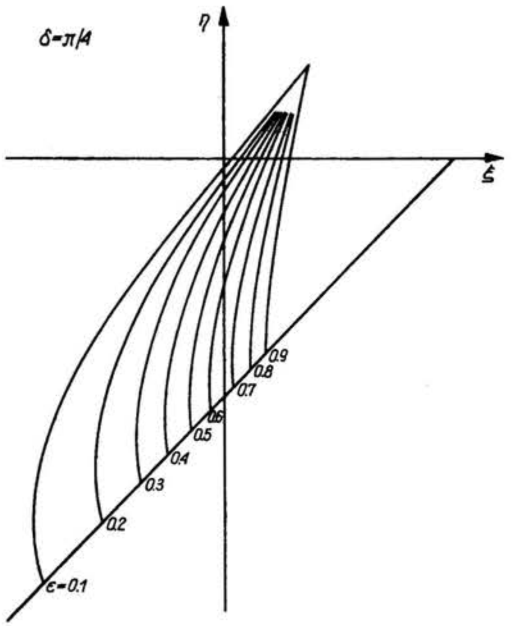

FIG. 2a.

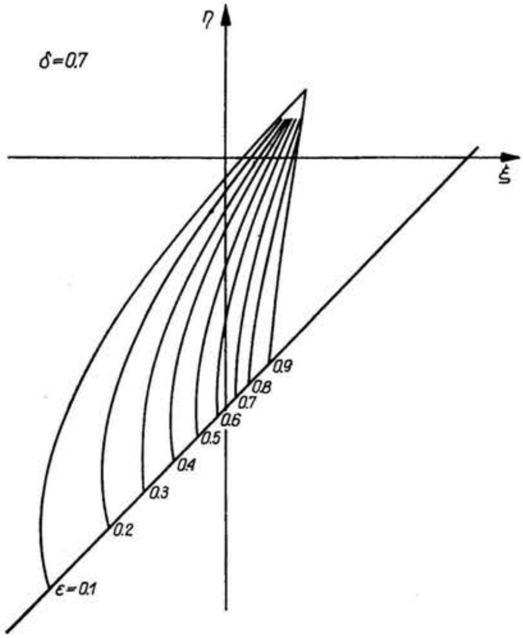

FIG. 2b.

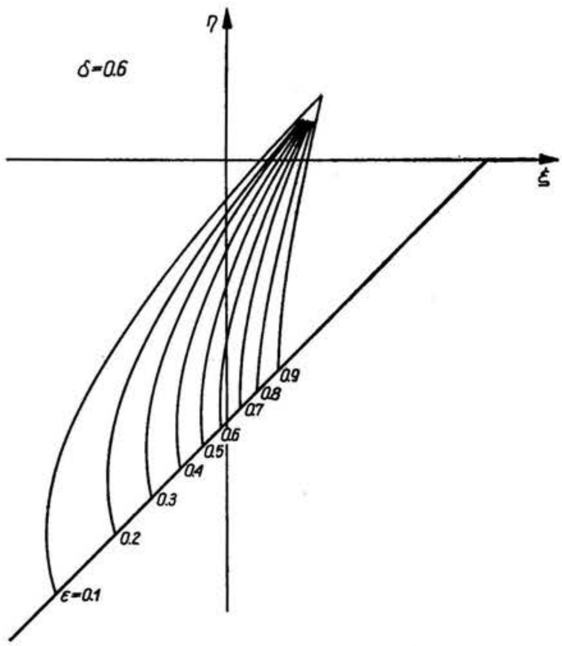

FIG. 2c.

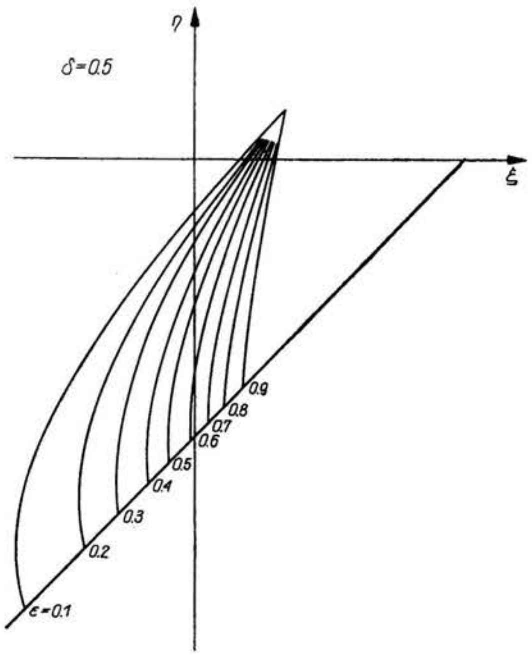

FIG. 2d.

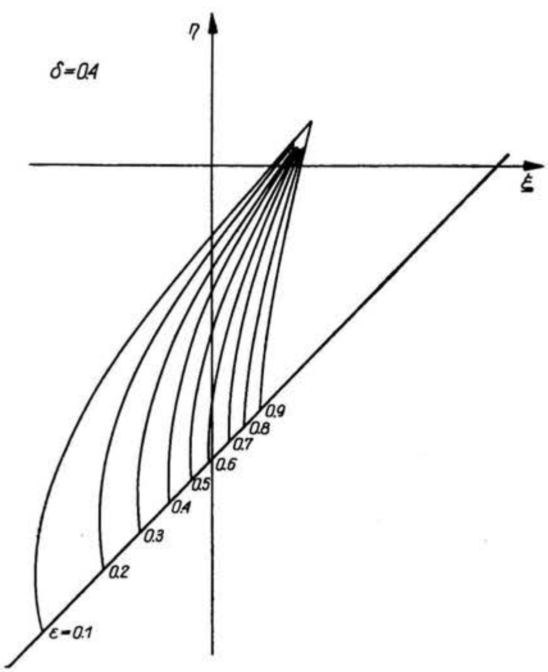

FIG. 2e.

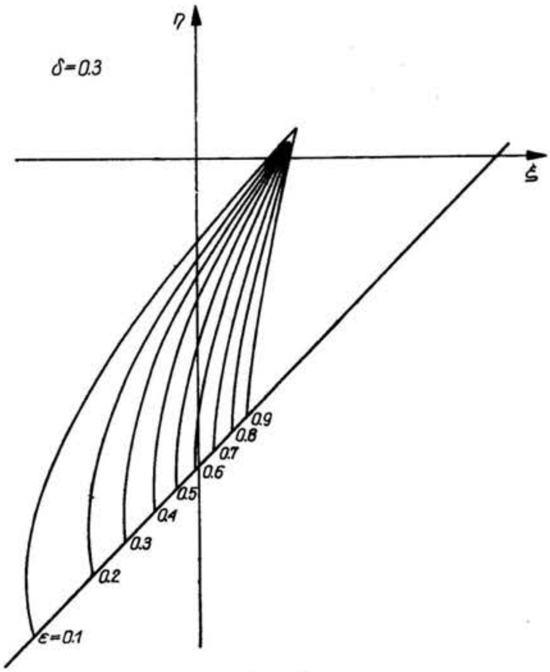

FIG. 2f.

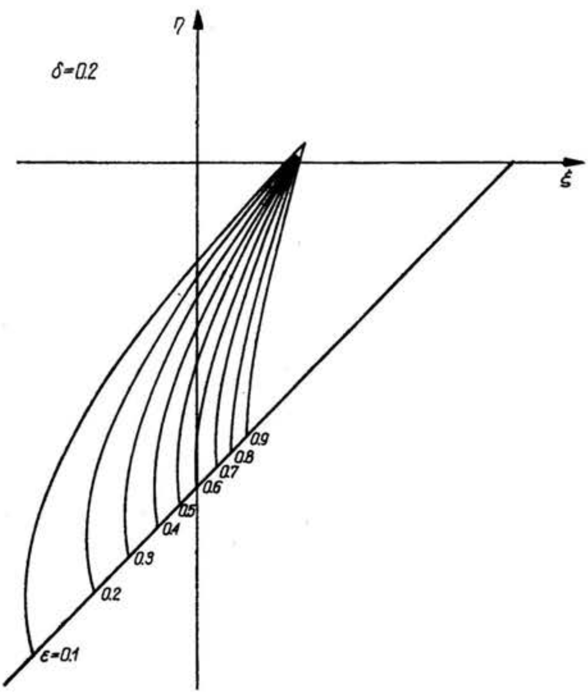

FIG. 2g.

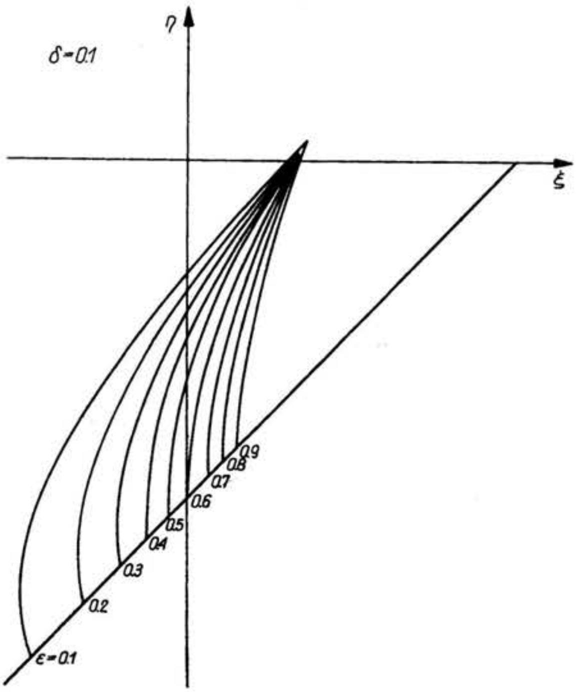

FIG. 2h.

3

Consider now the simplest case of constructing solutions of the problem considered. The results of Sec. 2 imply

COROLLARY 1. *If*

a) u, v, f, g *are of class* C^2, $0 < \delta \leqslant \pi/4$, $\delta = \text{const}$,

b) (2.9) *holds for* $(\mu^1, \mu^2) \in [0, l] \times [0, m]$,

c) $\xi(\mu^1) \nearrow$, $\quad \eta(\mu^2) \searrow$,

$\xi(l) = \xi^\delta_\varepsilon$, $\quad \eta(m) = \eta^\delta_\varepsilon$,

d) ε *is such that the Eq.* (2.26) *has solutions; then the set* τ_δ *defines a non-characteristic curve with ends at the points* $(0, m)$ *and* $(l, 0)$, *connecting these two points and situated wholly in tho rectangle* $[0, l] \times [0, m]$. *The function describing this curve* $\mu^2 = \psi_1(\mu^1)$, $0 \leqslant \mu^1 \leqslant l$ *is axactly decreasing* (Fig. 3a).

COROLLARY 2. *Let*

$$U_1(\mu^1, \mu^2) = (u(\mu^1, \mu^2), \ldots, \theta(\mu^1, \mu^2))$$

be a solution of the system (2.2), (2.3) *with the boundary conditions* (2.16), (2.17), *and assume that f and g satisfy the assumptions of Corollary* 1 *and assumption* d) *of Corollary* 1. *Then there exists the solution U of the system* (1.1) *with the hodograph defined by* U_1. *In other words, there exists a solution of the form*:

$$U(x, y) = U_1\big(\mu^1(x, y), \mu^2(x, y)\big),$$

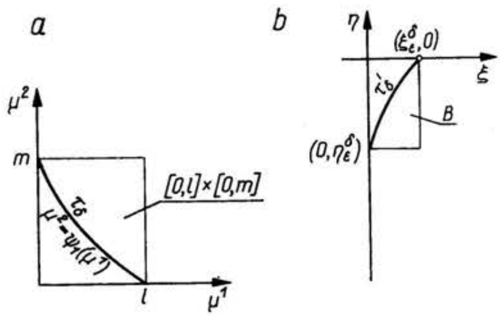

FIG. 3.

where $\mu^1(x,y)$, $\mu^2(x,y)$ give the solution of the system (2.4) *which satisfies the conditions* 1–4 *and constitutes a regular double wave in the closure of the region ALM, including the points L and M.*

This result can be derived as follows. On the plane E^2, we prescribe the curve $\varkappa$ (the contour of the die) by means of the function:

$$\varkappa = (x(\varphi), y(\varphi)), \quad \varphi_L \leqslant \varphi \leqslant \varphi_M,$$

$$\varphi_L = \frac{1}{2}[g(l)-f(0)]+\delta-\frac{\pi}{4}, \tag{3.1}$$

$$\varphi_M = \frac{1}{2}[g(0)+f(m)]+\delta-\frac{\pi}{4},$$

(δ — the friction angle)

$$\frac{dx}{d\varphi} = \omega(\varphi), \quad \frac{dy}{d\varphi} = \omega(\varphi)\operatorname{tg}\varphi, \tag{3.2}$$

where $\omega \in C^1$, $\omega(\varphi) > 0$, $\dot{\omega}(\varphi) > 0$. It follows from (3.2) that the contour of the die is uniquely determined by $\omega(\varphi)$, $\varphi_L \leqslant \varphi \leqslant \varphi_M$ and, conversely, the contour of the tool determines uniquely $\omega(\varphi)$.

Let us now formulate the initial problem for the system (2.4):

$$\mu^1(x(\varphi), y(\varphi)) = a_1, \quad \mu^2(x(\varphi), y(\varphi)) = a_2,$$

$$(a_1, a_2) \in \tau_\delta, \quad \frac{1}{2}[g(a_1)-f(a_2)]+\delta-\frac{\pi}{4} = \varphi. \tag{3.3}$$

The formulae (3.3) associate uniquely with every point of the curve $\varkappa$ a value of the functions μ^1 and μ^2 at this point, since along τ_δ the parameter φ varies monotonically and to every value φ on τ_δ there corresponds only one point (a_1, a_2); see Figs. 3a, 3b, 4a, 4b.

The Cauchy problem (2.4), (3.3) has a unique solution of class C^1: $\mu^1(x,y)$, $\mu^2(x,y)$ defined for (x,y) belonging to the closure of the region bounded by the curve $\varkappa$ and the characteristics C^+ and C^- emanating from the points L and M. The solution $U(x,y)$ of the problem considered is defined as follows:

$$U(x,y) = U_1(\mu^1, \mu^2), \quad \mu^1 = \mu^1(x,y), \mu^2 = \mu^2(x,y).$$

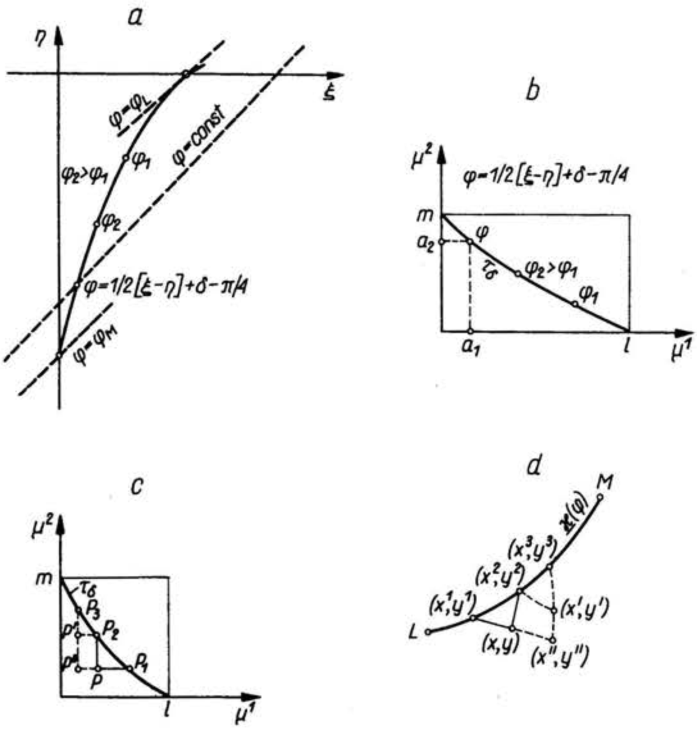

FIG. 4.

A method of determination of the slip lines consists in applying the method of characteristics [2, 1]. Thus we proceed as follows: we seek the solution in the form already presented; since U_1 is known, it is sufficient to determine the function $\mu(x, y) = (\mu^1(x, y), \mu^2(x, y))$. To this end, we introduce in the rectangle $[0, l] \times [0, m]$ a characteristic net, so that on the curve τ_δ the points $P_1, P_2, \ldots$ are chosen, and two characteristics drawn through them (Fig. 4c). If the values of φ at points P_1, P_2 are φ_1 and φ_2, respectively, then on the basis of (3.2) we find on the curve points (x^1, y^1) and (x^2, y^2) such that (see Fig. 4d)

$$(x^1, y^1) = \varkappa(\varphi_1), \quad (x^2, y^2) = \varkappa(\varphi_2).$$

For these points, we assume that

$$\mu(x^1, y^1) = P_1, \quad \mu(x^2, y^2) = P_2.$$

We now demonstrate the method of determining points such that

$$\mu(x, y) = P,$$
$$\mu(x', y') = P', \quad \mu(x'', y'') = P''$$

(Figs. 4c, 4d): we solve the system of algebraic equations

$$(3.4) \qquad \begin{aligned} &(x-x^1)\sin\theta(P) - (y-y^1)\cos\theta(P) = 0, \\ &(x-x^2)\cos\theta(P) + (y-y^2)\sin\theta(P) = 0, \end{aligned}$$

for (x, y), where $\theta(P)$ is the value of the function θ at point P, obtained by solving (2.2), (2.3), (2.16), (2.17). The solution (x, y) of this system yields a new point of the plastic zone and the values of the functions μ^1 and μ^2 at this point are assumed to be equal to the coordinates of P. To find the points (x', y') and (x'', y'') we replace in (3.4) successively (x^1, y^1) by (x^2, y^2) and (x^2, y^2) by (x^3, y^3) to construct point (x', y'), and (x^1, y^1) by (x, y), (x^2, y^2) by (x', y'). Applying this method step by step, we obtain the plasticity zone and the values of the function μ, and therefore also the solution U.

In the course of the numerical determination of the solution it can be verified that the additional condition

$$j = \frac{\partial(\mu^1, \mu^2)}{\partial(x, y)} \neq 0$$

is satisfied for (x, y) for which the solution is determined, since the procedure described above can be pursued to the end for all points of the characteristic net. The solution μ^1, μ^2 of the problem (2.4), (3.3) establishes a one to one correspondence between the points (x, y), for which the solution is defined, and the points (μ^1, μ^2) belonging to the set defined by the curve τ_δ and the axes μ^1 and μ^2.

We shall now present a modified solution, the difference being that a double wave is not a regular solution at point M.

COROLLARY 3. *Let*

$$U_1' = U_1 |_{[0, l'] \times [0, m]}$$

be a restriction of the solution in Corollary 2 *to the rectangle* $[0, l'] \times [0, m]$, *where* $0 < l' < l$, *and*

$$\varkappa^1(\varphi) = \big(x(\varphi), y(\varphi)\big)$$

$$\frac{1}{2}[g(l') - f(m')] + \delta - \frac{\pi}{4} \leqslant \varphi \leqslant \frac{1}{2}[g(0) - f(m)] + \delta - \frac{\pi}{4} \tag{3.5}$$

$$m' = \psi_1(l')$$

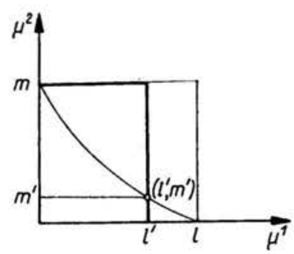

FIG. 5.

a curve on the plane E^2, *satisfying* (3.2) (Fig. 5). *Then there exists a solution* (1.1) *satisfying the conditions* 1–4, *defined by*

$$U(x, y) = U_1'\big(\mu^1(x, y), \mu^2(x, y)\big)$$

$\mu^1(x, y)$, $\mu^2(x, y)$ *is a solution of* (2.4) *which outside point* M *is everywhere a regular double wave.*

This solution has a characteristic fan C^+ *emanating from point* L.

We seek the solution in the form:

$$U(x, y) = U_1'\big(\mu^1(x, y), \mu^2(x, y)\big).$$

The function $\mu = \big(\mu^1(x, y), \mu^2(x, y)\big)$ is first defined in the region G_1 and simultaneously we construct this region. This is effected by means of the method of characteristics in such a way that $\mu(G_1) = M_1$ (see Figs. 6a, 6b) and $U(x, y)$ for $(x, y) \in G_1$ is a solution. Next, we define $\mu(x, y)$ in G_2 in a similar manner, constructing G_2 in such a way that

$$\mu(G_2) = M_2 \quad \text{and} \quad U(x, y) \quad \text{for} \quad (x, y) \in G_2$$

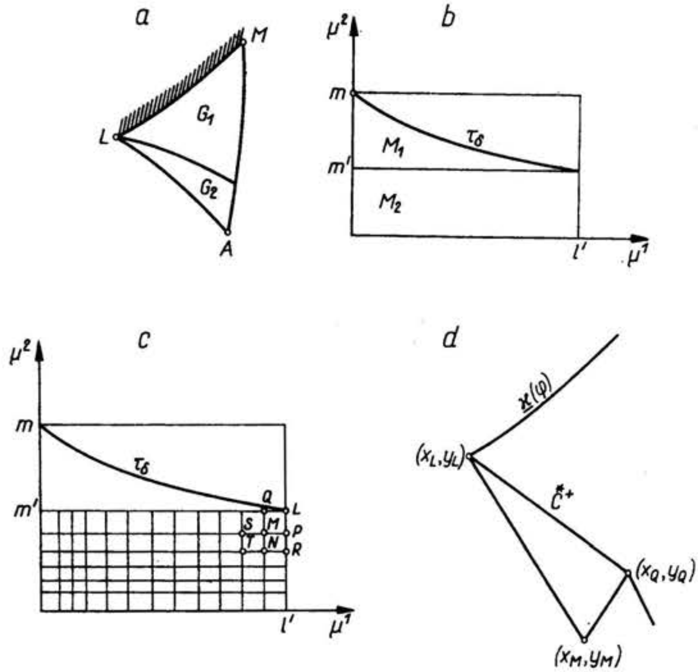

FIG. 6.

is a solution. The construction of μ in G_2 is as follows: as in (3.3) we state the Cauchy problem for (2.4)

$$\begin{gathered}\mu^1\big(\varkappa^1(\varphi)\big) = a_1, \quad \mu^2\big(\varkappa^1(\varphi)\big) = a_2, \\ (a_1, a_2) \in \tau_\delta, \quad \frac{1}{2}[g(a_1) - f(a_2)] + \delta - \frac{\pi}{4} = \varphi;\end{gathered} \tag{3.6}$$

φ belongs to the interval defined in (3.5). The Cauchy problem (2.4), (3.6) has again a unique solution (this follows from the same reasoning as in Corollary 2). The region G_1 in which it is defined, and $U(x, y)$ for $(x, y) \in G_1$, are deduced by a successive application of (3.4)—see Corollary 2. Next, we introduce at M_2 a characteristic net (Fig. 6c). Now we shall demonstrate the construction of the characteristic fan C^+—i.e. the points (see Fig. 6d) $(x_N, y_N), (x_M, y_M) \in G_2$ such that $\mu(x_N, y_N) = N$, $\mu(x_M, y_M) = M$. Since $U(x, y)$ for $(x, y) \in G_1$ is already known, we know also (x_Q, y_Q) such that

$$\mu(x_Q, y_Q) = Q$$

and (x_L, y_L) such that

$$\mu(x_L, y_L) = L.$$

Solving the algebraic system

$$(3.7)\qquad \begin{aligned}(x-x_L)\sin\theta(M)-(y-y_L)\cos\theta(M) &= 0,\\ (x-x_Q)\cos\theta(M)+(y-y_Q)\sin\theta(M) &= 0,\end{aligned}$$

we obtain the coordinates (x, y) of the point M. The point (x_N, y_N) is found by replacing in (3.7) $\theta(M)$ by $\theta(N)$ and (x_Q, y_Q) by (x_M, y_M), etc. The remaining points (x_S, y_S), (x_T, y_T) are deduced by a successive application of (3.4). Thus we obtain the characteristic fan G_2 and $U(x, y)$ defined for $(x, y) \in G_2$. If now

$$(3.8)\qquad \bar{G}_1 \cap \bar{G}_2 = \overset{*}{C}{}^+,$$

where $\overset{*}{C}{}^+$ is the characteristic C^+ obtained in constructing G_1 such that

$$U(\overset{*}{C}{}^+) \subset U_1'\big([(0, m'), (l', m')]\big),$$

where $[(0, m'), (l', m')]$ is the segment on the plane of the variables μ^1, μ^2, connecting the points $(0, m')$, (l', m'), then the construction described above yields the solution of (1.1), which satisfies the conditions 1–4.

The fact that the condition (3.8) is satisfied follows from the fact that a) holds in a sufficiently small vicinity of point L, in view of the assumption on the functions f and g, and b) the solution is of class C^1 (an additional assumption). Moreover, for a given die δ, ε, V the characteristic $\overset{*}{C}{}^+$ is determined uniquely and this implies the uniqueness of the solution $U(x, y)$ for $(x, y) \in G_2$. Thus $U(x, y)$ is determined uniquely for $(x, y) \in G_1 \cup G_2$.

In the course of the numerical solution, it can also be verified that

$$j = \frac{\partial(\mu^1, \mu^2)}{\partial(x, y)} \neq 0 \quad \text{for} \quad (x, y) \in G_1 \cup G_2 \setminus (x_L, y_L).$$

A further modification of the solution of Corollary 3 would consist in a solution with two characteristic fans C^+ and C^-. Then, instead of U we should consider:

$$U_1'' = U_1|_{[0, l'] \times [0, m']}, \quad 0 < m' < m.$$

Unfortunately, the assumptions concerning f and g lead to the conclusion that a counterpart of the condition (3.8) for the characteristic fan C^- does not hold, and this means that such solutions do not exist. The assumptions concerning f and g, and the above remark imply also that the above method cannot yield solutions of the problem considered if the following assumptions are satisfied:

a) f and g are of class C^2, $0 < \delta \leqslant \pi/4$, $\delta = \text{const}$,

b) (2.9) holds for $(\mu^1, \mu^2) \in [0, l] \times [0, m]$,

c) $\xi(\mu')\nearrow$, $\xi(l) = \xi^\delta_\varepsilon$,

d) ε is such that the Eq. (2.26) has no solutions. Solutions of the type appearing in Corollaries 2 and 3 are presented in Figs. 7 and 8.

The results of Sec. 2 lead to

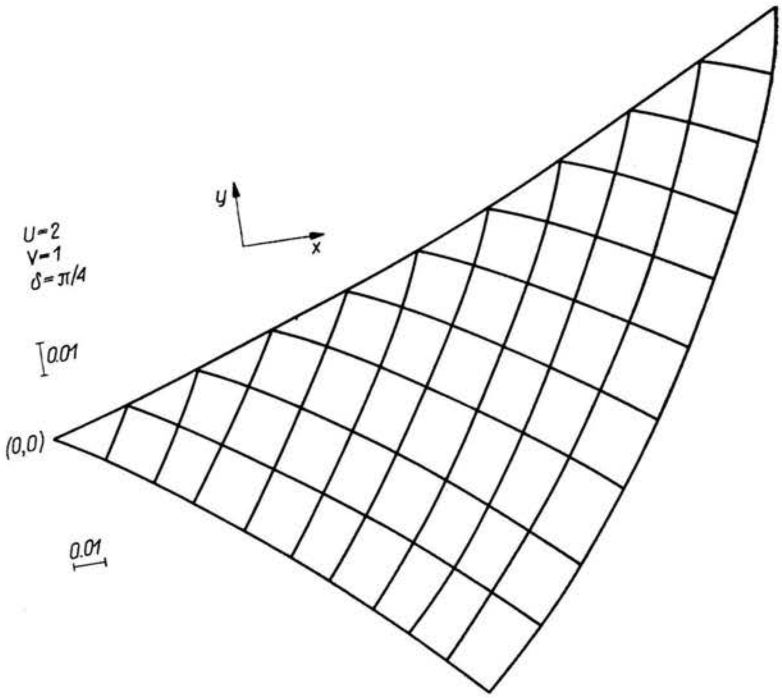

FIG. 7.

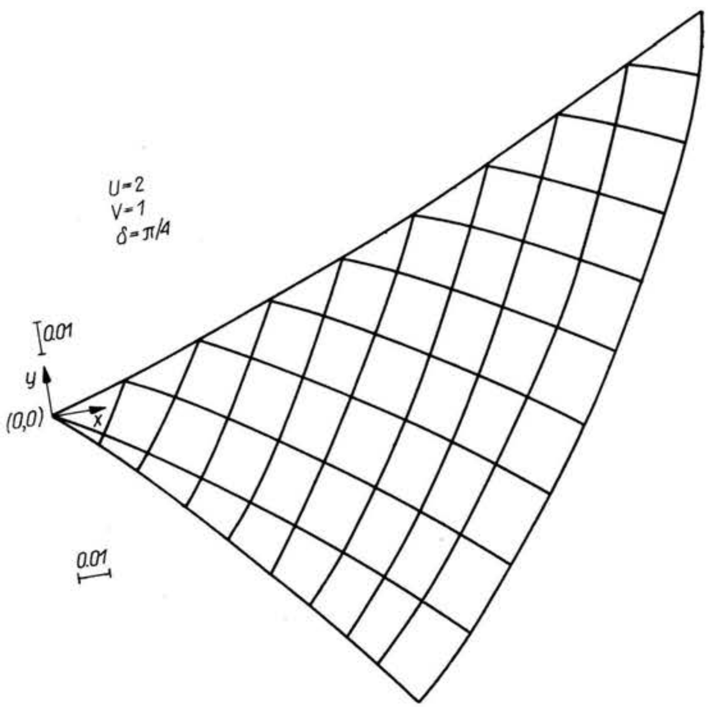

FIG. 8.

4

COROLLARY 4. *If*

a) u, v, f, g *are of class* C^2, $0 < \delta \leqslant \pi/4$, $\delta = \text{const}$,

b) (2.9) *holds for* $(\mu^1, \mu^2) \in [0, l] \times [0, m]$,

c) $\xi(\mu^1)\searrow$, $\eta(\mu^2)\searrow$, $F_\varepsilon^\delta(\xi(l), \eta(m)) = 0$,

$\eta(m') = \eta_\varepsilon^\delta$, $0 < m' < m$,

d) *the segment considered of the curve* F_ε^δ *is not tangent to the straight line* $\xi = \text{const}$.

e) ε *is such that the Eq.* (2.26) *has a solution, then there exists a non-characteristic curve* τ_δ *with ends at the points* $(0, m')$ *and* (l, m), *situated wholly in* $[0, l] \times [0, m]$. *The function* $\mu^2 = \psi_2(\mu^1)$ *describing this curve is exactly increasing* (Fig. 9a).

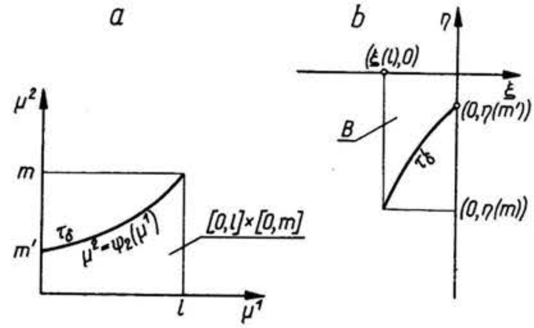

FIG. 9.

Consider now the method of constructing the solution of the system (1.1), satisfying the conditions 1–4, for which the variety $\mathscr{G}_2$ is determined by the solution of (2.2), (2.3), (2.16), (2.17) and the assumptions of Corollary 4 are satisfied. A derivation of this solution as in Sec. 3—i.e., satisfying the condition (2.9)—is impossible, in view of the behaviour of the curve τ_δ—see Corollary 4. Thus, if there exists such a solution, then it does not satisfy (2.9). We shall prove below that this is true; now we proceed to examine a certain property of the system (2.2), (2.3).

Observe that if $U_1(\mu^1, \mu^2) = \big(u(\mu^1, \mu^2), v(\mu^1, \mu^2), \sigma(\mu^1, \mu^2), \theta(\mu^1, \mu^2)\big)$ is a solution of the system (2.2), (2.3), which does not have to satisfy (2.9) and belongs to class C^2, then $U_1\big(\omega_1(\mu^1), \omega_2(\mu^2)\big)$ is also a solution of the system (2.2), (2.3) for arbitrary functions ω_1, ω_2 of class C^1 [2].

Assume that we know the solution U_1 of class C^1 of the system (2.2), (2.3), defined in the rectangle $P_1 = [0, l] \times [0, m]$, which satisfies the conditions (in the sense of one-sided derivatives):

$$\begin{aligned} &\theta_{,\mu^1}(l, \mu^2) = 0, \qquad 0 \leqslant \mu^2 \leqslant m,\\ &\theta_{,\mu^2}(\mu^1, m) = 0, \qquad 0 \leqslant \mu^1 \leqslant l, \end{aligned} \tag{4.1}$$

i.e., $dg(l)/d\mu^1 = df(m)/d\mu^2$. Then the function $U_1^N(\mu^1, \mu^2)$ defined for $(\mu^1, \mu^2) \in P'$, $P' = [0, 2l] \times [0, 2m]$ by the formulae (Fig. 10)

$$U_1^N(\mu^1, \mu^2) = \begin{cases} U_1(\mu^1, \mu^2) & \text{for} \quad (\mu^1, \mu^2) \in [0, l] \times [0, m] = P,\\ U_1(\mu^1, 2m-\mu^2) & \text{for} \quad (\mu^1, \mu^2) \in [0, l] \times [m, 2m] = P_2,\\ U_1(2l-\mu^1, \mu^2) & \text{for} \quad (\mu^1, \mu^2) \in [l, 2l] \times [0, m] = P_1,\\ U_1(2l-\mu^1, 2m-\mu^2) & \text{for} \quad (\mu^1, \mu^2) \in [l, 2l] \times [m, 2m] = P_3 \end{cases}$$

is a solution of class C^1 of (2.2), (2.3) in the rectangle P', and the assumption (2.9) is not satisfied on the two characteristics of the system (2.2), (2.3) emanating from the points $(l, 0)$ and $(0, m)$. Let us apply this result to the solution U_1 of the system (2.2), (2.3) satisfying (2.16), (2.17), assumptions of Corollary 4 and the conditions (4.1). Then in every rectangle P_1, P_2, P_3 there exists a curve satisfying (2.18), created by a symmetric reflection of the curve τ_δ from the rectangle P (Fig. 10). The assumption (2.9) is not satisfied only on the two characteristics of the system (2.2), (2.3) emanating from the points $(l, 0)$

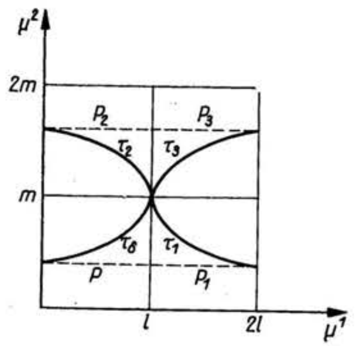

FIG. 10.

and $(0, m)$. We shall now prove that there exists a solution of the system (1.1) satisfying the conditions 1–4 with hodographs determined by the following solutions of the system (2.2), (2.3):

(4.2) $$U' = U_1^N|_{[0,2l]\times[0,m]},$$

(4.3) $$U'' = U_1^N|_{[0,l]\times[0,2m]}.$$

Let τ_1 denote the set $(\mu^1, \mu^2)\in P_1$ for which (2.18) holds in the case of the solution (4.2) and τ_2 the set of these $(\mu^1, \mu^2)\in P_2$ for which (2.18) holds in the case of the solution (4.3). Assume that we are faced with the case (4.2). On the plane E^2 we prescribe the curve $\varkappa^1$ (the contour of the die) by means of the function:

(4.4) $$\varkappa^1 = (x(\varphi), y(\varphi)) \qquad \varphi_L \leqslant \varphi \leqslant \varphi_M,$$
$$\varphi_L = \frac{1}{2}[g(2l)-f(m')]-\frac{\pi}{4}+\delta,$$
$$\varphi_M = \frac{1}{2}[g(l)-f(m)]-\frac{\pi}{4}+\delta;$$

(4.5) $$\frac{dx}{d\varphi} = \omega(\varphi), \qquad \frac{dy}{d\varphi} = \omega(\varphi)\operatorname{tg}\varphi,$$

where $\omega\in C^1$, $\omega(\varphi) > 0$, $\dot\omega(\varphi) > 0$. It follows from (4.4) and (4.5) that the contour of the die is convex. We again seek the solution in the form:

$$U(x, y) = U'\left(\mu^1(x, y), \mu^2(x, y)\right)$$

(see Figs. 11a, 11c). The function $\mu(x,y) = (\mu^1(x,y), \mu^2(x,y))$ is first defined in the region D_1; we construct also at the same time this region. This is effected by means of the method of characteristics, so that $\mu(D_1) = M_1$ and $U(x,y)$ for $(x,y) \in D_1$ is a solution of the system (1.1). Next $\mu(x,y)$ is defined in a similar way in D_2, and D_2 constructed at the same time, so that $\mu(D_2) = M_2$ and $U(x,y)$ for $(x,y) \in D_2$ is a solution (Figs. 11a,

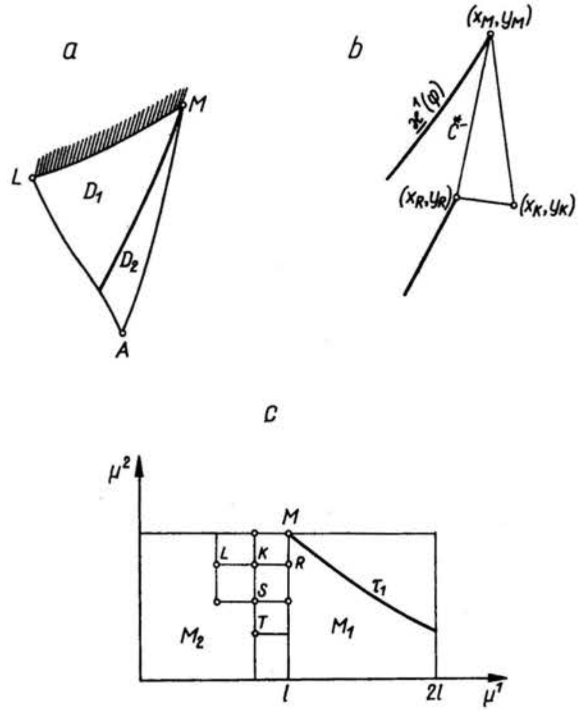

Fig. 11.

11c). The construction of the region D_1, the solution $U(x,y)$ and $(x,y) \in D_1$ is begun by prescribing the initial values of U on the curve $\varkappa^1$:

$$U(x(\varphi), y(\varphi)) = U'(\mu^1, \mu^2),$$

(4.6)
$$(\mu^1, \mu^2) \in \tau_1, \quad \varphi = \frac{1}{2}[g(\mu^1) - f(\mu^2)] + \delta - \frac{\pi}{4}.$$

It is readily observed that the entire reasoning following Corollaries 2 and 3 can now be repeated for $(\mu^1, \mu^2) \in [l, 2l] \times [0, m]$. Thus, we obtain the region D_1. In the region M_2 we introduce the characteristic net (Fig. 11c). We shall now demonstrate the method of constructing the points (x_K, y_K), $(x_L, y_L) \in D_2$ (Figs. 11b, 11c) such that $\mu(x_K, y_K) = K$, $\mu(x_L, y_L) = L$. This construction leads to the characteristic fan C^-. Since $U(x,y)$ for $(x,y) \in D_1$ is already known, we also know the values (x_M, y_M), (x_R, y_R) such that

$$\mu(x_M, y_M) = M, \quad \mu(x_R, y_R) = R.$$

Solving the algebraic system

$$(4.7)\qquad \begin{aligned} &(x-x_R)\sin\theta(K)-(y-y_R)\cos\theta(K)=0,\\ &(x-x_M)\cos\theta(K)+(y-y_M)\sin\theta(K)=0, \end{aligned}$$

we obtain the values of the coordinates (x, y) of the point (x_K, y_K). The values of the coordinates of the point L are deduced by replacing in (4.7) x_R by x_K, y_K by y_K and $\theta(K)$ by $\theta(L)$, etc. For nods S, T, etc. we use (3.4). Thus, we find D_2 and the solution U of the system (1.1), defined for $(x, y) \in D_2$.

It can readily be verified that

$$(4.8)\qquad \bar{D}_1 \cap \bar{D}_2 = \overset{*}{C}{}^-,$$

where $\overset{*}{C}{}^-$ is a characteristic C^- such that

$$U(\overset{*}{C}{}^-) \subset U'([(l, 0), (l, m)])$$

[the latter is the counterpart of the condition (3.8)]. The fulfilment of the condition (4.8) leads to the statement that we have obtained a solution $U(\mu(x, y))$ of the system (1.1) satisfying the conditions 1–4. The uniqueness of the solution is proved as before in Corollaries 2 and 3 (for given δ, ε, U and the contour of the profile).

Let us now investigate the case (4.3). On the plane E^2 we prescribe the curve $\varkappa^2$ (the contour of the tool) by means of the function:

$$(4.9)\qquad \varkappa^2(\varphi) = \big(x(\varphi_M)-x(\varphi), y(\varphi_M)-y(\varphi)\big),$$

where φ_M is defined by (4.4) and $x(\varphi)$, $y(\varphi)$, $x(\varphi_M)$, $y(\varphi_M)$ satisfy (4.4) and (4.5). Observe that the curve (4.9) is no longer convex, but "concave". The construction of the solution is the same as in the case (4.2), except that (4.6) is replaced by

$$(4.10)\qquad \begin{aligned} &U\big(\varkappa^2(\varphi)\big) = U''(\mu^1, \mu^2),\\ &(\mu^1, \mu^2) \in \tau_2, \quad \varphi = \frac{1}{2}[g(\mu^1)-f(\mu^2)]+\delta-\frac{\pi}{4}. \end{aligned}$$

In the course of the construction, we should again verify the condition analogous to (4.8). Thus we have

COROLLARY 5. *Let U be a solution of* (2.2), (2.3), (2.16), (2.17), *and let the functions f and g in* (2.16) *and* (2.17) *satisfy* (4.1) *and the assumptions of Corollary* 4. *Furthermore, let*

$$(4.11)\qquad U' = U_1^N|_{[0,2l]\times[0,m]},$$

$$(4.12)\qquad U'' = U_1^N|_{[0,l]\times[0,m]}.$$

Then, in both cases there exists a solution U of the system (1.1) *satisfying the conditions 1–4 defined by $U(x, y) = U'\big(\mu^1(x, y), \mu^2(x, y)\big)$ in the case* (4.11), and by $U(x, y) = U''\big(\mu^1(x, y), \mu^2(x, y)\big)$ *in the case* (4.12).

The first solution has two characteristic fans, while the second—the fan C^+ only. Figure 12 constitutes an example for (4.11).

So far we have considered only solutions for which the segment of the curve F_ε^δ was situated in one quarter only of the plane ξ, η. Now, we can easily remove this restriction.

Consider the solution of the system (2.2), (2.3), denoted by $\overline{U}$ and defined in the rectangle $[0, l'] \times [0, m]$, where $l' > 2l$ and l has the same value as in (4.2) (Fig. 13)

$$\overline{U}|_{[0,2l]\times[0,m]} = U' \tag{4.15}$$

(U' here is the same solution as in (4.2)). $U|_{[l,l']\times[0,m]}$ is the solution with the boundary conditions prescribed on the segments $[2l, l']$ and $[(2l, 0), (2l, m')]$, as in Corollary 1—i.e.,

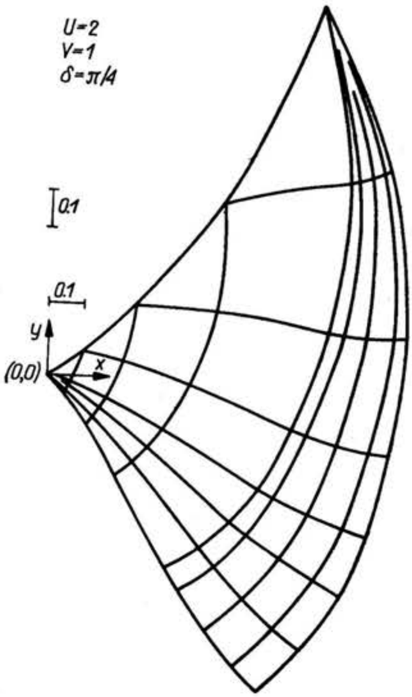

FIG. 12.

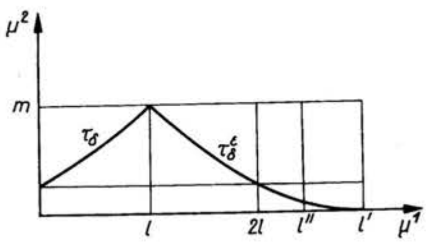

FIG. 13.

a) δ, ε have exactly the same values as in (4.15),

b) $f, g \in C^2$ on the segments considered,

c) $\xi(\mu^1) \nearrow$, $\xi(l') = \xi_\varepsilon^\delta$,

d) the function $\eta(\mu^2)$ on the segment $[(2l, 0), (2l, m')]$ has the same values as the function η appearing in Corollary 1 on the segment $[0, l]$,

e) u, v are defined by (2.16), (2.17) on the segments considered,

f) $\left.\dfrac{dg(2l)}{d\mu^1}\right|_{+0} = \left.\dfrac{dg(2l)}{d\mu^1}\right|_{-0}$.

In other words, the solution $\overline{U}$ is a smooth combination of the solution U' of (4.2) with the solution from Corollary 2.

The results of Secs. 2 and 3 indicate the manner of deducing the solution. We begin by prescribing on the plane E^2 the curve $\varkappa$,

$$\varkappa = (x(\varphi), y(\varphi)), \quad \varphi_L \leqslant \varphi \leqslant \varphi_M,$$

$$\varphi_L = \frac{1}{2}[g(l') - f(0)] + \delta - \frac{\pi}{4}, \quad \varphi_M = \frac{1}{2}[g(l) - f(m)] + \delta - \frac{\pi}{4},$$

satisfying (4.5). We assume that $U(x, y)$, the required solution of the problem for (1.1) satisfies the relation $U(x(\varphi), y(\varphi)) = \bar{U}(\mu^1, \mu^2) \in \tau_\delta^\varepsilon$, where $(\mu^1, \mu^2) \in \tau_\delta^\varepsilon$. The curve τ_δ^ε is smooth and situated in the rectangle $[l, l'] \times [0, m]$ (Fig. 13). The construction of the solution is the same as in the case (4.2). The characteristic net derived has one characteristic fan C^-. Outside the point M the solution is a regular double wave. It is readily observed that there exists a modified solution with two characteristic fans. Then, as the point of departure we take the following solution of the system (2.2), (2.3):

$$\bar{U}|_{[0, l''] \times [0, m]},$$

$\bar{U}$ having been already defined; l'' satisfies the inequality $2l < l'' < l'$ (Fig. 13). In this case, the existence of the characteristic fan C^+ follows from Sec. 3, Corollary 3. The type, of characteristic nets are in this case presented in Figs. 14 and 15. Thus, we have

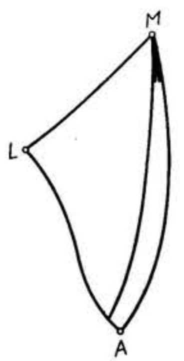

FIG. 14.

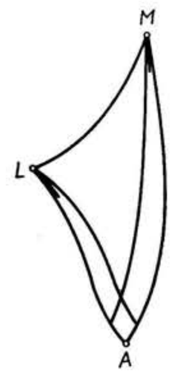

FIG. 15.

COROLLARY 6. *For every δ and ε, such that the Eq.* (2.26) *has a solution, and for every segment of the curve F_ε^δ not tangent at any point to the straight line ξ =* const *and situated in the third and fourth quarters, it is possible to construct a solution of the problem formulated for the system* (1.1).

For a given contour of the die (corresponding to the above segment of the curve F_ε^δ) ε, δ, $U((-U, 0)$ is the exit velocity of the material from the die) the solution considered is unique.

5

We now proceed to the case in which the curve F is such that it is tangent to the straight line ξ = const. This problem is described by

COROLLARY 7. *If*

a) *u, v, f, g are of class* C^2, $0 < \delta \leqslant \pi/4$, δ = const,

b) (2.9) *holds for* $(\mu^1, \mu^2) \in [0, l] \times [0, m]$,

c) $\xi(\mu^1) \nearrow$, $\eta(\mu^2) \searrow$,

d) ε *is such that the curve* F_ε^δ *is tangent to the straight line* $\xi = l$ *at the point* $(\xi(l), \eta(m''))$, $0 < m'' < m$, *then the set* τ_δ *defines a curve situated in the rectangle* $[0, l] \times [0, m]$ *and behaving as shown in Fig.* 16, *which can be described by two functions*

$$\psi'(\mu^1), \quad 0 \leqslant \mu^1 \leqslant l \quad \textit{exactly increasing},$$
$$\psi''(\mu^1), \quad l' \leqslant \mu^1 \leqslant l \quad \textit{exactly decreasing}.$$

The curve τ_δ has a tangent with the equation $\mu^1 = l$ at the point (l, m''). It is now readily observed that if we confine ourselves to the part of the curve described by the function

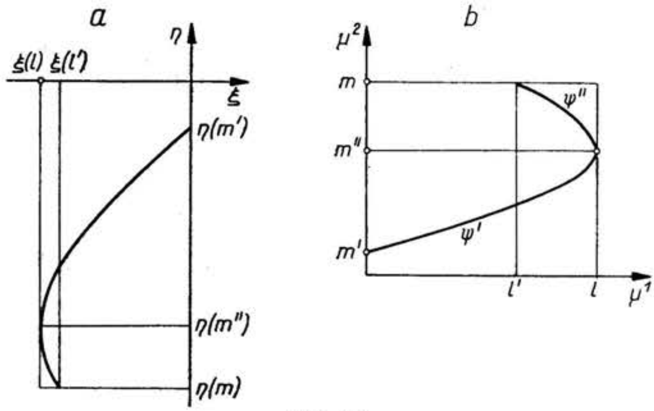

FIG. 16.

ψ'', then there exist solutions of the problem considered satisfying (2.9), provided that the existence of two characteristic fans C^+ and C^- is possible. Let $U_1(\mu^1, \mu^2)$, $(\mu^1, \mu^2) \in [0, l] \times [0, m]$ be a solution of the problem (2.2), (2.3), (2.16), (2.17) and, moreover, let the assumptions b, c, d of Corollary 1 be satisfied, $f, g \in C^2$, $\delta = \text{const}$, $0 < \delta \leqslant \pi/4$

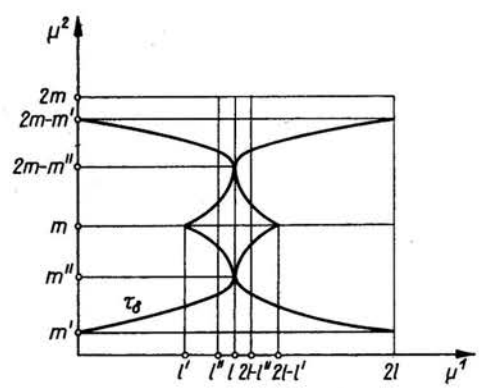

FIG. 17.

and (4.1) holds. Then U_1^N (see Sec. 4) is also a solution of (2.2), (2.3) in the rectangle P', and l_δ is parametrized by (μ^1, μ^2) constituting mirror images of the set τ_δ of Corollaty 7 (see Fig. 17). In this case, if we confine ourselves only to the behaviour of the curves parametrizing l_δ in $[0, 2l] \times [0, 2m]$, then from this point of view the following curves τ_δ

are suitable for constructing the problem stated in the introduction, as curves defining the hodographs of the tool:

—the curve passing through the points $(2l, m')$, (l, m''), (l', m),

—the curve passing through the points $(2l-l', m)$, $(l, 2m-m'')$, $(0, 2m-m')$,

—the curves constituting parts of one of the above curves.

Let us now verify whether all the above possibilities lead to a solution of the problem. Observe that the angle of inclination of the contour of the die to the x-axis has the value $\varphi = \theta+\delta$, $0 < \delta \leqslant \pi/4$. The vector tangent to the contour of the tool $\widetilde{LM}$ is $(\cos(\theta+\delta), \sin(\theta+\delta))$ and is never identical with the vector $\mathbf{c}^+$ or $\mathbf{c}^-$. Thus, the contour of the die is a non-characteristic curve, and if U denotes the solution of the problem considered, then $U(\widetilde{LM})$ is not tangent at any point to the characteristic curve $\overset{(+)}{\Gamma}$ or $\overset{(-)}{\Gamma}$—i.e., the vector tangent to the curve $U(\widetilde{LM})$ at no point has the direction of the vector $\overset{(+)}{\gamma}$ or $\overset{(-)}{\gamma}$,

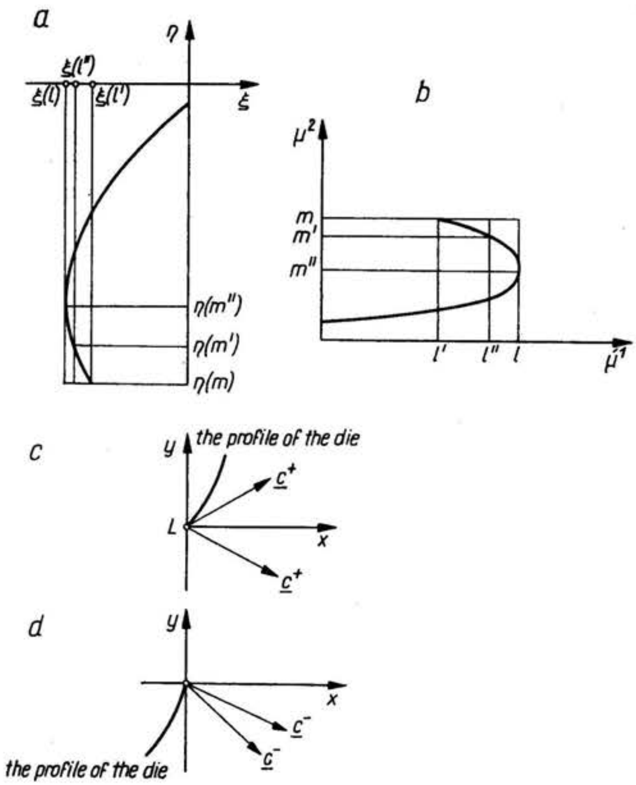

FIG. 18.

where $\overset{(+)}{\gamma}$, $\overset{(-)}{\gamma}$ are characteristic vectors in the hodograph space. It follows immediately that the above defined $U_1(\mu^1, \mu^2)$ and $U_1^N(\mu^1, \mu^2)$ for (μ^1, μ^2) belonging to $[0, l] \times [0, m]$ and $[0, 2l] \times [0, 2m]$, respectively, cannot lead to a solution of the problem. Hence, in this case the curves defined by (2.18) correspond to the streamline but are not non-characteristic. They can therefore be used only after eliminating points at which they are tangent to the straight line $\xi = \text{const}$. Thus, it is sufficient to find whether there exists a solution of the problem considered, confining ourselves to the parts of the curves τ_δ situated between the points (l, m'') and (l', m), and $(2l-l', m)$ and $(l, 2m-m'')$, without the points (l, m'') and $(l, 2m-m'')$ (Fig. 17). In both cases, there should exist a possibility of constructing two characteristic fans C^+ and C^-. If we choose the part of the curve τ_δ between the points (l, m'') and (l', m) (Figs. 18a, 18b), then the directions of the characteristic vectors at the points L and M are shown in Figs. 18c, 18d; this means that there exists a solution of the problem considered. In a similar way, we can verify the existence of the solution in the case of the choice of the part of the curve τ_δ between the points $(2l-l', m)$ and $(l, 2m-m'')$. Then the construction is the following: instead of the solution defined above U_1 in $[0, l]\times[0, m]$, we take

$$U_1' = U_1|_{[0,l'']\times[0,m]}, \quad \text{where} \quad l' < l'' < l.$$

Next, in the manner described we obtain from this solution U_1 , Fig. 18. For the hodograph of the solution we take $U_1'^N|_{[0,2l'']\times[0,2m-m']}$. The contour of the die corresponds to the curve between the points $(2l''-l', m)$ and $(l'', 2m-m')$. The directions of the characteristic vectors in this case behave as before. Thus, there exists a solution of the problem with a "concave tool" with two characteristic fans. However, it does not satisfy the condition (2.9). Thus, we have —

COROLLARY 8. a) *There do not exist solutions of the problem considered with the contour of the die having at one of its points the inclination angle* $\varphi_{cr} = \theta_{cr}+\delta$, $0 < \delta \leqslant \pi/4$, *where* θ_{cr} *is the value of the function* θ *at the point* (μ^1, μ^2) *at which the curve* τ_δ *is tangent to the characteristic* $\mu^1 = \text{const}$. b) *There exist solutions of the problem considered with the contour of the die having the inclination angles in the interval*:

$$\varphi_{cr} < \varphi \leqslant \frac{\pi}{2}.$$

There exist two types of these solutions, namely

—with a convex contour of the die; then the condition (2.9) *is satisfied,*

—with a concave contour of the die; then (2.9) *does not hold.*

Both types have two characteristic fans. For a fixed die, the hodograph of which is determined by the curve τ_δ *of Corollary* 7, ε, U *and* δ, *the above two solutions are uniquely determined.*

6

Consider now the case of a part of the curve F_ε^δ situated wholly in the first quarter

COROLLARY 9. *If*

a) $u, v, g, f \in C^2$, $0 < \delta \leqslant \pi/4$, $\delta = \text{const}$,

b) (2.9) *holds for* $(\mu^1, \mu^2) \in [0, l]\times[0, m]$,

c) $\xi(\mu^1)\nearrow$, $\eta(\mu^2)\nearrow$, $\xi, \eta \geqslant 0$,

then the set τ_δ *determines a curve situated in* $[0, l]\times[0, m]$ *and behaving as shown in Fig.* 19a. *The curve* τ_δ *in non-characteristic.*

Let U_1 denote a solution of (2.2), (2.3), (2.16), (2.17) with the boundary conditions satisfying the assumptions of this Corollary. The behaviour of the curve τ_δ does not make it possible to construct a solution of the problem satisfying (2.9). If we apply the method of continuation of the solution of (2.2), (2.3) described in Sec. 4, then the curves contained between the points $(2l-l', 0)$ and (l, m), and (l, m) and $(l', 2m)$, have the

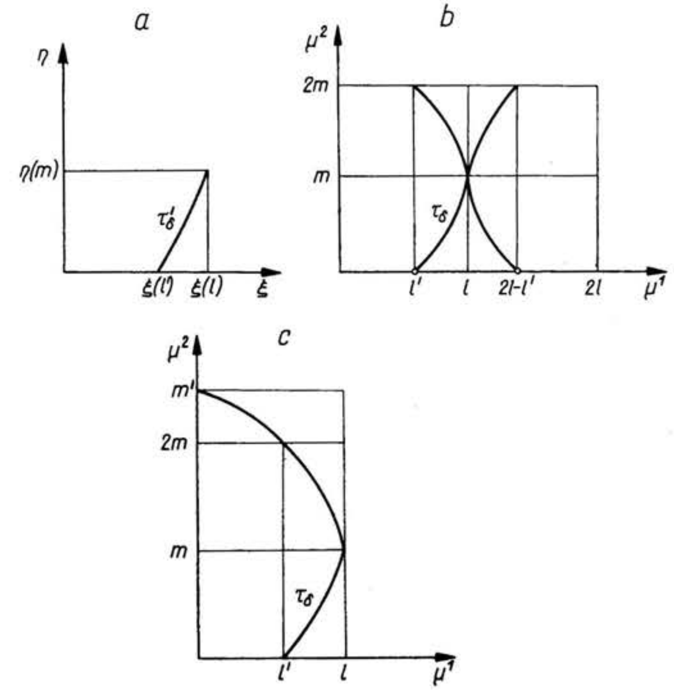

FIG. 19.

required behaviour. It can readily be verified that in both cases it is impossible to construct the characteristic fan C^-. Thus we have

COROLLARY 10. *There do not exist solutions of the problem considered with the contour of the die having inclination angles in the interval* $0 \leqslant \varphi \leqslant \varphi_P$, $\varphi_P = \theta_P + \delta$, *where* $\theta_P = \frac{1}{2}\xi_\delta^\varepsilon - \frac{\pi}{4}$ *and* ξ_δ^ε *is a solution of* (2.25).

The question arises whether there exists a solution of the problem with the contour of the die with inclination angles in the interval $0 \leqslant \varphi \leqslant \varphi_M$, where $\varphi_M > \varphi_P$ (φ_P is defined as in Corollary 10). The hodograph of this solution should be as follows (Figs. 19b, 19c).

In the rectangle $[0, l] \times [0, 2m]$, the solution U of the system (2.2) should have the form:

$$U|_{[0,l]\times[0,2m]} = U_1^N|_{[0,l]\times[0,2m]}$$

where U_1 is a solution of (2.2), (2.3), (2.16), (2.17) satisfying the assumptions of Corollary 9 in the rectangle $[(0, 2m), (l, 2m)] \times [(0, 2m), (0, m')] = P''$. The restriction $U|_{P''}$ is identical with the solution of (2.2), (2.3), (2.16), (2.17) defined in Sec. 3; this solution satisfies the assumptions of Corollary 2. An additional assumption should concern the smoothness of the functions f and g on $[0, l]$ and $[0, m']$. It is evident that there will exist a solution of the problem, provided it is possible to construct the characteristic fan C^+. The characteristic directions for this fan are defined by the values of the functions f and g on the segment $[(l, 0), (l, m)]$. Again, it can readily be verified that it is impossible to construct the fan C^+. Thus, we have

COROLLARY 11. *There do not exist solutions of the problem considered with convex contours of the die with inclination angles belonging to the interval* $0 \leqslant \varphi \leqslant \varphi_M$, $\varphi_M > \varphi_P$.

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